Conservation Laws and Finite Volume Methods

AMath 586
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Burgers' equation and Riemann problems

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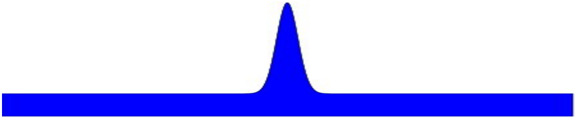
Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



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Computational challenges!

Need to capture sharp discontinuities.

PDE breaks down, standard finite difference approximation to $q_t + f(q)_x = 0$ can fail badly: nonphysical oscillations, convergence to wrong weak solution.

Characteristics for a scalar problem

$q_t + f(q)_x = 0 \implies q_t + f'(q) q_x = 0$ (if solution is smooth).

Characteristic curves satisfy $X'(t) = f'(q(X(t),t)), \;\; X(0) = x_0$.

How does solution vary along this curve?

$$
\begin{aligned}
\frac{d}{dt} q(X(t),t) &= q_x(X(t),t) X'(t) + q_t(X(t),t) \\
&= q_x(X(t),t) f(q(X(t),t)) + q_t(X(t),t) \\
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$q(X(t), t) =$ constant $\implies X'(t)$ is constant on characteristic,
so characteristics are straight lines!

Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $u_t + uu_x = 0.$

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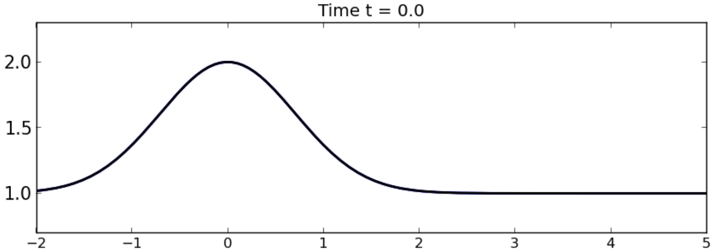
This looks like an advection equation with u advected with speed u.

True solution: u is constant along characteristic with speed $f'(u) = u$ until the wave "breaks" (shock forms).

Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

The solution is constant on characteristics so each value advects at constant speed equal to the value...



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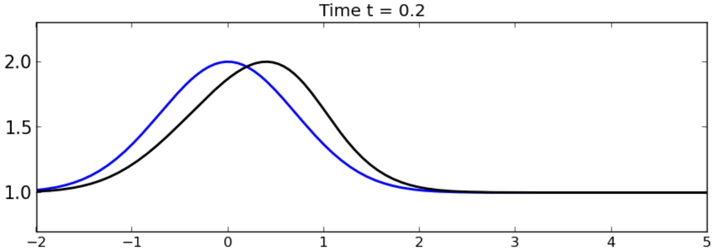

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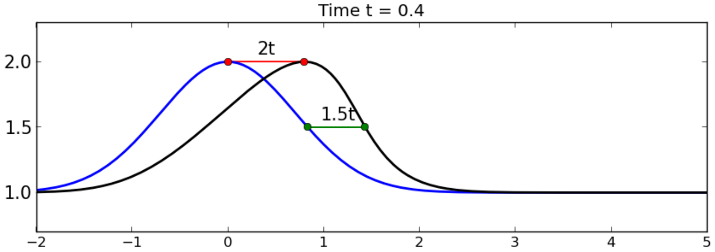

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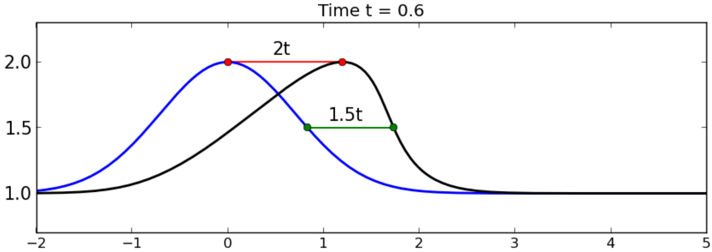

