

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see <https://canvas.uw.edu/courses/1352870/assignments/5268734>

Problem 1.

Suppose $A \in \mathbb{R}^{m \times m}$ is strictly row diagonally dominant, i.e.,

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|, \quad \text{for } i = 1, 2, \dots, m. \quad (1)$$

where the sum goes from $j = 1$ to m omitting the diagonal term.

Let G be the Jacobi iteration matrix for this matrix A .

(a) Show that $\|G\|_\infty < 1$ and hence the Jacobi iteration converges, since $\|e_k\|_\infty \leq \|G\|_\infty^k \|e_0\|_\infty$. (Here and below subscript k refers to the k th iteration).

(b) Use the Gershgorin Theorem of Appendix C.8 to show that we also have $\rho(G) < 1$ in the case when (1) holds.

(c) Suppose G is a normal matrix ($G^T G = G G^T$) and suppose something stronger than (1) holds, namely,

$$\sum_{j \neq i} |a_{ij}| \leq \beta |a_{ii}|, \quad \text{for } i = 1, 2, \dots, m \quad (2)$$

for $\beta = 1/2$. Show that in this case Jacobi would reduce the 2-norm of the error $\|e_k\|_2$ by a factor of at least 10^6 in only 20 iterations, i.e., $\|e_k\|_2 \leq 10^{-6} \|e_0\|_2$.

(d) More generally, suppose the factor β in (2) is some value satisfying $\beta < 1$, and suppose the matrix is not necessarily normal but we have an upper bound $\tilde{\kappa}$ on the 2-norm condition number of R , the matrix of right eigenvectors of G . In terms of β and $\tilde{\kappa}$, what is the maximum number of iterations that would be required to reduce the 2-norm error by a factor of 10^6 ? (In exact arithmetic.)

Problem 2.

Based on Problem 1, you might think that the more diagonally dominant a matrix is, the better in terms of convergence rate. However, consider these two matrices:

$$A_1 = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -3/4 \\ -1/12 & 1 \end{bmatrix}.$$

Determine the values β for each case (as in (2)), and also the asymptotic convergence rates $\rho(G_1)$ and $\rho(G_2)$ in each case, where G_i is the Jacobi iteration matrix for A_i .

Problem 3. Suppose A is a *singular* matrix and we do a splitting of the form $A = M - N$ in such a way that M is nonsingular (e.g. Jacobi iteration in a case where the diagonal elements of A are all nonzero).

Show that in spite of M being nonsingular, the iteration matrix $G = M^{-1}N$ can never satisfy $\rho(G) < 1$.

Thus we cannot expect an iterative method for such a system to converge in general, which makes sense if the matrix is singular. However, see the next problem...

Problem 4. Suppose we want to solve the boundary value problem $u''(x) = f(x)$ on $0 \leq x \leq 1$ with *periodic boundary conditions*: $u(0) = u(1)$. Recall that this has no solution unless a certain condition is satisfied by $f(x)$, in which case it has infinitely many solutions.

Recall also that if we discretize this with the standard centered second-order approximation, using a uniform grid with $h = 1/(m + 1)$, we get a tridiagonal matrix with additional corner terms from the periodic boundary conditions. The matrix is singular and so the discrete problem has an analogous solvability condition.

(a) Suppose we use Jacobi iteration to solve this problem, in a case where the discrete solvability condition is satisfied. What is the iteration matrix G for this problem?

(b) Determine the eigenvalues and eigenvectors of G . Appendix C.7 of the text might be useful.

(c) You should find that one eigenvalue of G is equal to 1, and hence the Jacobi iteration does not appear to converge according to the theory of Section 4.2 (and consistent with Problem 3). But if the solvability condition is satisfied **and m is even**, then in practice the method does converge to one of the infinitely many solutions of the linear system. Explain in what sense this is true and how the particular solution obtained is related to the initial guess $u^{[0]}$ used for the Jacobi iteration. **Hint:** Express the initial error relative to some particular solution as a linear combination of the eigenvectors and then observe the effect of iterating with the iteration matrix G .

(d) In the case it does converge, as described in (c), what is the expected convergence rate? (Note that since $\rho(G) = 1$, this is not it.)

Problem 5.

Consider the problem

$$\frac{d}{dx}(\kappa(x)u'(x)) = f(x)$$

on $0 \leq x \leq 1$ with $\kappa(x) > 0$ everywhere.

(a) Determine the solution of this problem for the case of $f(x) = 0$ and a discontinuous $\kappa(x)$ given by

$$\kappa(x) = \begin{cases} 0.1 & \text{if } x < 0.5, \\ 1 & \text{if } x > 0.5. \end{cases}$$

with boundary conditions $u(0) = 0$ and $u(1) = 1$. The solution should be continuous and piecewise linear with a discontinuity in slope at $x = 0.5$ (Note this could model steady steady heat flow through a material that conducts heat better on the right half than on the left.)

(b) Now suppose we discretize using the symmetric matrix A of (2.73) in the text. Choose $m = 19$ (odd, so there is a grid point exactly at $x = 0.5$, in which case the solution to the discrete system should agree with the exact solution of the differential equation).

Write a computer code to solve this system using the SOR method for an arbitrary value of ω . Note that setting $\omega = 1$ should just reduce to the Gauss-Seidel method for comparison.

Test this code and estimate the rate of convergence with $\omega = 1$ and with $\omega = 0.7$ **and $\omega = 1.7$** , by doing a least squares fit of the convergence history as demonstrated in the notebook `IterativeMethods.ipynb`.