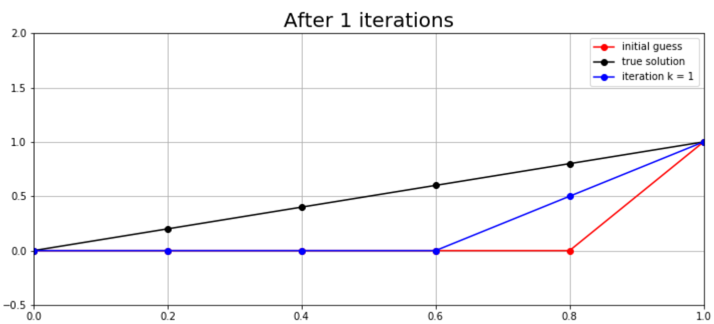
AMath 585
Homework #6
Due Thursday, March 12, 2020

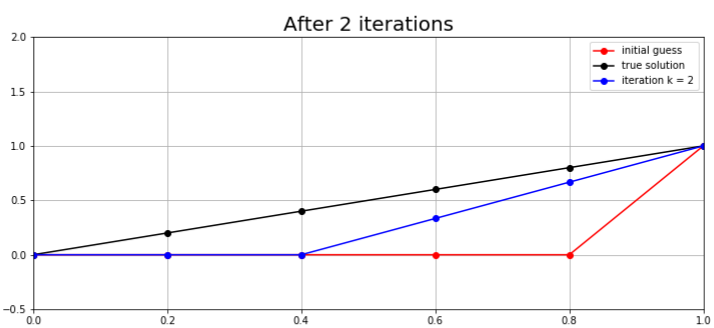
Homework is due to Canvas by 11:00pm PDT on the due date.

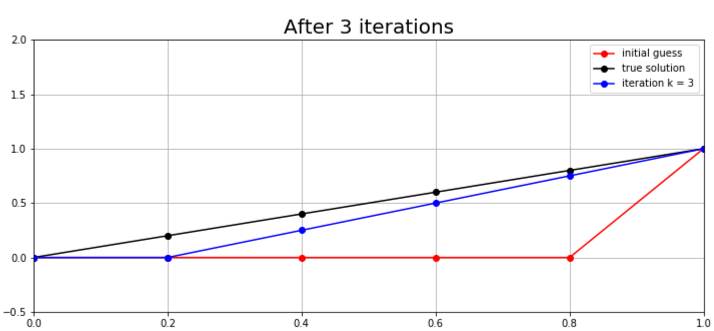
To submit, see `https://canvas.uw.edu/courses/1352870/assignments/5284853`

Problem 1. Consider the BVP $u''(x) = 0$ on $0 \le x \le 1$ with Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 1$. The exact solution is $u(x) = x$.

Discretize with the standard centered approximation using m equally spaced interior points. If we apply the Conjugate-Gradient method with initial data $u_i^{[0]} = 0$ for $i = 1,\ 2,\ \ldots,\ m$ then we see the sort of behavior that is illustrated in the plots below for the case $m = 4$. For $k < m$ the approximate solution is always piecewise linear and has $u_i^{[k]} = 0$ for $i \le m - k$. After m iterations, $u^{[m]}$ is equal to the exact solution.







(a) For the case $m = 3$, work through the C-G algorithm by hand to explicitly calculate the vectors $r^{[k]}$, $b^{[k]}$, and $u^{[k]} \in \mathbb{R}^3$ in each iteration. This should help you see why the behavior seen in the plots makes sense.

(b) To show this behavior is seen for general m, show by induction that each residual $r^{[k]}$ is a unit vector (all zeros except in one element). Hint: Use the fact that we know that all the residuals generated in C-G are pairwise orthogonal to one another, and that the only elements that can change from one iteration to the next are those in which the search direction $b^{[k]}$ has nonzero components, which can also be determined in general.

(c) Explain how the result of (b) implies the behavior seen in the plots.

Problem 2. Consider a linear system $Au = f$ in which the matrix A is **not** symmetric positive definite, so C-G cannot be applied directly.

