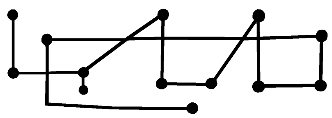
Chloe's Friends (A Symposium About Music and Mathematics)



John Rahn

Setting: Athens, 5th century BC

Personae:

Chloe, a very smart woman in her late twenties
Xanthippe, an underestimated housewife (known to her friends as "Xp")
Hermione, a hermaphrodite in love with Chloe
Xanthus, a fun young man with nice buns, and a musician
Iesus, an immigrant from the Near East, in his forties, from the School of Parmenides
Megakephalos ("Meg"), a senior Academician and a music theorist

Xp: What a nice house you have, Chloe—it's cozy without being cluttered.

Iesus: I love the wall paintings. So cool, so classically Greek.

Chloe: Well, thanks, but where are Hermione and Xanthus, did we lose them on the way back from the lecture?

Hermione and Xanthus: [knock and enter] Here we are, Chloe! We got distracted by some jugglers.

Chloe: So, what did you think of Megakephalos's lecture, everybody?

Xanthus: Way cool. I mean, who could have guessed that music and math connect so weirdly? And the math itself was such fun. A lot of it was stuff I've never heard of.

Chloe: Wait a minute—[to her slaves]—Could we have a bowl of wine in here, please? [to her guests] Sorry, please go ahead.

Xp: Actually, Xanthus, I thought it was a mixed bag.

Xanthus: What do you mean?

Xp: Well, it seems to me there were several kinds of confusion in the talk. For one thing, there were those ratios of string lengths which were supposed to extrapolate to all sorts of comic, I mean cosmic extremes—I think he even talked about modes of vibrations of atoms, or parts of atoms, as if they were sounds, and there was some obscure reference to cosmic strings vibrating in more than the usual number of physical dimensions. Teeny, teeny cheerios humming away, so to speak.[1] Now, I'm tempted to say, they don't hum for me, but that would be facile. For all I know, they really do snap, crackle, and pop. My point is, the connection with music is broken somewhere—we don't actually hear those teeny cheerios.

Hermione: Right on, Xp. And all that stuff about pulses in the air, that's not what we hear, either. Chloe, don't you think that nasty man Aristoxenos is right, that the ratios should measure what we hear, not what in the physical world causes what we hear? That changes everything about the application, and about the generalization, but not about the actual mathematics of the ratios, the math of harmonics. That's what is so interesting. The math stays the same, when one way to paste it on the world is wrong, and another different one might be right.

Chloe: [to the slave] Just put the Crater here, please, and the water there. [to guests] Help yourselves, math is thirsty work!

Iesus: Thanks, I will. [helps himself to wine] But let's be fair, you guys. You keep harping on the differences. There really is something grand and mysterious about the same mathematics of harmonics applying to the physical sounds that cause musical sounds, and maybe the most profound structures of the physical universe. If you insist on limiting your theory strictly to what you perceive musically, we can still get some mystery.

Aristoxenos only talked about ratios of perceived intervals of pitch, and of time, which can get pretty complicated in themselves the deeper you go—the deeper you go, the higher you go, that's one great thing about it. All that stuff about continued fractions was pretty dazzling, in a relatively pedestrian sort of way.[2] And all that is directly about the scales that underlie the music you really hear.

But the more important thing here is not the ratios and so on. Remember when Meg was talking about those teeny cheerios, as Xp put it so well, he was talking about Space. He had to make up weird spaces in order to make any sense of the vibes of the cheerios. He had to talk about pushing spaces through other spaces, and so on. Now, that is what I find exciting. When I took my last sabbatical from the school of Parmenides, I studied in Tibet. They had a very refined concept of Space which was a lot like Meg's concepts. I don't remember everything my teacher said, but I did memorize a few things. For example: "everything is of one basic space, like waves on water," and

