AMath 586 / ATM 581
Homework #1
Due Thursday, April 11, 2019

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/1271892/assignments/4773426`

If you haven't done so already, clone the class repository and read about how to use it on the class webpage `http://staff.washington.edu/rjl/classes/am586s2016/class_repos.html`.

Also figure out how to use Jupyter notebooks from the `notebooks` directory of the repository. See `http://staff.washington.edu/rjl/classes/am586s2016/code.html`.

Problem 1.

Check that the solution $u(t)$ given by (5.8) in the textbook satisfies the ODE (5.6) and initial condition. Hint: To differentiate the matrix exponential you can differentiate the Taylor series (D.31) (in Appendix D) term by term.

Problem 2.

Consider the IVP

$$\begin{aligned} u_1' &= 2u_1, \\ u_2' &= 3u_1 - u_2, \end{aligned}$$

with initial conditions specified at time $t = 0$. Solve this problem in two different ways:

(a) Solve the first equation, which only involves u_1, and then insert this function into the second equation to obtain a nonhomogeneous linear equation for u_2. Solve this using (5.8). Check that your solution satisfies the initial conditions and the ODE.

(b) Write the system as $u' = Au$ and compute the matrix exponential using (D.30) to obtain the solution.

Problem 3.

Consider the IVP

$$\begin{aligned} u_1' &= 2u_1, \\ u_2' &= 3u_1 + 2u_2, \end{aligned}$$

with initial conditions specified at time $t = 0$. Solve this problem in two different ways:

(a) Solve the first equation, which only involves u_1, and then insert this function into the second equation to obtain a nonhomogeneous linear equation for u_2. Solve this using (5.8). Check that your solution satisfies the initial conditions and the ODE.

(b) Write the system as $u' = Au$ and compute the matrix exponential using (D.35) to obtain the solution. (See Appendix C.3 for a discussion of the Jordan Canonical form in the defective case.)

Problem 4.

Section 2.16 of the textbook describes the ODE modeling a simple pendulum, given in (2.74):

$$\theta''(t) = -(g/L)\sin(\theta(t)). \tag{1}$$

with initial data on the angle $\theta(0) = \theta_0$ and the angular velocity $\theta'(0) = v_0$.

Recall that θ is the angle from vertical, with $\theta = 0$ (or more generally $\theta = 2\pi N$ for any integer N) corresponding straight down. With velocity $v = 0$ these points are stable equilibria. The values $\theta = (2N+1)\pi$ correspond to straight up and are unstable equilibria is $v = 0$. (Stable in the sense of ODE theory — if the data is perturbed by a small amount then the solution stays close to the original solution.)

For $\theta(0)$ and $\theta'(0)$ sufficiently small this can be approximated by the linearized equation

$$\theta''(t) = -(g/L)\theta(t).$$

Rewrite this as a first order system of two equations and solve using the matrix exponential approach. (You probably already know what the solution looks like in terms of sine and cosine, and you should see this come out of complex exponentials since the matrix has imaginary eigenvalues in this case.)

Problem 5.

Rewrite the nonlinear pendulum equation (1) as a first order system of two equations and set up a solver for this problem in a Jupyter notebook using `scipy.integrate.solve_ivp`, following the example in the notebook `Example_using_solve_ivp` from the class repository.

Develop the notebook to illustrate the following tests, including some commentary about what you observe.

Use values $g = 9.81$ m/s^2 and $L = 1$ meter.

(a) Show that for $\theta(0)$ sufficiently small and $\theta'(0) = 0$ the solution computed with `solve_ivp` agrees well with the exact solution of the linearized equation, as a test that you have implemented things properly.

(b) Try $\theta(0) = 3.1$, $\theta'(0) = 0$ and solve up to time $T = 15$ using the `RK45` method with the default error tolerance. Note that the pendulum is close to vertical initially. Comment on whether the solution seems reasonable.

(c) Try other values of $\theta(0)$ that are close to π (also values slightly larger than π) and comment on your results.

(d) Try setting `rtol` to a smaller value than the default and comment on whether you can get reasonable solutions with sufficiently small tolerance.

Problem 6.

Determine the coefficients $\beta_0,\ \beta_1,\ \beta_2$ for the third order, 2-step Adams-Moulton method. Do this in two different ways:

(a) Using the expression for the local truncation error in Section 5.9.1,

(b) Using the relation

$$u(t_{n+2}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f(u(s))\,ds.$$

Interpolate a quadratic polynomial $p(t)$ through the three values $f(U^n),\ f(U^{n+1})$ and $f(U^{n+2})$ and then integrate this polynomial exactly to obtain the formula. The coefficients of the polynomial will depend on the three values $f(U^{n+j})$. It's easiest to use the "Newton form" of the interpolating polynomial and consider the three times $t_n = -k$, $t_{n+1} = 0$, and $t_{n+2} = k$ so that $p(t)$ has the form

$$p(t) = A + B(t+k) + C(t+k)t$$

where $A,\ B$, and C are the appropriate divided differences based on the data. Then integrate from 0 to k. (The method has the same coefficients at any time, so this is valid.)