Conservation Laws and Finite Volume Methods

AMath 586
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Notation and Derivation

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For more about these methods...

- AMath 574, Winter 2017
- Book on Finite Volume Methods for Hyperbolic Problems, www.clawpack.org/book.html
- Clawpack Software (Conservation Laws Package)
  www.clawpack.org
  www.clawpack.org/gallery

Note different notation...

Solution = $q(x,t)$,

Advection velocity $= u$,

Advection equation: $q_t + uq_x = 0$,

Linear hyerbolic system: $q_t + Aq_x = 0$

Nonlinear hyerbolic system: $q_t + f(q)_x = 0$

Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = F_1(t) - F_2(t)$$

where

$$F_j = f(q(x_j,t)), \qquad f(q) = \text{flux function.}$$

Derivation of Conservation Laws

If q is smooth enough, we can rewrite

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

as

$$\int_{x_1}^{x_2} q_t\,dx = -\int_{x_1}^{x_2} f(q)_x\,dx$$

or

$$\int_{x_1}^{x_2} (q_t + f(q)_x)\,dx = 0$$

True for all $x_1,\ x_2 \implies$ differential form:

$$q_t + f(q)_x = 0.$$

Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0 \end{aligned}$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Linearized shallow water equations

$h(x,t) = h_0 + \tilde{h}(x,t)$ (with $|\tilde{h}| \ll h_0$)
$u(x,t) = 0 + \tilde{u}(x,t)$ (linearized about ocean at rest)

Insert into the nonlinear equations

$$h_t + (hu)_x = 0$$
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Then ignore quadratic terms like $\tilde{u}\tilde{h}_x$ to obtain:

$$\tilde{h}_t + h_0\,\tilde{u}_x = 0$$
$$h_0\,\tilde{u}_t + gh_0\,\tilde{h}_x = 0$$

$$\Longrightarrow \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix}_t + \begin{bmatrix} 0 & h_0 \\ g & 0 \end{bmatrix} \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix}_x = 0.$$ Eigenvalues: $\pm\sqrt{gh_0}$

Same structure as linear acoustics.

Compressible gas dynamics

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

Same as shallow water if $P(\rho) = \frac{1}{2} g \rho^2$ (with $\rho \equiv h$).

Isothermal: $P(\rho) = a^2 \rho$ (since T proportional to p/ρ).

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

Linear acoustics

Example: Linear acoustics in a 1d gas tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity} \end{array}$$

Equations:

$$\begin{aligned} p_t + \kappa u_x &= 0 \qquad \text{Change in pressure due to compression} \\ \rho u_t + p_x &= 0 \qquad \text{Newton's second law, } F = ma \end{aligned}$$

where $K =$ bulk modulus, and $\rho =$ unperturbed density of gas.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Linear hyperbolic systems

Linear system of m equations: $q(x,t) \in \mathbb{R}^m$ for each (x,t) and

$$q_t + Aq_x = 0, \qquad -\infty < x, \infty, \;\; t \geq 0.$$

A is $m \times m$ with eigenvalues λ^p and eigenvectors r^p,
for $p = 1,\ 2,\ ,\ldots,\ m$:

$$Ar^p = \lambda^p r^p.$$

Combining these for $p = 1,\ 2,\ ,\ldots,\ m$ gives

$$AR = R\Lambda$$

where

$$R = [r^1 \;\; r^2 \;\; \cdots \;\; r^m], \qquad \Lambda = \text{diag}(\lambda^1,\ \lambda^2,\ \ldots,\ \lambda^m).$$

The system is hyperbolic if the eigenvalues are real and
R is invertible. Then A can be diagonalized:

$$R^{-1}AR = \Lambda$$

Linear hyperbolic systems

Let R be matrix of right eigenvectors and $v(x,t) = R^{-1}q(x,t)$.
Multiply system $q_t + Aq_x = 0$ by R^{-1} on left to obtain

$$R^{-1}q_t + R^{-1}AR\,R^{-1}q_x = 0$$

Since $R^{-1}AR = \Lambda$, this diagonalizes the system:

$$w_t + \Lambda w_x = 0.$$

This is a system of m decoupled advection equations

$$w_t^p + \lambda^p w_x^p = 0.$$

So

$$w^p(x,t) = w^p(x - \lambda^p t, 0)$$

where $w(x,0) = R^{-1}q(x,0) = R^{-1}\eta(x)$.