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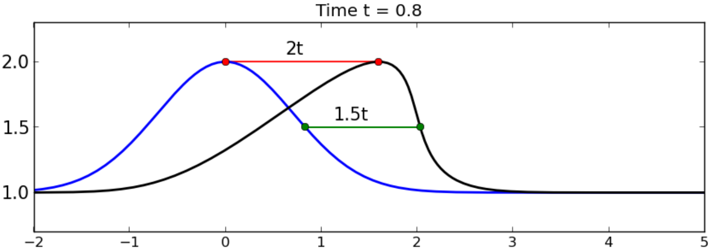

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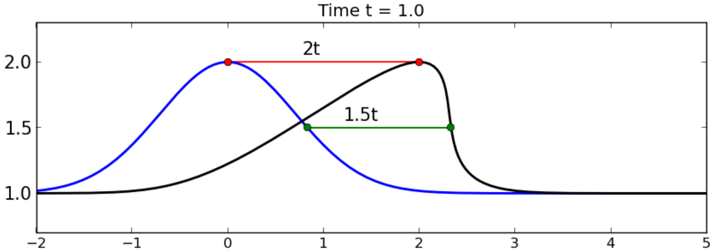

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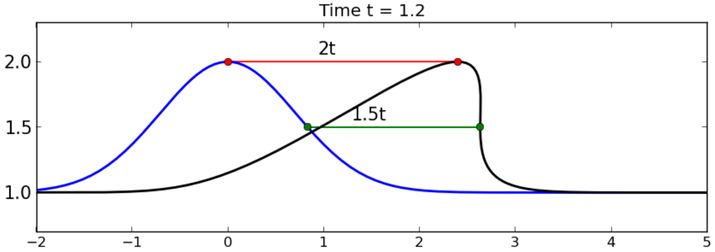

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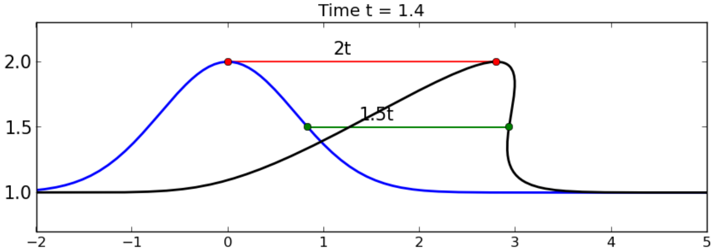

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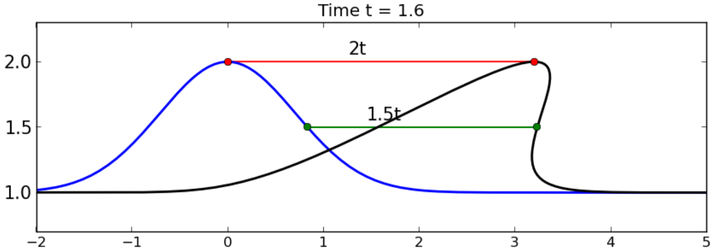

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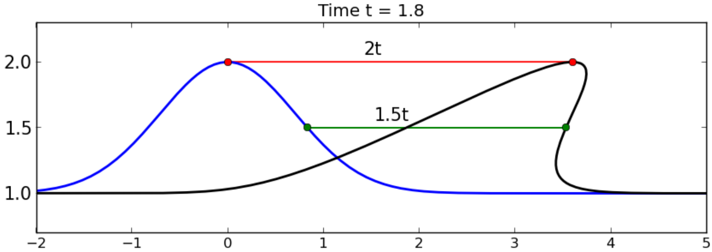