> immersion in genuine being in the past, present, and future—
> is the single basic space of enlightened intent, the uninterrupted nature of phenomena.
> Masters of awareness share a dimension of experience equal to that of all victorious ones.
> The noncomposite expanse—unchanging and indivisible.
> The expanse of naturally occurring timeless awareness—beyond effort or achievement.
> The expanse in which all phenomena are mere names—beyond imagination and expression.
> Within this wholly positive realm, in which nothing need be done, regardless of what manifests there is still wholly positive basic space.[3]

[there is a grand pause as the company considers all this]

Chloe: Wow, Iesus. That's amazing. But you know, when you talk about dimension and space in this way—or when your Tibetan masters did—did they mean what we mean by it, what Euclid meant by these terms?

Iesus: Obviously they had not studied Euclid. So if you want to be picky—and Chloe, I know you are always picky!—maybe not—how could the terms be exactly equivalent unless the theories that they are embedded in are also equivalent? But what I find most suggestive, is the connection between—let's get back to dear old Meg and his music—the concept of space in both realms.

XANTHUS: That's what Hermione was saying, right? The same math pasties on different, er, realms.

[Chloe, Hermione, and Xp throw pillows at Xanthus for this]

XP: Iesus, you'll have to explain yourself better than Xanthus did here. What do you mean, the concept of space connecting music and . . . whatever?

CHLOE: I think I get it. When Meg talked about those cheerios, he kept having to up the number of dimensions just to make the description simpler and clearer and more regular. So you could infer a principle here. I don't know if it would really generalize, but—when your description has too many bells and whistles, like one of those cute art-machines in the museum, and seems inscrutable, you can judiciously add dimensions to the space you're describing within until it looks simple, or at least, possible. A bit like in a formal theory, you trade off the number of axioms for the number of inference rules, and so on—just a metaphor. Instant theory, just add dimensions. Housewife's dream, Xp.

XP: Don't know if my hubby would go for that one.

HERMIONE: We know your hubby, Xp.

XANTHUS: Girls, girls. Please. Let's not get personal.

HERMIONE: What do you mean, "girls"? [more pillows]

CHLOE: OK, OK. But Iesus, what do you mean—expatiate!

IESUS: Well, I have to admit I am getting over my head here. But I think you are right on track, Chloe, in what you said. Meg did not go into it very much, but isn't it possible that the same kind of abstract algebraic geometries he alluded to in his cheerio theory also apply in music theory? And that, in fact, it is the same strategy—when dazed and confused, make it simpler by adding to the number of dimensions?

CHLOE: How many dimensions, for music?

XANTHUS: Well, I'm a composer so I should know. Without getting too fancy, at least 22.

EVERYONE ELSE IN CHORUS: 22!?

HERMIONE: Why not? It's a nice number.

IESUS: That's more than the number of dimensions in the universe.

[general laughter]

HERMIONE: Are you just making fun with Chloe's good point? Just add dimensions, until it looks simple?

XANTHUS: Not at all, I'm serious. Anyone who composes will recognize that composers play with all sorts of independently variable quantities. It's totally practical. I won't even use the auloi for an example, that would be too easy; after all they are two pipes at once both continuously inflectable in pitch and so on. Take just the classic kithara. A few strings tuned in advance, but you strum them and stop them in combinations and mute them to different degrees in combinations and so on, and then of course you are singing with that, and the relation of your vocal tune to the strings, and everything else to the poetry, well, I needn't go into that. Why else do we have genuine kithara stars playing to large audiences. It's hard, it's complex. Besides, they keep the true musicality of the Greek language alive. Without the kithara players, people would probably forget the nuances of how to speak with correct stress, pitch, quantity, elision, and so on. I can see it happening in spite of this. But, don't get me started. The fact is, in my own compositional praxis, I do happen to use 22 parameters for the simpler events.

But, I still don't know how what Meg was saying fits into what I do, I fact, I still don't really understand what Meg was saying!