Linear acoustics

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

This has the form $q_t + Aq_x = 0$ with

eigenvalues: $\lambda^1 = -c, \qquad \lambda^2 = +c,$

where $c = \sqrt{\kappa/\rho}$ = speed of sound.

eigenvectors: $r^1 = \begin{bmatrix} -Z \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} Z \\ 1 \end{bmatrix}$

where $Z = \rho c = \sqrt{\rho\kappa}$ = impedance.

$$R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z}\begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix}.$$

Linear acoustics

$$\begin{aligned} p_t + \kappa u_x &= 0 \\ \rho u_t + p_x &= 0 \end{aligned}$$

Change in pressure due to compression
Newton's second law, $F = ma$

This is a first-order hyperbolic system $q_t + Aq_x = 0$.

Second-order form:
Can combine equations to obtain wave equation:

$$p_{tt} = c^2 p_{xx}$$

since

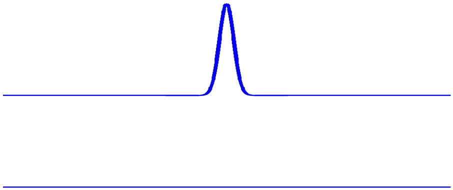
$$\begin{aligned} p_{tt} &= -\kappa u_{xt}, \\ u_{tx} &= -1/\rho \, p_{xx} \end{aligned}$$

and so

$$p_{tt} = -\kappa(-1/\rho)p_{xx} = c^2 p_{xx}.$$

Acoustic waves

$$
\begin{aligned}
q(x,0) = \begin{bmatrix} p_0(x) \\ 0 \end{bmatrix} &= -\frac{p_0(x)}{2Z} \begin{bmatrix} -Z \\ 1 \end{bmatrix} + \frac{p_0(x)}{2Z} \begin{bmatrix} Z \\ 1 \end{bmatrix} \\
&= w^1(x,0)r^1 + w^2(x,0)r^2 \\
&= \begin{bmatrix} p_0(x)/2 \\ -p_0(x)/(2Z) \end{bmatrix} + \begin{bmatrix} p_0(x)/2 \\ p_0(x)/(2Z) \end{bmatrix}.
\end{aligned}
$$



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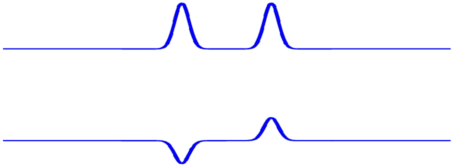

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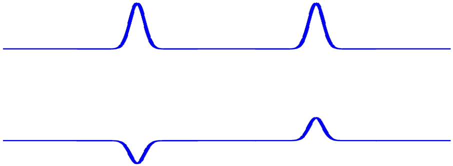

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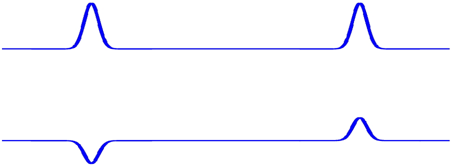

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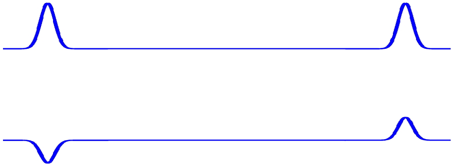
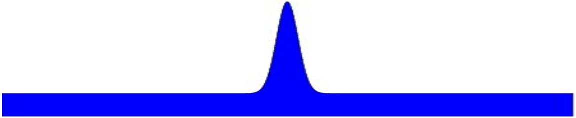


Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



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Computational challenges!

Need to capture sharp discontinuities.

PDE breaks down, standard finite difference approximation to $q_t + f(q)_x = 0$ can fail badly: nonphysical oscillations, convergence to wrong weak solution.