AMath 574 February 25, 2011

Today:

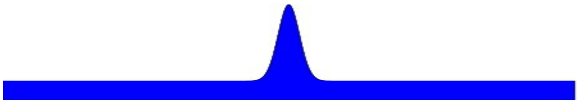
- Integral curves
- Simple waves
- Rarefaction waves
- Genuine nonlinearity
- Linear degeneracy

Monday:

- Finite volume methods
- Approximate Riemann solvers

Reading: Chapter 15

Simple waves



After separation, before shock formation:

Left- and right-going waves look like solutions to scalar equation.

Simple waves: q varies along an integral curve of $r^p(q)$.

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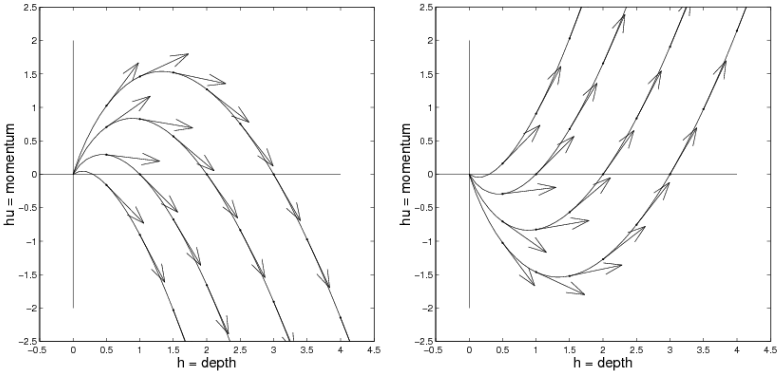
Integral curves of r^p

Curves in phase plane that are tangent to $r^p(q)$ at each q.



$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfying $\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

1-waves: integral curves of r^1

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Choose $\alpha(\xi) \equiv 1$ and obtain

$$\begin{bmatrix} (\tilde{q}^1)' \\ (\tilde{q}^2)' \end{bmatrix} = \tilde{q}'(\xi) = r^1(\tilde{q}(\xi)) = \begin{bmatrix} 1 \\ \tilde{q}^2/\tilde{q}^1 - \sqrt{g\tilde{q}^1} \end{bmatrix}$$

This is a system of 2 ODEs

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Require $\tilde{q}^2(h_*) = h_* u_* \implies$

$$\tilde{q}^2(\xi) = \xi u_* + 2\xi\left(\sqrt{gh_*} - \sqrt{g\xi}\right).$$

1-wave integral curves of r^p

So

$$\tilde{q}^1(\xi) = \xi,$$
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and hence

$$hu = hu_* + 2h\left(\sqrt{gh_*} - \sqrt{gh}\right).$$

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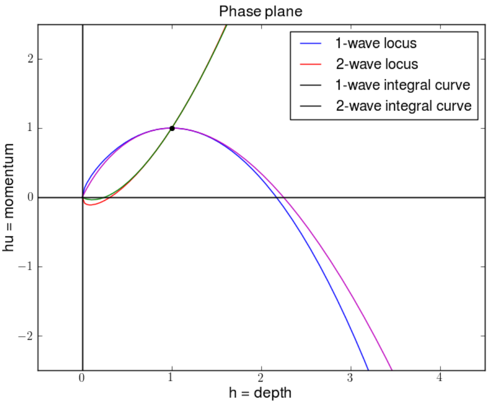
Similarly, 2-wave integral curves satisfy

$$hu = hu_* - 2h\left(\sqrt{gh_*} - \sqrt{gh}\right).$$

Integral curves of r^p versus Hugoniot loci



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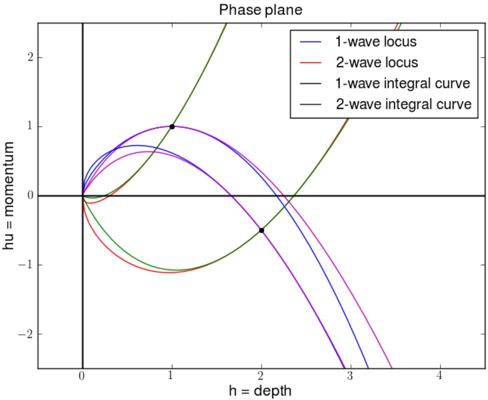

Solution to Riemann problem depends on which state is q_l, q_r.

Riemann invariants

Along a 1-wave integral curve,

$$u = u_* + 2\left(\sqrt{gh_*} - \sqrt{gh}\right)$$

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$$u + 2\sqrt{gh} = u_* + 2\sqrt{gh_*}.$$

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So at every point on the integral curve through $(h_*, h_* u_*)$

$$w^1(q) = u + 2\sqrt{gh}$$

has the constant value $w^1(q) \equiv w^1(q_*) = u + 2\sqrt{gh}$.

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The function $w^1(q)$ is a 1-Riemann invariant for this system.

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2-Riemann invariants:

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Rarefaction waves

Centered rarefaction wave:

Similarity solution with piecewise constant initial data:

$$q(x,t) = \begin{cases} q_l & \text{if } x/t \le \xi_1 \\ \tilde{q}(x/t) & \text{if } \xi_1 \le x/t \le \xi_2 \\ q_r & \text{if } x/t \ge \xi_2, \end{cases}$$

where q_l and q_r are two points on a single integral curve with $\lambda^p(q_l) < \lambda^p(q_r)$.