[silence while everyone takes this in]

Hermione: But, Xanthus, you didn't answer my question, really, entirely. I accept that you use 22 dimensions or so in your musical practice, and Xp will agree with me that there is nothing shabby about practice. Practice can teach us things too. Theory alone does not have the last word.

Xp: I do agree.

Hermione: But my point was not so much that you use many dimensions. I am wondering about the strategy Chloe articulated, Just add dimensions. OK, OK, I accept you need your prosaic 22 or whatever, just to compose. But what about theory? What is the relation between practical musicianship, composing and so on, and theory, and would music theory possibly fall into the same strategy as cheerio theory? That is, Just add dimensions until it looks simple?

Xp: Actually, we don't know that that is a bad strategy. Just because Chloe made it sound like a cake mix . . . it might be a perfectly sound way to do things.

Chloe: Hey, everyone, this is getting serious. Here, have some more wine [pours around]. [knocks are heard at the door] That must be Meg! [runs to door and opens it]

[enter Megakephalos with garland, already a bit high]

Meg: Carrion crows! I just know you are picking me apart and feeding on my abundantly fleshy liver, I mean, lecture!

Xp: Pack it in, Meg, and cool your head with some more wine. But, you're right!

Chloe: Here, sit down, you're just in time, Meg. We were just choking on various morsels of your talk, and maybe you can help us get them down. But first, have a drink! [pours]

Meg: That's better. I swear, after the talk, all the Sweater Sets of Athens surrounded me like Harpies, they and the Strigil Sluts. Nothing in any of their noodles, I'm afraid. But you can tell me where and how I went off the rails, and I can tell you whether I meant it—if I know!

Chloe: You're too modest, Meg. You know we know you don't know what you're talking about! But neither do we. So, from this, good conversation is born.

All: Hear, hear!

Chloe: So, here's where we were, I think, Meg. Xp pointed out that there's a mixed realm problem between physics and music—just the kind of problem that young Aristotle is so hard on. But Iesus pointed out that mixed, schmixed, if the math is the same—I think Hermione called it pasting the same math on different things, and of course Xanthus had his own idea of things—if the math is the same, maybe something really is the same. But Iesus pushed it way farther, all the way to Tibet; you'll have to ask him to explain that as I don't understand it, but the idea is about space as a medium of explanation. So, as Hermione conjectured, you can maybe just add dimensions until the world looks simple enough. Got infinity? No problem, got infinite space. But that's going too far: more to the point for us poor finite creatures, there is a strategic or ethical issue. Faced with some complex of things to explain, is it fair, right, just, or correct, to just add dimensions to the space within which you explain them until they look simple within that space?

Xanthus: Don't forget the cheerios.

Xp: Those teeny teeny cheerios!

Hermione: Just add sugar!

Meg: Cheerios?

Chloe: Strings . . .

Meg: O, aha, ha ha. Not bad, except cheerios are too friable to vibrate while retaining their integrity, but you probably take this as yet another symptom of the inevitable pomposity of an academic.

Xp: Damp cheerios.

Hermione: With sugar on top.

Meg: I give up. But let's talk about you. Xp, you got the mixed realm thing right. But Iesus got the part I was trying to put across, about a unity transcending the mixedness of the realms—while recognizing the separateness of the realms. It's tricky because people can easily get the wrong idea about it—simple people unlike the present company. If I say that the mathematics of modes of vibration are identical, or extrapolating to more dimensions and so on, for a vibrating kithara string, and a Epicurean probability wave for a subatomic particle,[4] and a teeny cheerio—actually an eleven-dimensionally vibrating toroid—in a fabric of teeny cheerios which is physical space, I am not saying that the kithara string is in any other way like the wave or the cheerio. (You see, Xp, I am using your disgusting simile.) Or like the planets for that matter, which also go round and round. Or like the months of gestation! [pompously] So much for those dunderheads that stain the tradition of Pythagoras with their muddled ontologies. Let's put all that aside.

