AMath 586 / ATM 581
Homework #4
Due Wednesday, May 13, 2015

Name: Your name here

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see https://canvas.uw.edu/courses/962872/assignments/2860108

Problem 1

Consider the following method for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}).$$

(a) Determine the formal order of accuracy of this method (in both space and time) based on computing the local truncation error.

(b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. Show that this method fails to be Lax-Richtmyer stable for any choice of α.

Do this in two ways:

- Consider the MOL interpretation and the stability region of the time-discretization being used.
- Use von Neumann analysis and solve a quadratic equation for $g(\xi)$.

(c) What if we take $k = \alpha h^3$ for some fixed $\alpha > 0$ and refine the grid. Would this method be stable?

Problem 2

Consider the PDE

$$u_t = \kappa u_{xx} - \gamma u, \tag{1}$$

which models diffusion combined with decay provided $\kappa > 0$ and $\gamma > 0$. Consider methods of the form

$$U_j^{n+1} = U_j^n + \frac{k\kappa}{2h^2}[U_{j-1}^n - 2U_j^n + U_{j+1}^n + U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}] - k\gamma[(1-\theta)U_j^n + \theta U_j^{n+1}] \tag{2}$$

where θ is a parameter. In particular, if $\theta = 1/2$ then the decay term is modeled with the same centered-in-time approach as the diffusion term and the method can be obtained by applying the Trapezoidal method to the MOL formulation of the PDE. If $\theta = 0$ then the decay term is handled explicitly. For more general reaction-diffusion equations it may be advantageous to handle the reaction terms explicitly since these terms are generally nonlinear, so making them implicit would require solving nonlinear systems in each time step (whereas handling the diffusion term implicitly only gives a linear system to solve in each time step).

(a) By computing the local truncation error, show that this method is $\mathcal{O}(k^p + h^2)$ accurate, where $p = 2$ if $\theta = 1/2$ and $p = 1$ otherwise.

(b) Using von Neumann analysis, show that this method is unconditionally stable if $\theta \geq 1/2$.

(c) Show that if $\theta = 0$ then the method is stable provided $k \leq 2/\gamma$, independent of h.

Problem 3

(a) The m-file `heat_CN.m` from the book repository http://faculty.washington.edu/rjl/fdmbook/ solves the heat equation $u_t = \kappa u_{xx}$ (with $\kappa = 0.02$) using the Crank-Nicolson method. A Python version is available in `$AM586/codes/heat_CN.py`.

Run this code, and by changing the number of grid points, confirm that it is second-order accurate. (Observe how the error at some fixed time such as $T = 1$ behaves as k and h go to zero with a fixed relation between k and h, such as $k = 4h$.)

Note that in order for the time step to evenly define the time interval the way this code is set up, you need to specify values such as $m = 19, 39, 79$. (The number of interior grid points.)

Produce a log-log plot of the error versus h.

(b) Modify this code to produce a new version that implements the TR-BDF2 method on the same problem. Test it to confirm that it is also second order accurate. Explain how you determined the proper boundary conditions in each stage of this Runge-Kutta method.

(c) Modify the code to produce a new m-file or Python code `heat_FE` that implements the forward Euler explicit method on the same problem. Test it to confirm that it is $\mathcal{O}(h^2)$ accurate as $h \to 0$ provided when $k = 24h^2$ is used, which is within the stability limit for $\kappa = 0.02$. Note how many more time steps are required than with Crank-Nicolson or TR-BDF2, especially on finer grids.

(d) Test `heat_FE` with $k = 26h^2$, for which it should be unstable. Note that the instability does not become apparent until about time 4.5 for the parameter values $\kappa = 0.02$, $m = 39$, $\beta = 150$. Explain why the instability takes several hundred time steps to appear, and why it appears as a sawtooth oscillation.

Hint: What wave numbers ξ are growing exponentially for these parameter values? What is the initial magnitude of the most unstable eigenmode in the given initial data? The expression (E.30) for the Fourier transform of a Gaussian may be useful.

Problem 4

(a) Modify `heat_CN.m` (or `heat_CN.py`) to solve the heat equation for $-1 \le x \le 1$ with step function initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0. \end{cases} \tag{3}$$

With appropriate Dirichlet boundary conditions, the exact solution is

$$u(x,t) = \frac{1}{2}\,\text{erfc}\left(x/\sqrt{4\kappa t}\right) \tag{4}$$

for $t > 0$, where erfc is the *complementary error function*

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\, dz.$$

Note that `erfc` is a built-in Matlab function and in Python you could use `scipy.special.erfc`.

(i) Use $\kappa = 0.02$ and test this method using $m = 39$ and $k = 10h$ for $-1 \le x \le 1$. Note that there is an initial rapid transient decay of the high wave numbers that is not captured well with this size time step.

(ii) How small do you need to take the time step to get reasonable results? For a suitably small time step, explain why you get much better results by using $m = 38$ than $m = 39$. What is the observed order of accuracy as $k \to 0$ when $k = \alpha h$ with α suitably small and m even?

Hint: You might also see what you get if you redefine the initial data so that $u(0, 0) = 1/2$ with $u(x, 0)$ the same everywhere else, rather than $u(0, 0) = 0$.

(b) Modify your TR-BDF2 solver from Problem 3 to solve the heat equation for $-1 \leq x \leq 1$ with step function initial data as above. Test this routine using $k = 10h$ and estimate the order of accuracy as $k \to 0$ with m even. Why does the TR-BDF2 method work better than Crank-Nicolson?