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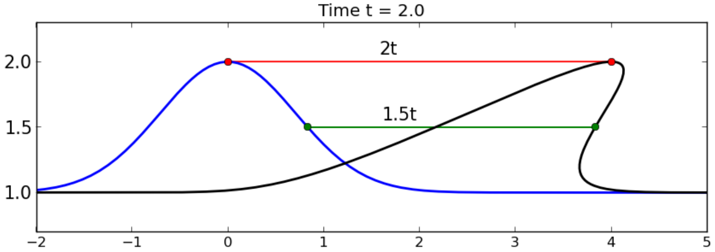
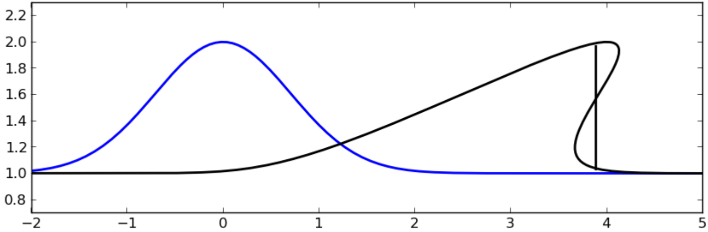


Burgers' equation

Equal-area rule:

The area "under" the curve is conserved with time,

We must insert a shock so the two areas cut off are equal.



Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}$.

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

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Why try to solve hyperbolic equation?

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Why try to solve hyperbolic equation?

- Solving parabolic equation requires implicit method,
- Often correct value of physical "viscosity" is very small, shock profile that cannot be resolved on the desired grid
  $\Longrightarrow$ smoothness of exact solution doesn't help!

The Riemann problem for advection

The Riemann problem for the advection equation $q_t + uq_x = 0$ with

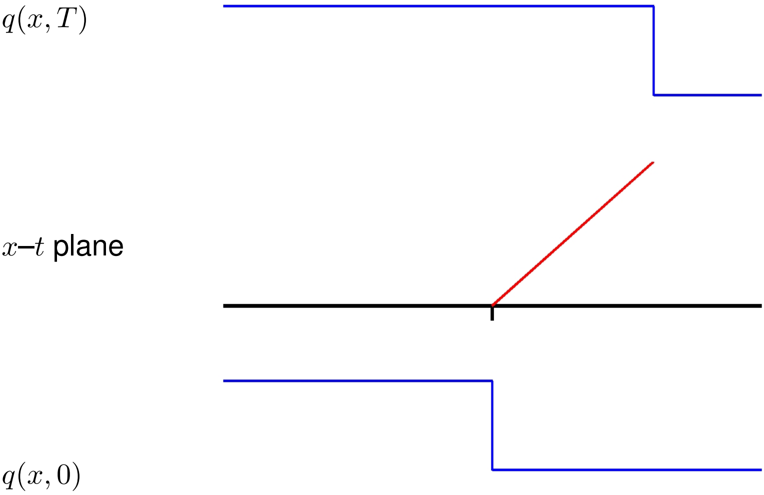
$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

has solution

$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x \geq ut \end{cases}$$

consisting of a single wave of strength $\mathcal{W}^1 = q_r - q_l$ propagating with speed $s^1 = u$.

Riemann solution for advection



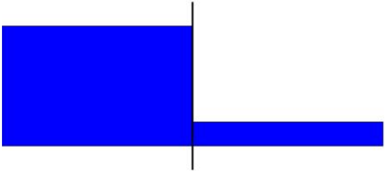
Riemann Problem

Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram

Pressure:



Acoustic waves propagate with speeds $\pm c$.

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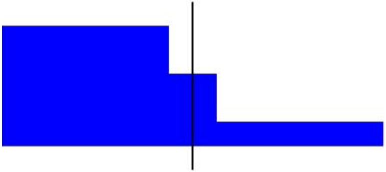

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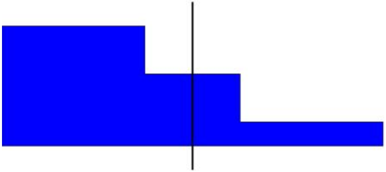

