AMath 483/583 — Lecture 20

Outline:

- Adaptive quadrature, recursive functions
- Load balancing with OpenMP
- nested forking

Code:

- `$UWHPSC/codes/adaptive_quadrature`

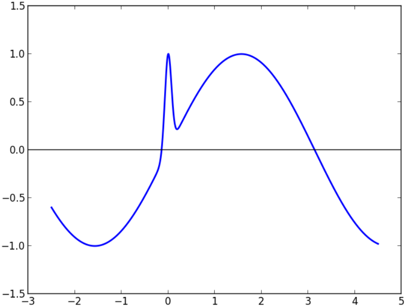
Adaptive quadrature

Problem: Approximate

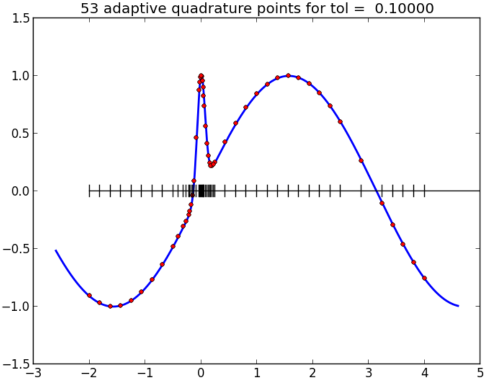
$$\int_{-2}^{4} e^{-\beta^2 x^2} + \sin(x)\,dx = \left[\frac{\sqrt{\pi}}{2\beta}\text{erf}(\beta x) - \cos(x)\right]_{-2}^{4}$$

where erf is the error function.

$\beta = 10$:



Adaptive quadrature



Idea: Subdivide into subintervals and apply Trapezoid or Simpson's Rule on each.

Use larger intervals where $f(x)$ is smoother. Automate!

Adaptive quadrature

Ideas:

- $\displaystyle\int_a^b f(x)\,dx = \int_a^{(a+b)/2} f(x)\,dx + \int_{(a+b)/2}^b f(x)\,dx.$

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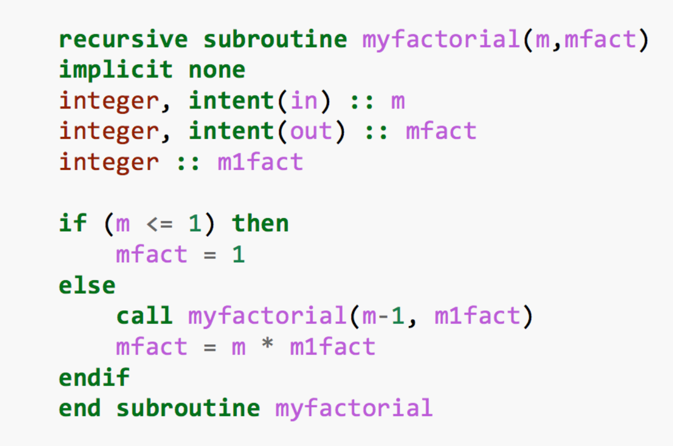
- If the error estimate on either half is greater than `tol/2`, then recursively subdivide that interval in half.

Recursive subroutine example

Compute $m!$ recursively,

Using $m! = m(m-1)(m-2)\cdots 3\cdot 2\cdot 1 = m\;(m-1)!$



```fortran
recursive subroutine myfactorial(m,mfact)
implicit none
integer, intent(in) :: m
integer, intent(out) :: mfact
integer :: m1fact

if (m <= 1) then
    mfact = 1
else
    call myfactorial(m-1, m1fact)
    mfact = m * m1fact
endif
end subroutine myfactorial
```

`$UWHPSC/adaptive_quadrature/factorial_example.f90`

Adaptive quadrature

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Adaptive Quadrature

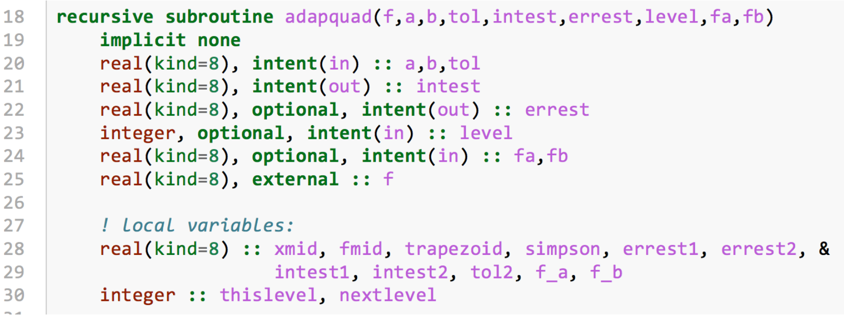
See codes in `$UWHPSC/codes/adaptive_quadrature`

- `../serial`: Serial code with recursive subroutine
- `../openmp1`: OpenMP splitting into two pieces
- `../openmp2`: OpenMP with nested forks

Adaptive quadrature — recursion

Selected lines from

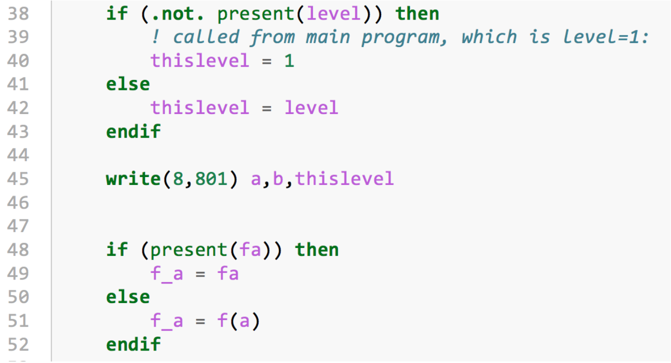
`$UWHPSC/codes/adaptive_quadrature/serial/adapquad_mod.f9`



```
recursive subroutine adapquad(f,a,b,tol,intest,errest,level,fa,fb)
    implicit none
    real(kind=8), intent(in) :: a,b,tol
    real(kind=8), intent(out) :: intest
    real(kind=8), optional, intent(out) :: errest
    integer, optional, intent(in) :: level
    real(kind=8), optional, intent(in) :: fa,fb
    real(kind=8), external :: f

    ! Local variables:
    real(kind=8) :: xmid, fmid, trapezoid, simpson, errest1, errest2, &
                    intest1, intest2, tol2, f_a, f_b
    integer :: thislevel, nextlevel
```

Adaptive quadrature — recursion

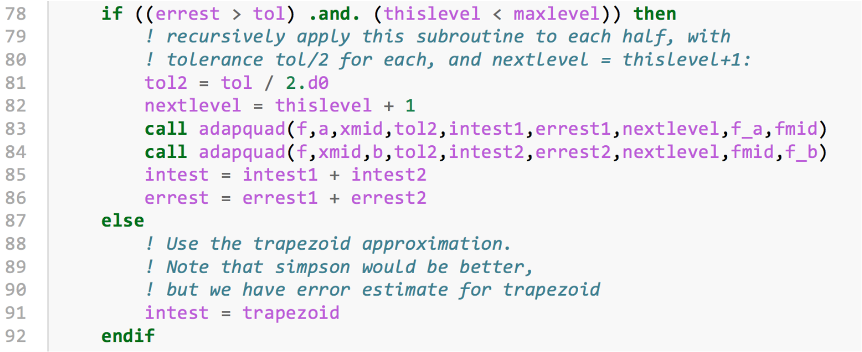
Using optional subroutine parameters in Fortran 90:



```
    if (.not. present(level)) then
        ! called from main program, which is level=1:
        thislevel = 1
    else
        thislevel = level
    endif

    write(8,801) a,b,thislevel


    if (present(fa)) then
        f_a = fa
    else
        f_a = f(a)
    endif
```

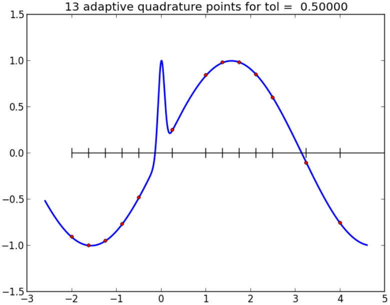
Adaptive quadrature — recursion

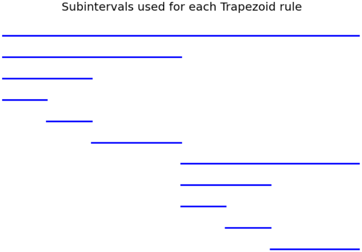
Main recursion step:



```
    if ((errest > tol) .and. (thislevel < maxlevel)) then
      ! recursively apply this subroutine to each half, with
      ! tolerance tol/2 for each, and nextlevel = thislevel+1:
      tol2 = tol / 2.d0
      nextlevel = thislevel + 1
      call adapquad(f,a,xmid,tol2,intest1,errest1,nextlevel,f_a,fmid)
      call adapquad(f,xmid,b,tol2,intest2,errest2,nextlevel,fmid,f_b)
      intest = intest1 + intest2
      errest = errest1 + errest2
    else
      ! Use the trapezoid approximation.
      ! Note that simpson would be better,
      ! but we have error estimate for trapezoid
      intest = trapezoid
    endif
```

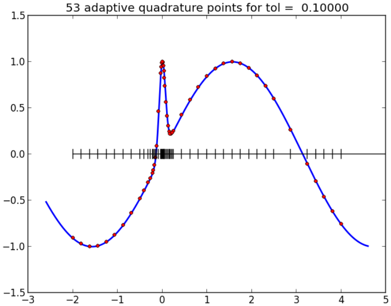
Adaptive quadrature with `tol = 0.5`

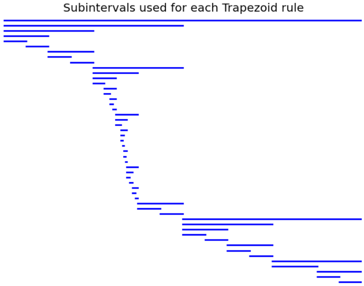




```
approx =  0.1982448782099E+00
true =    0.4147421694070E+00
error =  -0.216E+00
errest = -0.415E-01
```

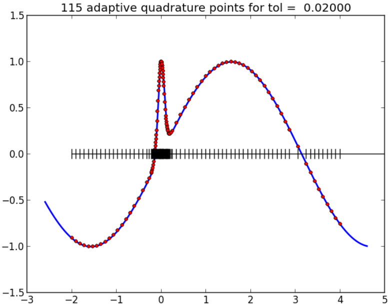
Adaptive quadrature with `tol = 0.1`

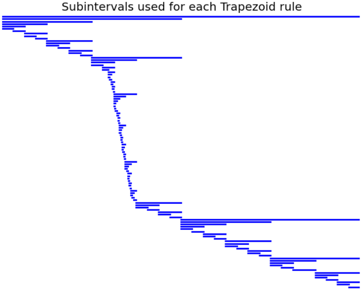




```
approx =  0.407416798536 7E+00
true =    0.414742169407 0E+00
error =  -0.733E-02
errest = -0.730E-02
g was evaluated        53 times
```

Adaptive quadrature with `tol = 0.02`





```
approx =  0.414474298092E+00
true =    0.414742169407E+00
error =  -0.268E-03
errest =  0.119E-01
g was evaluated       115 times
```

Adaptive quadrature — OpenMP

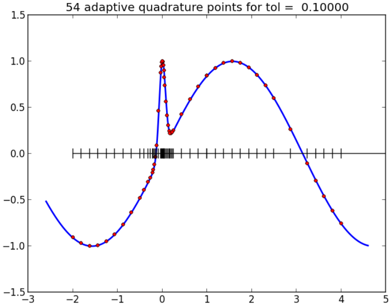
First attempt: split up original interval into 2 pieces in main program...

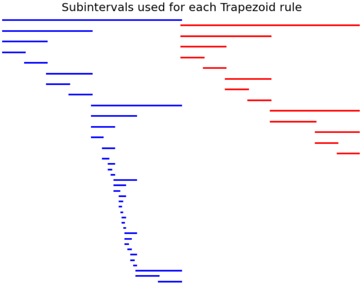
```
! $UWHPSC/codes/adaptive_quadrature/openmp1/testquad.f90

    xmid = 0.5d0*(a+b)
    tol2 = tol / 2.d0

    !$omp parallel sections
    !$omp section
        call adapquad(g,a,xmid,tol2,intest1,errest1)
    !$omp section
        call adapquad(g,xmid,b,tol2,intest2,errest2)
    !$omp end parallel sections

    int_approx = intest1 + intest2
    errest = errest1 + errest2
```

May exhibit poor load balancing if much more work has to be done in one half than the other.

Adaptive quadrature with `tol = 0.1`

Two threads, with OpenMP applied at top level only.



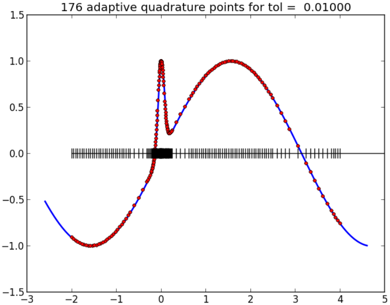


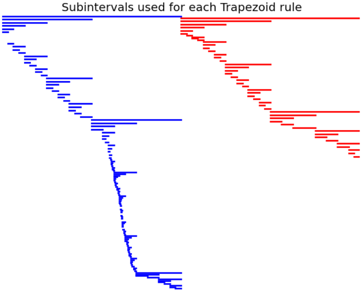
Thread 0 works only on left half,
Thread 1 works only on right half

Blue: Thread 0
Red: Thread 1

Adaptive quadrature with `tol = 0.01`

Two threads, with OpenMP applied at top level only.





Note that Thread 1 is done before Thread 0

Blue: Thread 0
Red: Thread 1

Poor load balancing if function is much smoother on one half of interval than the other!

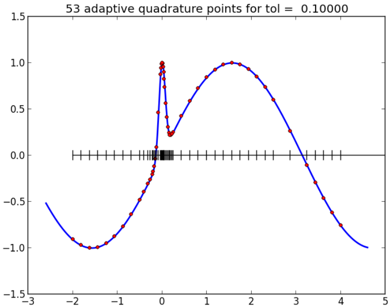
Adaptive quadrature — OpenMP

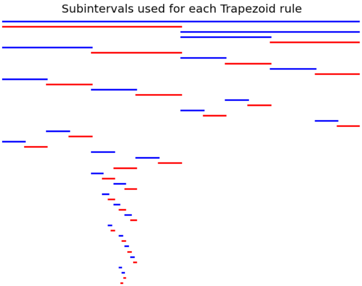
Better approach: Allow nested calls to OpenMP.

```
! $UWHPSC/codes/adaptive_quadrature/openmp2/testquad.f90

! Allow nested OpenMP threading:
!$ call omp_set_nested(.true.)

call adapquad(g, a, b, tol, int_approx, errest)

!============

! $UWHPSC/codes/adaptive_quadrature/openmp2/adapquad_mod.f90

if ((errest > tol) .and. (thislevel < maxlevel)) then
    ! recursively apply this subroutine to each half, with
    ! tolerance tol/2 for each, and nextlevel = thislevel+1:
    tol2 = tol / 2.d0
    nextlevel = thislevel + 1

    !$omp parallel sections
    !$omp section
        call adapquad(f,a,xmid,tol2,intest1,errest1,nextlevel,f_a,fmid
    !$omp section
        call adapquad(f,xmid,b,tol2,intest2,errest2,nextlevel,fmid,f_b
    !$omp end parallel sections
```

Adaptive quadrature with `tol = 0.1`

Two threads, with nested OpenMP calls





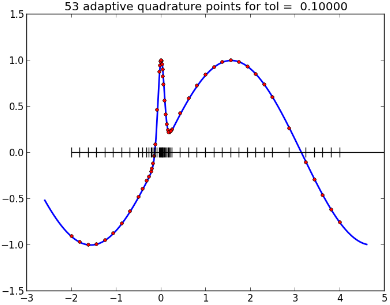
Next available thread takes each interval to be handled.

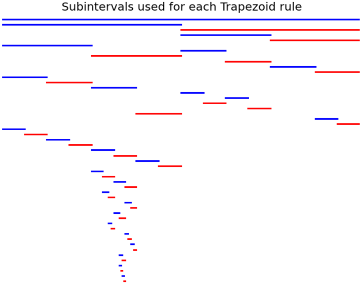
Blue: Thread 0

Red: Thread 1

Adaptive quadrature with `tol = 0.1`

Running same thing a second time gives different pattern:



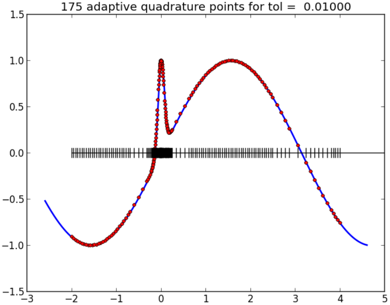


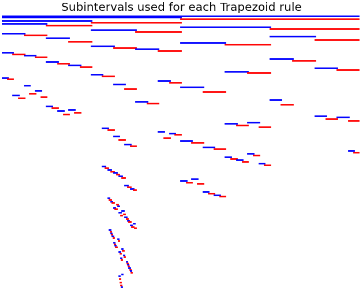
Next available thread takes each interval to be handled.

Blue: Thread 0
Red: Thread 1

Adaptive quadrature with `tol = 0.01`

Two threads, with nested OpenMP calls





Next available thread takes each interval to be handled.

Blue: Thread 0
Red: Thread 1

Software for adaptive quadrature

Much more sophisticated quadrature routines are available...

QUADPACK: Fortran 77

http://en.wikipedia.org/wiki/QUADPACK

SciPy: `scipy.integrate.quad` uses QUADPACK:

```
In [1]: from scipy import integrate as I
In [2]: beta = 10.
In [3]: f = lambda x: exp(-beta**2 * x**2) + sin(x)

In [4]: I.quad(f, -2., 4.)
Out[4]: (0.414742169407021, 8.440197311887498e-09)
```

Returns estimate of integral and of error.
Use `I.quad?` or `I?` to learn more.