XANTHUS: But Meg, it's not trivial what you paste it on. Or how you paste it on. You can take a perfectly good math pastie, for example, a kind of group theory—let's say (just at random) a sequence of wreath products . . .

CHLOE [interrupts]: Xanthus, where in the world did you come up with that example? I barely remember wreath products from my days auditing at the Academy, that's pretty obscure.

XANTHUS: It's something that came up in my work, believe it or not. The point is, you can use this construction for good or ill, so to speak. Or, you can use it well or poorly. The math remains the same, and the way you apply it makes all the difference. Applying math is a complex affair; naive notions of it won't do. Even it you delete the ontological fallacies of numerology and so on, application of math is an art in itself. Of course you can formalize it as model theory . . . but that only illumines the formal issues, not issues of good sense.

MEG: What's your point?

XANTHUS: How do you put all that aside, as you say? What makes good application of math, and bad application of math? In particular, application to music—how do we make it good rather than bad?

XP: Here we go, I knew we'd get to the good and the bad.

IESUS: And another thing, Meg. I was impressed how you made weird spaces in order to make any sense of the vibes of the cheerios. You talked about pushing spaces through other spaces, and so on. I find that kind of thinking intrinsically exciting, but, taking Xanthus's point too, how do I know I am not just being seduced by the math until I find myself using cool math to make some badly fitting application?

CHLOE: And let's not forget the issue of Just add dimensions. Maybe if you add enough dimensions, anything would look simple (though the space might be hard to grasp)? So would that be a defensible move toward explanation? Have another drink, Meg.

MEG: Well, I've already had enough wine during the post-lecture frenzy, but the way things are going here, maybe I don't want to put too fine a point on it.

First, let me say I think you are all too hung up on my cheerios. I was supposed to be talking about math and music, and those cheerios are just physics. I did want to make Iesus's point about the oneness of it all, but without falling into the trap of spreading the oneness of the math over the bumpiness of the different ontologies. Reality is diverse. Mathematical structure is not substance or essence. OK?

Now, Chloe, I don't know why you are so bothered by adding dimensions.

Iesus: I met a hermit in Persia who kept muttering something about not multiplying entities beyond necessity, making do with less, and so on.

Meg: Yes, fine, but are dimensions entities? I agree, don't use more than you need. But why not use as many as you need? We are not talking about ontology here, I hope this is clear now. The dimensions are in the description, the theory, and who knows what the reality being described might ultimately be. You could even think about dimensionality flexibly. If you need a grand theory of everything physical, such that the rules aren't too baroque, you may need eleven, or 111, dimensions. But you can take any projection onto any subspace . . . that's probably not a good example. Say you want a theory of wrestling. You don't need more than three or four dimensions for that, I suppose, unless some superstar wrestler comes along who can tie his opponents in seven-dimensional knots. Then you'd have to extend your theory. So, to each domain its own theory, with some appropriate space for the theory. I'm not just talking about subspaces, or [gesturing widely] restrictions of

functors, sheaves, and so on, subtheories or cohomologies. It's good practice to use the simplest theory you can to get the job done that you need to do. You don't want to try to use Epicurean quantum mechanics to describe the motions of the planets, or of the wrestlers.

Iesus: That sounds plausible, Meg, but I wonder. For example, I've had to do a lot of sailing during my travels. The math involved could be just simple trigonometry, vector space. But when we're in a long chase with some pirate behind us and it's vital to get every bit of speed out of the boat, I find my mind wandering to fluid dynamics, and so on. The more intensely one thinks about something, the more complex the math is likely to get.

Meg: Yes, as I said: whatever you need to get the job done. When the job gets harder, you may need something more powerful and comprehensive, but if not, you can make do with less, as the hermit said. In fact, here's a better example: those kithara strings are physical objects, so theoretically their behavior is describable by 11-dimensionally vibrating cheerios, but in musical practice, the theory is much simpler harmonics with no reference to cheerios.

