Conservation Laws and Finite Volume Methods

AMath 586
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Godunov and limiter methods

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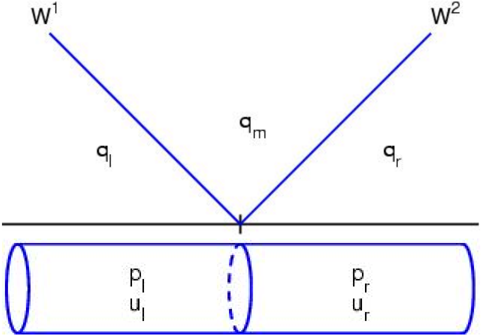
Riemann Problem for acoustics

Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

The Riemann problem for advection

The Riemann problem for the advection equation $q_t + uq_x = 0$ with

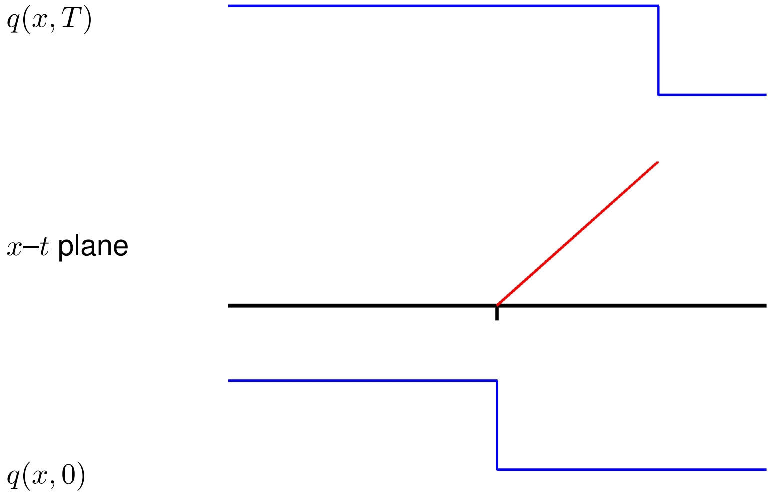
$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

has solution

$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x \geq ut \end{cases}$$

consisting of a single wave of strength $\mathcal{W}^1 = q_r - q_l$ propagating with speed $s^1 = u$.

Riemann solution for advection



Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

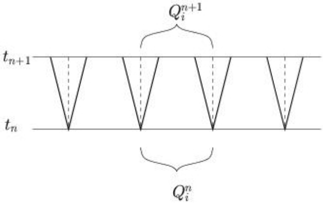
Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

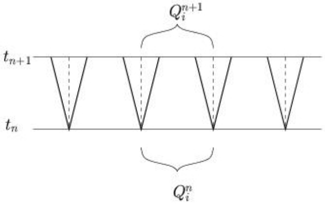
Godunov's Method for $q_t + f(q)_x = 0$



1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

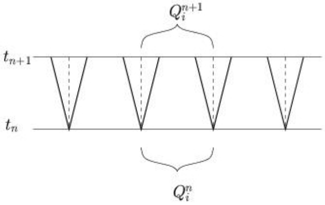
Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},

Godunov's Method for $q_t + f(q)_x = 0$

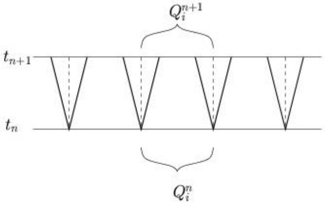


Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

First-order REA Algorithm

1. **Reconstruct** a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

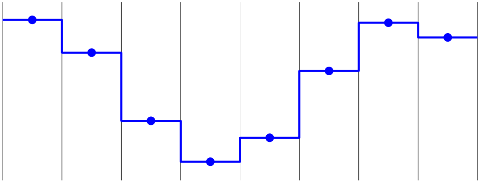
2. **Evolve** the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. **Average** this function over each grid cell to obtain new cell averages

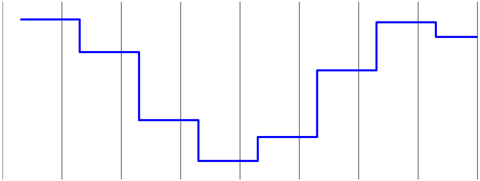
$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

First-order REA Algorithm

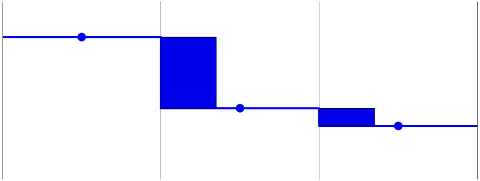
Cell averages and piecewise constant reconstruction:



After evolution:



Cell update



The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n).$$

Second-order REA Algorithm

1. Reconstruct a piecewise linear function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i) \quad \text{for all } x \in \mathcal{C}_i.$$

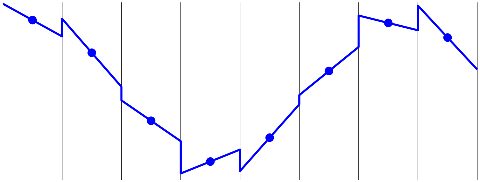
2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

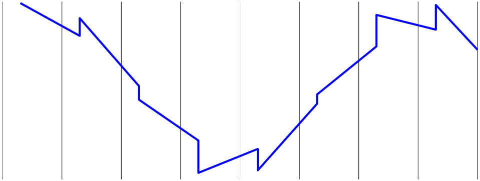
$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

