AMath 586 / ATM 581
Homework #4
Due Tuesday, May 10, 2016

Name: Your name here

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see https://canvas.uw.edu/courses/962872/assignments/3270587

Problem 1

Consider the following method for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}).$$

(a) Determine the formal order of accuracy of this method (in both space and time) based on computing the local truncation error.

(b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. Show that this method fails to be Lax-Richtmyer stable for any choice of α.

Do this in two ways:

- Consider the MOL interpretation and the stability region of the time-discretization being used.
- Use von Neumann analysis and solve a quadratic equation for $g(\xi)$.

(c) What if we take $k = \alpha h^3$ for some fixed $\alpha > 0$ and refine the grid. Would this method be convergent?

Problem 2

(a) Suppose we want to approximate the fourth derivative u_{xxxx} numerically. One approach is to apply the second derivative operator twice, i.e., if

$$D_2 U_i = \frac{1}{h^2}(U_{i-1} - 2U_i + U_{i+1})$$

then use

$$u_{xxxx}(x_i) \approx \frac{1}{h^2}(D_2 U_{i-1} - 2D_2 U_i + D_2 U_{i+1}).$$

When you write these terms out and combine them, this gives a finite difference approximation that is a linear combination of five values $U_{i-2},\ U_{i-1},\ U_i,\ U_{i+1},\ U_{i+2}$. Write out this approximation.

(b) The notebook `notebooks/vxx.ipynb` from the class repository illustrates how to set up a matrix A corresponding to the centered second difference operator, both for Dirichlet boundary conditions and for periodic boundary conditions. Adapt the *periodic case* to test out this approximation to u_{xxxx}. Note that the matrix will now be pentadiagonal (5 nonzero diagonals) and will also have more terms in the corners that arise from periodicity. Test the accuracy on the periodic function $v(x) = \sin^2(2\pi x)$ for which it's not too bad to compute the exact fourth derivative. You should observe second order accuracy.

(c) Determine analytically the eigenvalues of this matrix. Recall that since it is a circulant matrix, the $(m+1)\times(m+1)$ version of this matrix with $h = 1/(m+1)$ will have eigenvectors u^p with components $u_j^p = e^{2\pi i p j h}$. Note that since the matrix is symmetric it should have real eigenvalues, so simplify the eigenvalue expressions to make this clear.

(d) Check that for fixed p the eigenvalue λ_p of the matrix agrees to $\mathcal{O}(h^2)$ with the eigenvalue $(2\pi p)^4$ of the differential operator ∂_x^4 on the interval $[0, 1]$ with periodic boundary conditions.

(e) Suppose we want to solve the equation $u_t = -\kappa u_{xxxx}$ for some $\kappa > 0$ and we use the difference approximation derived above for the spatial term but then apply Forward Euler in time to the resulting MOL system. What is the stability restriction relating k and h that would have to be satisfied for the method to be convergent?

Problem 4

(a) The notebook `notebooks/HeatEquation_CN.ipynb` from the class repository solves the heat equation $u_t = \kappa u_{xx}$ (with $\kappa = 0.02$) using the Crank-Nicolson method.

Modify the notebook to also produce a log-log plot of the error versus h with the timestep k chosen to be $k = 2h$ (this is slightly different than the test currently performed in the notebook).

(b) Modify this notebook to produce a new version `HeatEquation_TRBDF2.ipynb` implements the TR-BDF2 method on the same problem. Test it to confirm that it is also second order accurate. Explain how you determined the proper boundary conditions in each stage of this Runge-Kutta method.

(c) Modify the code to produce a new notebook `HeatEquation_FE.ipynb` that implements the forward Euler explicit method on the same problem. Test it to confirm that it is $\mathcal{O}(h^2)$ accurate as $h \to 0$ provided when $k = 24h^2$ is used, which is within the stability limit for $\kappa = 0.02$. Note how many more time steps are required than with Crank-Nicolson or TR-BDF2, especially on finer grids.

(d) Test the forward Euler method with $k = 26h^2$, for which it should be unstable. Note that the instability does not become apparent until about time 4.5 for the parameter values $\kappa = 0.02$, $m = 39$, $\beta = 150$. Explain why the instability takes several hundred time steps to appear, and why it appears as a sawtooth oscillation.

Hint: What wave numbers ξ are growing exponentially for these parameter values? What is the initial magnitude of the most unstable eigenmode in the given initial data? The expression (E.30) in the book for the Fourier transform of a Gaussian may be useful.