(a) Show that if A is nonsingular then the matrix $B = A^T A$ is symmetric positive definite.

(b) So one approach to solving $Au = f$ is multiply both sides by A^T to get $Bu = A^T f$ and then solve this system with C-G. The problem with this approach is that the condition number increases. Show that the 2-norm condition number of B is the square of the 2-norm condition number of A.

(c) On page 93 it is noted that applying C-G to the two-dimensional Poisson problem $Au = f$ on an m by m grid (with second order centered differencing) requires $O(m^3)$ work to converge to a fixed tolerance. Suppose we multiplied both sides by A^T as described above (even though not necessary here since A is already SPD) and solved the resulting system (which is still SPD) by C-G. What order of work would now be required to reach a fixed tolerance?

(d) Given that the global error for this discretization is $O(h^2) = O(1/m^2)$ for smooth solutions, it makes more sense to look at the work required to get the error in the C-G solution down to this level. How does this change the work estimates given above, both for solving $Au = f$ and $A^T Au = A^T f$?

Note: there are better approaches for nonsymmetric matrices than the approach described above that do not magnify the condition number, see Section 4.4 and other references.

Problem 3. As in the previous homework, consider the one-dimensional BVP

$$\frac{d}{dx}\left(\kappa(x)u'(x)\right) = 0$$

on $0 \le x \le 1$ with Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 1$, again discretizing this problem using the system (2.71) in the text. (Or negate it if you prefer, to make it positive definite.)

Consider the piecewise constant diffusivity

$$\kappa(x) = \begin{cases} \epsilon & \text{if } x < 0.5, \\ 1 & \text{if } x > 0.5. \end{cases}$$

where $\epsilon > 0$.

(a) Generalizing what you did in HW5, determine the exact solution, in terms of the parameter ϵ.

(c) Implement the conjugate gradient method for this problem. For the convergence test require $\|r^{[k]}\|_2 < 10^{-14}$. Allow more than m iterations, if necessary.

Make semilogy plots of the max-norm of the error and the 2-norm of the residual as a function of iteration k for the case $m = 19$ with $\epsilon = 0.1$. Also try $\epsilon = 10^{-3}$. You should observe that more than m iterations are required to get good results. Comment on the behavior of the iterates in each case.

(d) Implement the preconditioned C-G algorithm (PCG) using the diagonal preconditioner and observe that this greatly improves the convergence behavior.

Note: Make sure you do this in a way for which M is symmetric positive definite and not negative definite, as discussed in the notebook `PCG.ipynb` and video that goes with it. This also contains corrections to some typos in the PCG algorithm written on page 95.

The notebook `DarcyFlow.ipynb` provides an implementation of the PCG algorithm for the two dimensional version of this problem that may be useful to follow.

Problem 4. Consider the same problem as in Problem 3 but now on the interval $0 \leq x \leq 4$ with Dirichlet boundary conditions and with $m = 3$ internal grid points (so $h = 1$ for convenience). Now put the jump in κ at the midpoint $x = 2$:

$$\kappa(x) = \begin{cases} \epsilon & \text{if } x < 2, \\ 1 & \text{if } x > 2. \end{cases}$$

(a) Write out the 3×3 matrix A explicitly in this case.

(b) Write out the matrix M that would be used as the "diagonal preconditioner" in this case. Also compute $B = M^{-1}A$ and observe that it is not symmetric.

(c) In this case we can choose C to be $\text{diag}(\sqrt{M_{ii}})$. Write out the matrix $\tilde{A} = C^{-T}AC^{-1}$ in this 3×3 case and observe that it is symmetric.

(d) For the case $\epsilon = 10^{-4}$ compute the eigenvalues and 2-norm condition number of A and B (recall that those of $\tilde{A}$ agree with those of B, but B is easier to work with). You can use the `eig` function in Numpy or Matlab, or do it by hand.

(e) Note that as $\epsilon \to 0$ the matrix A approaches a singluar matrix and the condition number blows up. What does the condition number of B approach as $\epsilon \to 0$? (You should be able to compute this analytically by looking at the limiting matrix.)