AMath 574 February 11, 2011

Today:

- Scalar nonlinear conservation laws
- Shocks and rarefaction waves
- Lax-Wendroff theorem
- Entropy conditions

Monday:

- Numerical methods and entropy functions
- Start nonlinear systems

Reading: Chapters 12, 13

Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0$.

Quasilinear form: $u_t + uu_x = 0$.

These are equivalent for smooth solutions, not for shocks!

Nonlinear scalar conservation laws

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Upwind methods for $u > 0$:

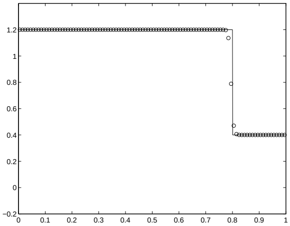
Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n)$.

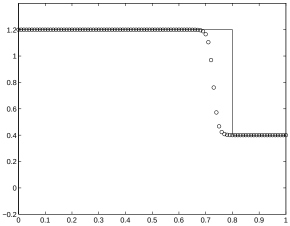
Ok for smooth solutions, not for shocks!

Importance of conservation form

Solution to Burgers' equation using conservative upwind:



Solution to Burgers' equation using quasilinear upwind:



Weak solutions depend on the conservation law

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions,

different shock speeds!

Weak solutions depend on the conservation law

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad \Longrightarrow \quad s = \frac{1}{2}\frac{u_r^2 - u_\ell^2}{u_r - u_l} = \frac{1}{2}(u_\ell + u_r).$$

whereas

$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0 \quad \Longrightarrow \quad s = \frac{2}{3}\frac{u_r^3 - u_\ell^3}{u_r - u_\ell}.$$

These speeds are different in general $\Longrightarrow$ different weak solutions.

Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \frac{\Delta t}{\Delta x}(F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

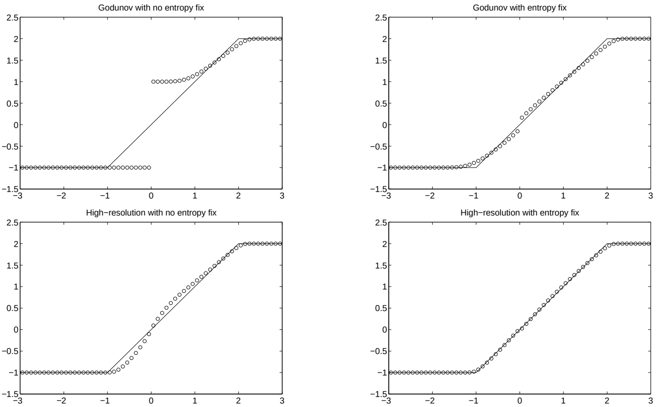
Two sequences might converge to different weak solutions.

Also need to satisfy an entropy condition.

Entropy-violating numerical solutions

Riemann problem for Burgers' equation at $t = 1$

with $u_\ell = -1$ and $u_r = 2$:



Non-uniqueness of weak solutions

For scalar problem, any jump allowed with speed:

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.$$

So even if $f'(q_r) < f'(q_l)$ the integral form of cons. law is satisfied by a discontinuity propogating at the R-H speed.

In this case there is also a rarefaction wave solution.

In fact, infinitely many weak solutions.

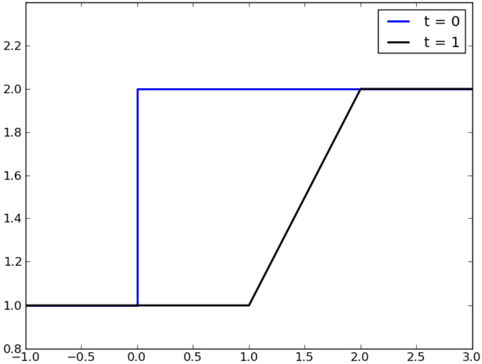
Which one is physically correct?

Weak solutions to Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

"Physically correct" rarefaction wave solution:



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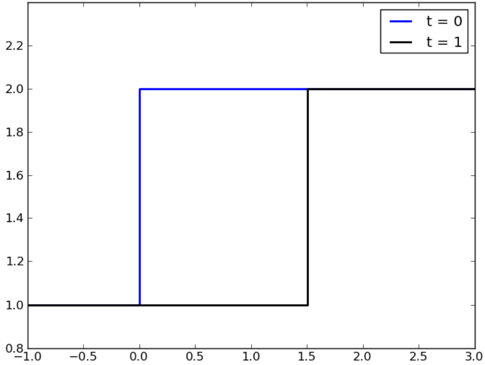
Entropy violating weak solution:



Weak solutions to Burgers' equation

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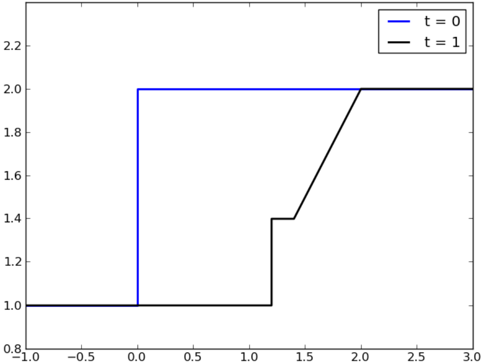
Another Entropy violating weak solution:



Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

$$q_t + f(q)_x = \epsilon q_{xx}.$$

This selects a unique weak solution:

- Shock if $f'(q_l) > f'(q_r)$,
- Rarefaction if $f'(q_l) < f'(q_r)$.