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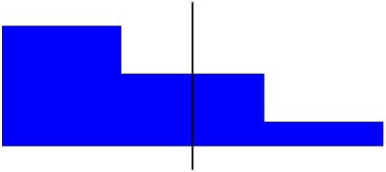

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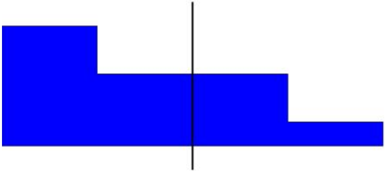

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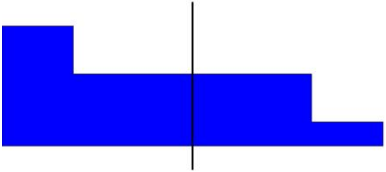

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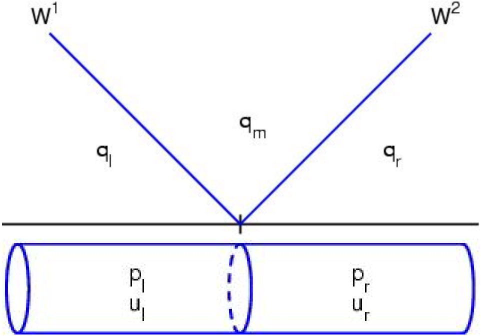
Riemann Problem for acoustics

Waves propagating in x–t space:



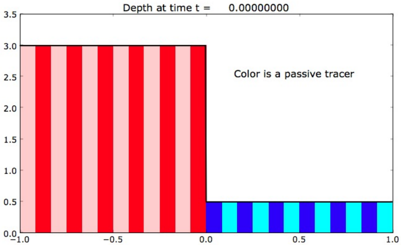
Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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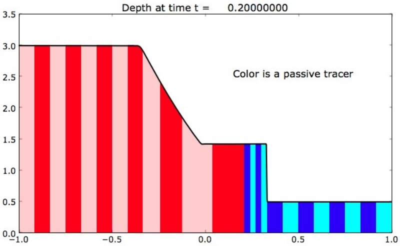

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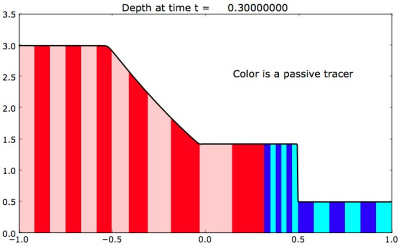

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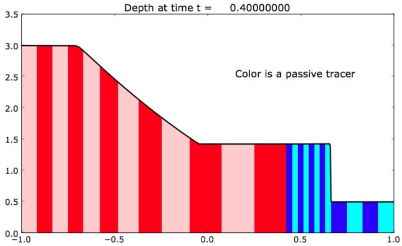

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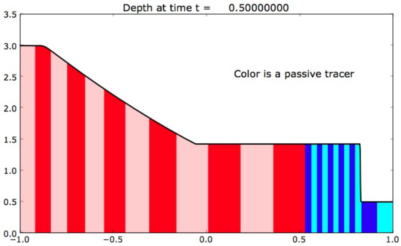
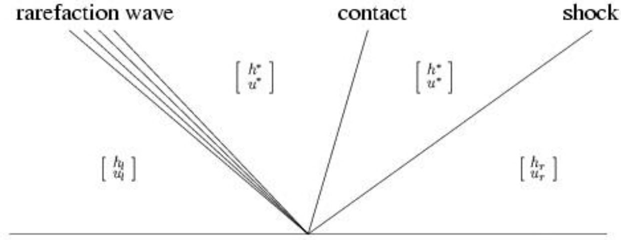


Riemann solution for the SW equations in x-t plane



Solution is constant on any ray: $q(x,t) = Q(x/t)$
A "similarity solution".

Riemann solution can be calculated for many problems.
Linear: Eigenvector decomposition. Nonlinear: more difficult.

In practice "approximate Riemann solvers" used numerically.