AMath 574 February 7, 2011

Today:

- Wave propagation for 2d acoustics
- 2d elasticity

Wednesday:

- Nonlinear scalar conservation laws

Reading: Chapter 11

Acoustics in heterogeneous media

$$q_t + A(x,y)q_x + B(x,y)q_y = 0, \qquad q = (p,\ u,\ v)^T,$$

where

$$A = \begin{bmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{bmatrix}.$$

Note: Not in conservation form!

Acoustics in heterogeneous media

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Wave propagation still makes sense. In x-direction:

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}.$$

Wave speeds: $s^1_{i-1/2,j} = -c_{i-1,j}, \;\; s^2_{i-1/2,j} = 0, \;\; s^3_{i-1/2,j} = +c_{ij}$.

Acoustics in heterogeneous media

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Decompose $\Delta Q = (\Delta p,\ \ \Delta u,\ \ \Delta v)^T$:

$$\begin{aligned} \alpha^1_{i-1/2,j} &= (-\Delta Q^1 + Z\Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\ \alpha^2_{i-1/2,j} &= \Delta Q^3, \\ \alpha^3_{i-1/2,j} &= (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}). \end{aligned}$$

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Fluctuations: (Note: $s^1 < 0,\ s^2 = 0,\ s^3 > 0$)

$$\mathcal{A}^-\Delta Q_{i-1/2,j} = s^1_{i-1/2,j}\mathcal{W}^1_{i-1/2,j},$$

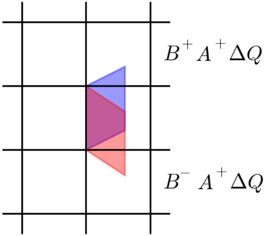
$$\mathcal{A}^+\Delta Q_{i-1/2,j} = s^3_{i-1/2,j}\mathcal{W}^3_{i-1/2,j}.$$

Acoustics in heterogeneous media

Transverse solver: Split right-going fluctuation

$$\mathcal{A}^+\Delta Q_{i-1/2,j} = s^3_{i-1/2,j}\mathcal{W}^3_{i-1/2,j}$$

into up-going and down-going pieces:



Decompose $\mathcal{A}^+\Delta Q_{i-1/2,j}$ into eigenvectors of B. Down-going:

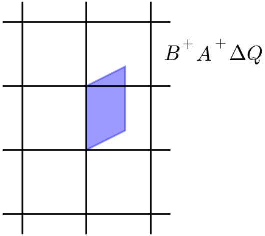
$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1\begin{bmatrix}-Z_{i,j-1}\\0\\1\end{bmatrix} + \beta^2\begin{bmatrix}0\\-1\\0\end{bmatrix} + \beta^3\begin{bmatrix}Z_{ij}\\0\\1\end{bmatrix},$$

Transverse solver for acoustics

Up-going part: $\mathcal{B}^+\mathcal{A}^+\Delta Q_{i-1/2,j} = c_{i,j+1}\beta^3 r^3$ from

$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{ij} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix},$$

$$\beta^3 = \left((\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{i,j+1}\right) / (Z_{ij} + Z_{i,j+1}).$$



Transverse Riemann solver in Clawpack

`rpt2` takes vector `asdq` and returns `bmasdq` and `bpasdq` where

`asdq` $= \mathcal{A}^* \Delta Q$ represents either
$\mathcal{A}^- \Delta Q$ if `imp = 1`, or
$\mathcal{A}^+ \Delta Q$ if `imp = 2`.

Returns $\mathcal{B}^- \mathcal{A}^* \Delta Q$ and $\mathcal{B}^+ \mathcal{A}^* \Delta Q$.

Transverse Riemann solver in Clawpack

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Returns $\mathcal{B}^-\mathcal{A}^*\Delta Q$ and $\mathcal{B}^+\mathcal{A}^*\Delta Q$.

Note: there is also a parameter `ixy`:

`ixy = 1` means normal solve was in x-direction,

`ixy = 2` means normal solve was in y-direction,
In this case `asdq` represents $\mathcal{B}^-\Delta Q$ or $\mathcal{B}^+\Delta Q$ and the routine must return $\mathcal{A}^-\mathcal{B}^*\Delta Q$ and $\mathcal{A}^+\mathcal{B}^*\Delta Q$.

Elasticity in 3d

Instead of pressure, there is a symmetric stress tensor

$$\sigma = \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{12} & \sigma^{22} & \sigma^{23} \\ \sigma^{13} & \sigma^{23} & \sigma^{33} \end{bmatrix}.$$

In a gas,

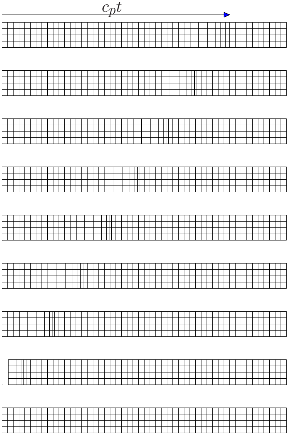
$$\sigma(x, y, z, t) = -p(x, y, z, t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This reduces to one variable, the pressure.

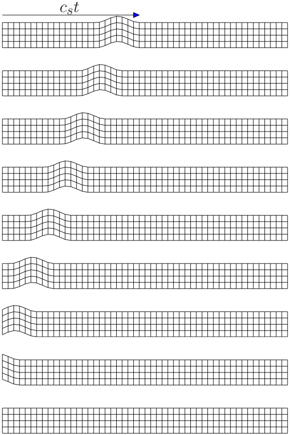
More generally compressional stress is not isotropic and there are also shear stresses that resist shear motions.

Elastic waves

P-waves



S-waves



Linear elasticity in 3d

Hyperbolic system $q_t + Aq_x + Bq_y + Cq_z = 0$ with

$$q = (\sigma^{11},\ \sigma^{22},\ \sigma^{33},\ \sigma^{12},\ \sigma^{23},\ \sigma^{13},\ u,\ v,\ w)^T$$

and, for example:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+2\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \end{bmatrix},$$

where $\rho(x,y) =$ density and $\lambda(x,y), \mu(x,y)$ are Lamé parameters that characterize the stiffness of material.

Linear elasticity in 3d

Hyperbolic system $q_t + Aq_x + Bq_y + Cq_z = 0$

The eigenvalues of $\breve{A} = n^x A + n^y B + n^z C$ are the same for any unit vector $\vec{n}$, and are given by

$$\begin{aligned} \lambda^1 &= -c_p, \quad \lambda^2 = c_p, \quad \text{P-waves} \\ \lambda^3 &= -c_s, \quad \lambda^4 = c_s, \quad \text{S-waves} \\ \lambda^5 &= -c_s, \quad \lambda^6 = c_s, \quad \text{S-waves} \\ \lambda^7 &= \lambda^8 = \lambda^9 = 0, \end{aligned}$$

P-waves: compression/expansion in direction $\vec{n}$ of propagation.

S-waves: motion in 2-dimensional plane orthogonal to $\vec{n}$.

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad > \qquad c_s = \sqrt{\frac{\mu}{\rho}}.$$