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Lax Entropy Condition:

A discontinuity propagating with speed s in the solution of a convex scalar conservation law is admissible only if $f'(q_\ell) > s > f'(q_r)$, where $s = (f(q_r) - f(q_\ell))/(q_r - q_\ell)$.

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Note: This means characteristics must approach shock from both sides as t advances, not move away from shock!

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Recall that for a linear system, $s^p = \lambda^p$ and waves $\mathcal{W}^p$ are eigenvectors.

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}^p_{i-1/2},$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}^p_{i-1/2},$$

For scalar advection $m = 1$, only one wave.
$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2} = Q_i - Q_{i-1}$ and $s = u$,

$$\mathcal{A}^-\Delta Q_{i-1/2} = u^- \mathcal{W}_{i-1/2},$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = u^+ \mathcal{W}_{i-1/2}.$$

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Define

$$\mathcal{A}^-\Delta Q_{i-1/2} = F_{i-1/2} - f(Q_{i-1}) \quad \text{left-going fluctuation}$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - F_{i-1/2} \quad \text{right-going fluctuation}$$

Then this reduces to:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right].$$

Riemann problem for scalar nonlinear problem

$q_t + f(q)_x = 0$ with data

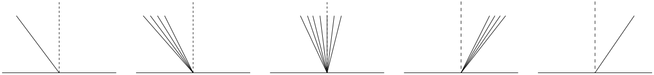
$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Piecewise constant with a single jump discontinuity.

For Burgers' or traffic flow with quadratic flux, the Riemann solution consists of:

- Shock wave if $f'(q_l) > f'(q_r)$,
- Rarefaction wave if $f'(q_l) < f'(q_r)$.

Five possible cases:



Riemann problem for scalar convex flux

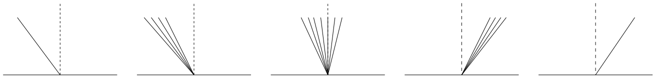
$q_t + f(q)_x = 0$ with $f''(q)$ of one sign, so $f'(q)$ is monotone.

Then f is called a convex flux function.

Then there is at most one point q_s where $f'(q_s) = 0$.

q_s is called the sonic point or stagnation point.

5 possible cases:



Case 3: $f'(q_l) < 0 < f'(q_r)$, so q_s lies between q_l and q_r.
This is a trans-sonic rarefaction.

Upwind wave-propagation algorithm

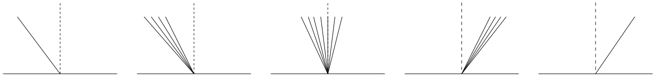
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

$$\mathcal{A}^-\Delta Q_{i-1/2} = F_{i-1/2} - f(Q_{i-1}) \quad \text{left-going fluctuation}$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - F_{i-1/2} \quad \text{right-going fluctuation}$$

For high-resolution method, we also need to define a wave $\mathcal{W}$ and speed s,

$$\mathcal{W}_{i-1/2} = Q_i - Q_{i-1},$$
$$s_{i-1/2} = \begin{cases} (f(Q_i) - f(Q_{i-1}))/(Q_i - Q_{i-1}) & \text{if } Q_{i-1} \neq Q_i \\ f'(Q_i) & \text{if } Q_{i-1} = Q_i. \end{cases}$$

Godunov flux for scalar problem



The Godunov flux function for the case $f''(q) > 0$ is

$$F^n_{i-1/2} = \begin{cases} f(Q_{i-1}) & \text{if } Q_{i-1} > q_s \text{ and } s > 0 \\ f(Q_i) & \text{if } Q_i < q_s \text{ and } s < 0 \\ f(q_s) & \text{if } Q_{i-1} < q_s < Q_i. \end{cases}$$

$$= \begin{cases} \displaystyle\min_{Q_{i-1} \le q \le Q_i} f(q) & \text{if } Q_{i-1} \le Q_i \\ \displaystyle\max_{Q_i \le q \le Q_{i-1}} f(q) & \text{if } Q_i \le Q_{i-1}, \end{cases}$$

Here $s = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}$ is the Rankine-Hugoniot shock speed.

Approximate Riemann solver

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

For scalar advection $m = 1$, only one wave.
$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2} = Q_i - Q_{i-1}$ and $s_{i-1/2} = u$,

$$\mathcal{A}^-\Delta Q_{i-1/2} = s_{i-1/2}^-\mathcal{W}_{i-1/2},$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = s_{i-1/2}^+\mathcal{W}_{i-1/2}.$$

For scalar nonlinear: Use same formulas with
$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2}$ and $s_{i-1/2} = \Delta F_{i-1/2}/\Delta Q_{i-1/2}$.

Need to modify these by an entropy fix in the trans-sonic rarefaction case.

Entropy fix

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Revert to the formulas

$$\mathcal{A}^-\Delta Q_{i-1/2} = f(q_s) - f(Q_{i-1}) \qquad \text{left-going fluctuation}$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - f(q_s) \qquad \text{right-going fluctuation}$$

if $f'(Q_{i-1}) < 0 < f'(Q_i)$.

For high-resolution method, can still define wave $\mathcal{W}$ and speed s by

$$\mathcal{W}_{i-1/2} = Q_i - Q_{i-1},$$
$$s_{i-1/2} = \begin{cases} (f(Q_i) - f(Q_{i-1}))/(Q_i - Q_{i-1}) & \text{if } Q_{i-1} \neq Q_i \\ f'(Q_i) & \text{if } Q_{i-1} = Q_i. \end{cases}$$

Entropy-violating numerical solutions

Riemann problem for Burgers' equation with $q_l = -1$ and $q_r = 2$: