Conservation Laws and Finite Volume Methods

AMath 574
Winter Quarter, 2017

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http://faculty.washington.edu/rjl/classes/am574w2017

Outline

- Linear acoustics
- Diagonalization of linear systems
- Meaning of eigenvectors
- Characteristic solution for acoustics
- Riemann problem for acoustics

Reading: Chapter 3

Linear acoustics

Example: Linear acoustics in a 1d gas tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{aligned} p(x,t) &= \text{pressure perturbation} \\ u(x,t) &= \text{velocity} \end{aligned}$$

Equations:

$$\begin{aligned} p_t + \kappa u_x &= 0 \qquad \text{Change in pressure due to compression} \\ \rho u_t + p_x &= 0 \qquad \text{Newton's second law, } F = ma \end{aligned}$$

where $K =$ bulk modulus, and $\rho =$ unperturbed density of gas.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Linear acoustics

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

This has the form $q_t + Aq_x = 0$ with

eigenvalues: $\lambda^1 = -c, \qquad \lambda^2 = +c,$

where $c = \sqrt{\kappa/\rho}$ = speed of sound.

eigenvectors: $r^1 = \begin{bmatrix} -Z \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} Z \\ 1 \end{bmatrix}$

where $Z = \rho c = \sqrt{\rho\kappa}$ = impedance.

$$R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z} \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix}.$$

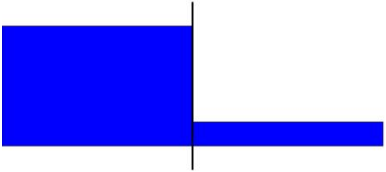
Riemann Problem

Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram ($u_l = u_r = 0$)

Pressure:



Acoustic waves propagate with speeds $\pm c$.

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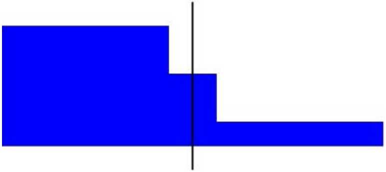

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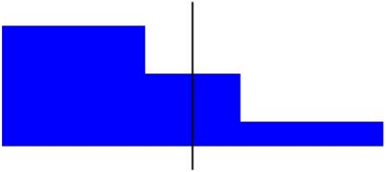

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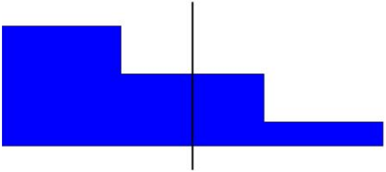

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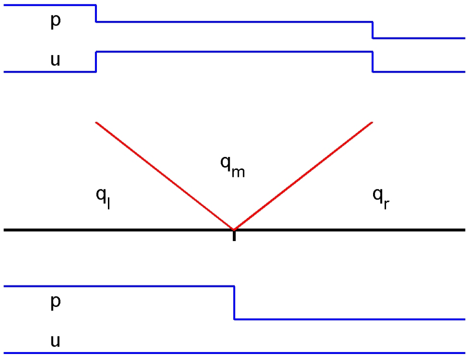
Riemann Problem for acoustics

Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Suppose hyperbolic:

- Real eigenvalues $\lambda^1 \leq \lambda^2 \leq \cdots \leq \lambda^m$,
- Linearly independent eigenvectors $r^1,\ r^2,\ \ldots,\ r^m$.

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Let $R = [r^1|r^2|\cdots|r^m] \quad m \times m$ matrix of eigenvectors.

Then $Ar^p = \lambda^p r^p$ means that $AR = R\Lambda$ where

$$\Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix} \equiv \text{diag}(\lambda^1, \lambda^2, \ldots, \lambda^m).$$

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$AR = R\Lambda \implies A = R\Lambda R^{-1}$ and $R^{-1}AR = \Lambda$.
Similarity transformation with R diagonalizes A.

Diagonalization of linear system

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Multiply system by R^{-1}:

$$R^{-1}q_t(x,t) + R^{-1}Aq_x(x,t) = 0.$$

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This decouples to m independent scalar advection equations:

$$w_t^p(x,t) + \lambda^p w_x^p(x,t) = 0. \qquad p = 1,\ 2,\ \ldots,\ m.$$

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$$w^p(x,t) = w^p(x - \lambda^p t, 0) = \overset{\circ}{w}{}^p(x - \lambda^p t).$$

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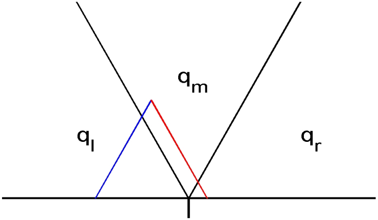
We can rewrite this as

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t)\, r^p = \sum_{p=1}^{m} \overset{\circ}{w}{}^p(x - \lambda^p t)\, r^p$$

Riemann Problem for acoustics



$$q_l = w_l^1 r^1 + w_l^2 r^2$$
$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

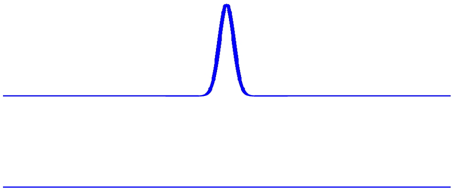
So the waves $\mathcal{W}^1$ and $\mathcal{W}^2$ are eigenvectors of A:

$$\mathcal{W}^1 = q_m - q_l = (w_r^1 - w_l^1) r^1$$
$$\mathcal{W}^2 = q_r - q_m = (w_r^2 - w_l^2) r^2.$$

$$q(x,0) = \begin{bmatrix} \overset{\circ}{p}(x) \\ 0 \end{bmatrix} = -\frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

$$= w^1(x,0)r^1 + w^2(x,0)r^2$$

$$= \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ -\overset{\circ}{p}(x)/(2Z_0) \end{bmatrix} + \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ \overset{\circ}{p}(x)/(2Z_0) \end{bmatrix}.$$



Acoustic waves

$$q(x,0) = \begin{bmatrix} \overset{\circ}{p}(x) \\ 0 \end{bmatrix} = -\frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

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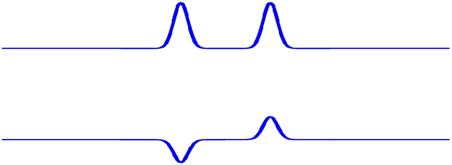

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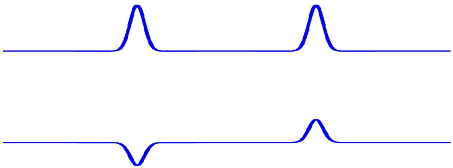

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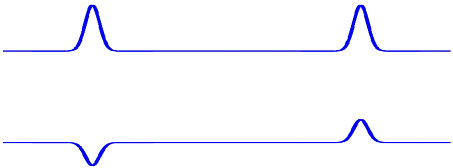

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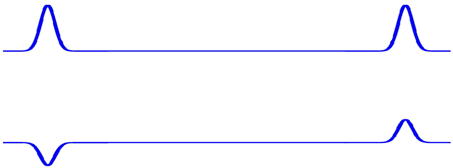

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

In a simple 1-wave (propagating at speed $\lambda^1 = -c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is $-Z_0$ times the velocity variation.

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Similarly, in a simple 2-wave ($\lambda^2 = c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is Z_0 times the velocity variation.

Riemann solution for a linear system

Linear hyperbolic system: $q_t + Aq_x = 0$ with $A = R\Lambda R^{-1}$.
General Riemann problem data $q_l,\ q_r \in \mathbb{R}^m$.

Decompose jump in q into eigenvectors:

$$q_r - q_l = \sum_{p=1}^{m} \alpha^p r^p$$

Note: the vector α of eigen-coefficients is

$$\alpha = R^{-1}(q_r - q_l) = R^{-1}q_r - R^{-1}q_l = w_r - w_l.$$

Riemann solution consists of m waves $\mathcal{W}^p \in \mathbb{R}^m$:

$$\mathcal{W}^p = \alpha^p r^p, \qquad \text{propagating with speed } s^p = \lambda^p.$$