Second-order REA Algorithm

Cell averages and piecewise linear reconstruction:



After evolution:



Choice of slopes

$$\tilde{Q}^n(x,t_n) = Q_i^n + \sigma_i^n(x - x_i) \qquad \text{for } x_{i-1/2} \le x < x_{i+1/2}.$$

Applying REA algorithm gives:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - \bar{u}\Delta t\right)\left(\sigma_i^n - \sigma_{i-1}^n\right)$$

Choice of slopes:

Centered slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x}$ (Fromm)

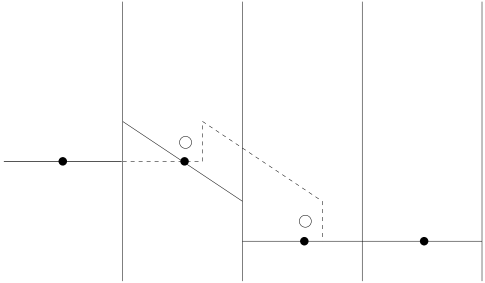
Upwind slope: $\sigma_i^n = \dfrac{Q_i^n - Q_{i-1}^n}{\Delta x}$ (Beam-Warming)

Downwind slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_i^n}{\Delta x}$ (Lax-Wendroff)

Oscillations

Any of these slope choices will give oscillations near discontinuities.

Ex: Lax-Wendroff:



High-resolution methods

Want to use slope where solution is smooth for “second-order” accuracy.

Where solution is not smooth, adding slope corrections gives oscillations.

Limit the slope based on the behavior of the solution.

$$\sigma_i^n = \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\Phi_i^n.$$

$\Phi = 1 \implies$ Lax-Wendroff,

$\Phi = 0 \implies$ upwind.

Minmod slope

$$\text{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |b| < |a| \text{ and } ab > 0 \\ 0 & \text{if } ab \leq 0 \end{cases}$$

Slope:

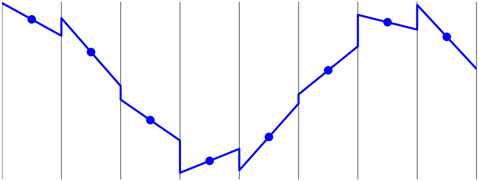
$$\begin{aligned} \sigma_i^n &= \text{minmod}((Q_i^n - Q_{i-1}^n)/\Delta x,\ \ (Q_{i+1}^n - Q_i^n)/\Delta x) \\ &= \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\Phi(\theta_i^n) \end{aligned}$$

where

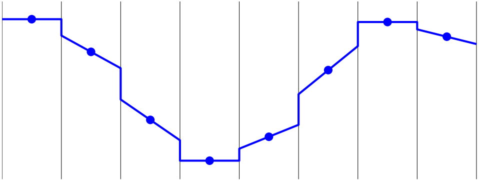
$$\begin{aligned} \theta_i^n &= \frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n} \\ \Phi(\theta) &= \text{minmod}(\theta, 1) \end{aligned}$$

Piecewise linear reconstructions

Lax-Wendroff reconstruction:



Minmod reconstruction:



TVD Methods

Total variation:

$$TV(Q) = \sum_i |Q_i - Q_{i-1}|$$

For a function, $TV(q) = \int |q_x(x)|\,dx$.

A method is Total Variation Diminishing (TVD) if

$$TV(Q^{n+1}) \le TV(Q^n).$$

If Q^n is monotone, then so is Q^{n+1}.

No spurious oscillations generated.

Gives a form of stability useful for proving convergence, also for nonlinear scalar conservation laws.

TVD REA Algorithm

1. Reconstruct a piecewise linear function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \quad \text{for all } x \in \mathcal{C}_i$$

   with the property that $TV(\tilde{q}^n) \leq TV(Q^n)$.

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1}) \, dx.$$

Note: Steps 2 and 3 are always TVD.

Some popular limiters

Linear methods:

$$\begin{aligned} \text{upwind}: &\quad \phi(\theta) = 0 \\ \text{Lax-Wendroff}: &\quad \phi(\theta) = 1 \\ \text{Beam-Warming}: &\quad \phi(\theta) = \theta \\ \text{Fromm}: &\quad \phi(\theta) = \frac{1}{2}(1+\theta) \end{aligned}$$

High-resolution limiters:

$$\begin{aligned} \text{minmod}: &\quad \phi(\theta) = \text{minmod}(1,\theta) \\ \text{superbee}: &\quad \phi(\theta) = \max(0,\ \min(1,2\theta),\ \min(2,\theta)) \\ \text{MC}: &\quad \phi(\theta) = \max(0,\ \min((1+\theta)/2,\ 2,\ 2\theta)) \\ \text{van Leer}: &\quad \phi(\theta) = \frac{\theta+|\theta|}{1+|\theta|} \end{aligned}$$

Extensions

These methods extend naturally to:

Linear systems of equations:

Solve Riemann problem to decompose each jump into waves,
Apply same technique to each wave.

Nonlinear problems:

Use approximate Riemann solver to decompose jump,
Apply same technique to each wave.

Multidimensional problems:

Waves propagate normal to interfaces,
Can add in transverse propagation.