AMath 586 / ATM 581
Homework #2
Due Thursday, April 16, 2015

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/962872/assignments/2845007`

Problem 1 (Corrected 11 April 2015)

The proof of convergence of 1-step methods in Section 6.3 shows that the global error goes to zero as $k \to 0$. However, this bound may be totally useless in estimating the actual error for a practical calculation.

For example, suppose we solve $u' = -10u$ with $u(0) = 1$ up to time $T = 10$, so the true solution is $u(T) = e^{-100} \approx 3.7 \times 10^{-44}$. Using forward Euler with a time step $k = 0.01$, the computed solution is $U^N = (.9)^{1000} \approx 1.75 \times 10^{-46}$, and so $E^N \approx 3.7 \times 10^{-44}$. Since $L = 10$ for this problem, the error bound (6.16) gives

$$\|E^N\| \le e^{100} \cdot 10 \cdot \|\tau\|_\infty \approx 2.7 \times 10^{44} \|\tau\|_\infty. \tag{1}$$

Here $\|\tau\|_\infty = |\tau^0| \approx 50k$, so this upper bound on the error does go to zero as $k \to 0$, but obviously it is not a realistic estimate of the error. It is too large by a factor of about 10^{88}.

The problem is that the estimate (6.16) is based on the Lipschitz constant $L = |\lambda|$, which gives a bound that grows exponentially in time even when the true and computed solutions are decaying exponentially.

(a) Determine the computed solution and error bound (6.16) for the problem $u' = 10u$ with $u(0) = 1$ up to time $T = 10$. *Correction: Ignore the following statement, since it's not correct...* Note that the error bound is the same as in the case above, but now it is a reasonable estimate of the actual error.

(b) A more realistic error bound for the case where $\lambda < 0$ can be obtained by rewriting (6.17) as

$$U^{n+1} = \Phi(U^n)$$

and then determining the Lipschitz constant for the function Φ. Call this constant M. Prove that if $M \le 1$ and $E^0 = 0$ then

$$|E^n| \le T\|\tau\|_\infty$$

for $nk \le T$, a bound that is similar to (6.16) but without the exponential term.

(c) Show that for forward Euler applied to $u' = \lambda u$ we can take $M = |1 + k\lambda|$. Determine M for the case $\lambda = -10$ and $k = 0.01$ and use this in the bound from part (b). Note that this is much better than the bound (1). *But note that it's still not a very sharp bound.*

Problem 2

Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

(a) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$

(b) $U^{n+1} = U^n$

(c) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$

(d) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

Problem 3

(a) Determine the general solution to the linear difference equation $2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = 0$.

Hint: One root of the characteristic polynomial is at $\zeta = 1$.

(b) Determine the solution to this difference equation with the starting values $U^0 = 11$, $U^1 = 5$, and $U^2 = 1$. What is U^{10}?

(c) Consider the LMM

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = k(\beta_0 f(U^n) + \beta_1 f(U^{n+1})).$$

For what values of β_0 and β_1 is local truncation error $\mathcal{O}(k^2)$?

(d) Suppose you use the values of β_0 and β_1 just determined in this LMM. Is this a convergent method?

Problem 4

Consider the midpoint method $U^{n+1} = U^{n-1} + 2kf(U^n)$ applied to the test problem $u' = \lambda u$. The method is zero-stable and second order accurate, and hence convergent. If $\lambda < 0$ then the true solution is exponentially decaying.

On the other hand, for $\lambda < 0$ and $k > 0$ the point $z = k\lambda$ is never in the region of absolute stability of this method (see Example 7.7), and hence it seems that the numerical solution should be growing exponentially for any nonzero time step. (And yet it converges to a function that is exponentially decaying.)

(a) Suppose we take $U^0 = \eta$, use Forward Euler to generate U^1, and then use the midpoint method for $n = 2,\ 3,\ \ldots$. Work out the exact solution U^n by solving the linear difference equation and explain how the apparent paradox described above is resolved.

(b) Devise some numerical experiments to illustrate the resolution of the paradox.

Problem 5

Perform numerical experiments to confirm the claim made in Example 7.10.