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Required so that characteristics spread out as time advances.

Also want $\lambda^p(q)$ monotonically increasing from q_l to q_r.

This genuine nonlinearity generalizes convexity of scalar flux.

Genuine nonlinearity

For scalar problem $q_t + f(q)_x = 0$, want $f''(q) \neq 0$ everywhere.

This implies that $f'(q)$ is monotonically increasing or decreasing between q_l and q_r.

Shock if decreasing,   Rarefaction if increasing.

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This requires $\nabla\lambda^p(q) \cdot r^p(q) \neq 0$.

Genuine nonlinearity of shallow water equations

1-waves: Requires $\nabla\lambda^1(q)\cdot r^1(q) \neq 0$.

$$\lambda^1 = u - \sqrt{gh} = q^2/q^1 - \sqrt{gq^1},$$

$$\nabla\lambda^1 = \left[\begin{array}{c} -q^2/(q^1)^2 - \frac{1}{2}\sqrt{g/q^1} \\ 1/q^1 \end{array}\right],$$

$$r^1 = \left[\begin{array}{c} 1 \\ q^2/q^1 - \sqrt{gq^1} \end{array}\right],$$

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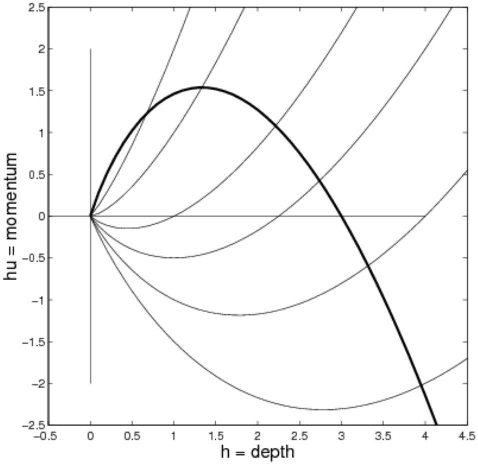
and hence

$$\nabla\lambda^1\cdot r^1 = -\frac{3}{2}\sqrt{g/q^1} = -\frac{3}{2}\sqrt{g/h}$$
$$< 0 \quad \text{for all} \;\; h > 0.$$

Genuine nonlinearity of shallow water equations

Integral curves (heavy line) and contours of λ^1:



Linearly degenerate fields

Scalar advection: $q_t + uq_x = 0$ with $u =$ constant.

Characteristics $X(t) = x_0 + ut$ are parallel.

Discontinuity propagates along a characteristic curve.

Characteristics on either side are parallel so not a shock!

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For system the analogous property arises if

$$\nabla\lambda^p(q) \cdot r^p(q) \equiv 0$$

holds for all q, in which case
λ^p is constant along each integral curve.

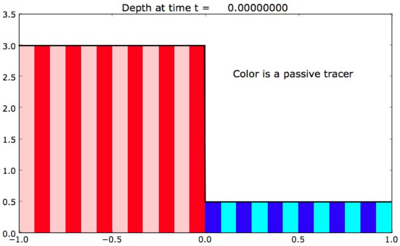
Then pth field is said to be linearly degenerate.

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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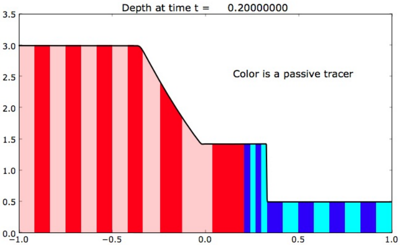

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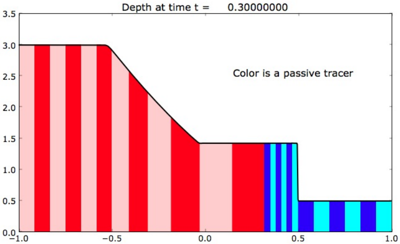

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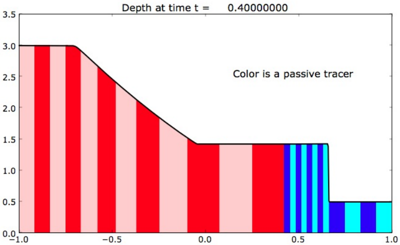

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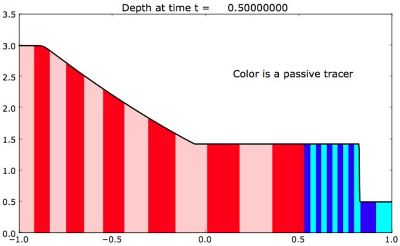

Shallow water with passive tracer

Let $\phi(x,t)$ be tracer concentration and add equation

$$\phi_t + u\phi_x = 0 \implies (h\phi)_t + (uh\phi)_x = 0.$$

Gives:

$$q = \begin{bmatrix} h \\ hu \\ h\phi \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ uh\phi \end{bmatrix} = \begin{bmatrix} q^2 \\ (q^2)/q^1 + \frac{1}{2}g(q^1)^2 \\ q^2q^3/q^1 \end{bmatrix}.$$

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Jacobian:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -u\phi & \phi & u \end{bmatrix}.$$

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$$\lambda^2 = u = (hu)/h \implies \nabla\lambda^2 = \begin{bmatrix} -u/h \\ 1/h \\ 0 \end{bmatrix} \implies \lambda^2 \cdot r^2 \equiv 0.$$

So 2nd field is linearly degenerate.

(Fields 1 and 3 are genuinely nonlinear.)