Xanthus: Yes, until you need to describe what happens to the string at the beginning and end of its sounds, or when you mute it a little with your other hand but let it ring in a stifled way . . . I see what you mean, Meg. The math only gets more complex when you need it to be more complex. And I've never needed cheerios.

But tell me, Meg, what you were saying that really excited my imagination wasn't the part about cheerios. I agree, that's been a red herring. You were talking about Space in a purely musical sense, a cognitive space for music theory. And frankly I did not follow what you were saying. There were algebraic things in it like groups—and we all know what a group is! But at the same time it was a space. It all sounded very ethereal, but very stimulating.

Hermione: Yes, and you talked about Categories . . . I didn't understand that at all. What is a Category[5]—is it related to Plato's Ideas, or Forms?[6]

Meg: You're right, this is the part I worked hardest on for the lecture, and I'm still not sure I've got it right. But before we go into all that, I've been thinking about Xanthus's earlier question, what makes a math application good or bad? You know, that may be the most interesting thing we can talk about.

Xp: Please tell me you are not going to begin by determining the nature of Good.

Meg: No fear, Xp. Though it might come to that. No, this is something I haven't really thought about before, yet it seems so basic. Perhaps we can feel our way forward here. A good application would have certain qualities. For one thing, the math itself would have to be OK.

Chloe: That seems obvious, Meg.

Meg: Then what? The principle of theoretical ascesis we were talking about? Don't use more than you need?

Hermione: Well, yes, but remember the boundary is flexible here. You might not think you needed it but then find out that you do need it. So it would be hard to judge until you'd seen the entire theory. And then you'd have to provide for extensions to the theory when you need more.

Meg: Agreed. And you know, I think I may be vulnerable here. For example, we'd all agree that representing the usual scale-pitch functions as residue classes mod n within Zn, and all the attendant group theory, is pretty basic and inevitable, right?

All: [noises of universal assent]

Meg: I'm not going to go into this, but in my talk, I reconstructed things like pitch classes in terms of denotators which basically involve the category of contravariant set-valued hom-functors for modules, that is, set-valued presheaves over modules.[7]

Xanthus and Chloe: Hunh?

MEG: Yes, well there is at least a question as to whether the question of theoretical ascesis should be raised here, but with the caveats given just now by Hermione.

HERMIONE: Didn't those Categories come into this?

MEG: Fundamentally. In fact the project I was talking about is a radical refoundation of music theory based on a radical refoundation of mathematics in terms of categories. It's based on some work by a mathematician from the mountains north of Sicily.[8] But let's get back to our theory of application. We've agreed that the math has to be OK, and that you shouldn't use more than you need, right?

HERMIONE: Right.

MEG: What else?

CHLOE: The mathematical structures, within the math you use, should fit nicely with the structures you are using the math to describe. I'd call this Good Fit. It's hard to specify this completely . . . Does anyone have a good example of Bad Fit? As I think Iesus said, using cool math to make some badly fitting application?

IESUS: How about, just numbering the pitch classes from one to *n*? That ruins a lot.

XP: Good example, Iesus. Here's another: name the pitch classes after the smallest *n* infinite numbers, starting with the number of natural numbers. Then you'd also have the problem of the Continuum Hypothesis, in spades, but the whole thing would be pointless. (So to speak.)

CHLOE: Xp, that is truly perverse.

MEG: Good, good. Math OK, ascesis, and Good Fit.

XANTHUS: But here's another Bad Fit, maybe: using sets at all, for pitch classes or whatever. [grandly] Why use sets? They are awful things, very artificial. Nothing is ever unordered.[9] That's why all the problems with the Axiom of Choice: can't tell them little buggers apart. So much for the foundation of mathematics. Is that what you were trying to get away from with your refoundation, Meg?

MEG: Whoosh, I'll have to think about that. But yes, in a way. There's a re-axiomatization in terms of categories rather than in terms of sets, underneath it all.

XANTHUS: I have a more extended example of application that might help us along further in our theory of Good Application. I mentioned wreath products?

HERMIONE: Assuming you're not talking about head ornaments.

CHLOE: I think we're all talking about head ornaments.

IESUS: Give the man a chance.

MEG: I was wondering about that, Xanthus. Do proceed, please.

XANTHUS: Well, the application I have in mind comes from a mathematician from νιορκ. He's an artist as well, and a machine theorist, and he's come up with a theory which he says explains the structure of perception, of cognition, really of the world as we take it in, and therefore also aesthetics, music, quantum mechanics, the structure of scientific theory, and so on. The primary areas he applied it to are visual perception and computer-aided design.[10]

The theory has two guiding principles called transfer and recoverability. The idea of transfer is that large, more complicated (and in some sense less symmetrical) structures are built up in levels from simpler ones which are "transferred" up. So it has the hierarchical level principle in common with music theory. Recoverability means that given a large, complicated structure, the generative history can be recovered—it can be parsed according to the levels of transfer from simpler structure (though the parsing is in fact not generally unique). This parsing models cognition, so a theory of construction can also serve as a theory of cognition—you get two for one.

CHLOE: Sounds ambitious.

XANTHUS: Yes, indeed. Now, our first criterion was, "Math OK?," and I think his math is fine. That makes it interesting, because it leaves ascesis and Good Fit. But first I'll describe the math. It's all sequences of wreath products of groups, so it's not too hard. You'll all remember about direct products of two groups?

ALL: [noises of assent]

XANTHUS: Well, then, do you all remember what a semi-direct product is?

CHLOE: Isn't that where the groups aren't necessarily abelian? So instead of a direct product of two groups, you take two groups such that the first is a normal subgroup of the product with respect to some particular homomorphism—you have to specify which one—and the intersection of the two components is just the identity, and when the two components are multiplied element-wise you get the resulting group? That's called a splitting extension of the first group by the second, right?

IESUS: Right, Chloe, I think. It's a clean way of doing a group extension. The reference homomorphism is a homomorphism from the second component into the automorphism group of the first, using conjugation in the first component group by the elements of the second component group. Conjugation by different elements will get the different homomorphisms.[11]

XANTHUS: Exactly, the idea of group extension is key. You build larger and larger groups to fit this idea of structural generation. But there's one more step from semi-direct product to wreath product: the first component is replaced in the semi-direct product by its direct product with itself some number of times. So, the first component of the underlying semi-direct product is $G1 \times G1 \times G1$... rather than just $G1$. So $G1$ w $G2$ means [$G1 \times G1 \times G1$...] semi-direct product with $G2$. Elements of the direct product group are vectors.

Now, for wreath products in this theory, the number of $G1$ copies in the direct product component is always equal to the order of $G2$, the second component. This allows us to index the direct product group elements by the elements of the second group. And the automorphism for the semi-direct product is always the same: a raised action on the indices in the direct product group, permuting the indices. It's called the transfer automorphism. This makes it all relatively simple.

CHLOE: But maybe not all that simple.

MEG: This seems pretty clear, though.

XANTHUS: So far we've just defined this particular kind of wreath product, $G1\ w\ G2$. The theory actually requires finite *w*-sequences—in fact, all of these groups should be viewed as finite, since this is about machine theory. A *w*-sequence is just something of the form $G1\ w\ G2\ w\ G3 \ldots w\ Gn$.

CHLOE: That makes it more interesting.

HERMIONE: Yes . . . hey, are these connector things associative; does this have a semigroup structure using *w* as the connector?

XANTHUS: Nope. Good question, but no such luck. In the constructions of this theory, at least, the last group on the right is always the $G2$ of a wreath product whose first component $G1$ is the result of evaluating the entire sequence to the left of the last group. So it's understood always as $((((G1\ w\ G2)\ w\ G3) \ldots)\ w\ Gn)$.[12]

Now, from a musician's perspective—and to illustrate the "good application" issue—I have a few questions about this approach, some of which are more music-application-oriented than others. I'm not sure whether they are all resolvable, but I think they are, in principle. I call them the Order Problem, the Model Problem, the Symmetry Problem, and the Occupancy Problem, which has two sub-problems: God's Hand and God's Switch, and a corollary problem which has to do with the lack of a theory of the structure of the Army of Occupation.

IESUS: We're not getting into any punitive religion thing here, are we, religious wars?

[Meg, Chloe, and Hermione throw pillows at Iesus.]

XANTHUS: [ignoring the byplay] I'll take them in that order, from least to most application-oriented. The Order Problem is just, for any given *w*-sequence, what governs the growth of a wreath sequence from shorter to longer? You might think it just grows from left to right, since it always parses from left to right, but that's not the case. So if your sequence has *N* components, there are *N* factorial different ways to grow it from scratch, given no constraints. What are the constraints?

MEG: Why do we need to know this? What difference does it make? If you have a sequence, why do you need to construe it as having grown step by step?

XANTHUS: It's a crucial difference in the theory, as the generative stages of the sequence are supposed to model a trace history of the cognitive process. In fact, there is a predilection for the groups on the left (as it turns out) to be all of a kind, namely all iso-regular groups. Informally, an iso-regular group is a control-nested hierarchy of repetitive isometries. The component isometry groups are either cyclic groups (if discrete), or one-parameter Lie groups (if continuous). [13]

CHLOE: I can see how that could plug into a theory of perception. After all, isometries can model measurable relationships among objects. So cyclic ones would make great building blocks for finitely described objects of perception, and hierarchical structures often work for perception. But I see why you'd need the generative order, if you mean to trace an order in the cognitive process.

XANTHUS: Yes, but as I say, there aren't any rules about how to grow it—although there are guidelines—and there isn't any separate, orthogonal abstract generative order attached to each sequence—in fact, you'd need a separate such specification for each different application of a given sequence.

HERMIONE: If you're talking about generative rules for sequences, how about a grammar? Works for sentences.

XANTHUS: Yes, Hermione, that might be the form a solution takes, or a grammar might be too restrictive even in general. We'll just have to see what happens as we develop it further.

CHLOE: The next one was the Model Problem, right? What's that? It sounds like an application problem.

XANTHUS: It is, really, but all of these relate to all the others. I'll just mention two aspects of this one. The first is a practical point about the size of the groups. The order of $G1 \ w \ G2$ is $((|G1|**|G2|)*|G2|)$. So the order of the resulting group from a *w*-sequence approximates the iterative exponential of the orders of the component groups. This is a ferocious order of growth. For smallish, finite group components, as you grow the *w*-sequence you'd pretty soon get a resulting group whose size would exceed the number of chin whiskers you could pack into the volume of the universe. Even if the machine is theoretically possible, it could never be built. But models can be useful without translating literally into practical computability.

The second model problem is that the *w*-sequence is a model of a universe of structural possibilities and does not pick out any one structure. In fact you'd need some agent to decide, at each generative stage, which particular structure is being modelled. You'd also need a way to do the picking out.[15] The way would be something I'll call God's Switch, and the agent's action is God's Hand. God's Switch is supposed to be provided by something called occupancy subgroups.

MEG: I begin to see why this may be a good example for our theory of application.

XANTHUS: Let me just finish up here. The Symmetry Problem is maybe not a problem. The idea of the iso-regular groups is that shapes are characterized by symmetry groups of isometries. This is the basic idea motivating group theory, right? Groups characterize symmetries and invariance. But somewhere along the generative sequence of a *w*-sequence model, the components stop being iso-regular, which complicates the picture. It might not have been clear before, but every *w*-sequence is a group which is a symmetry group of its data. The generative process in this theory is one in which the key notion of recoverability depends on symmetry breaking (or asymmetry building) from state to state in the generative process.[16]

What's puzzling is that as the *w*-sequence grows, so does the degree of symmetry of whatever it describes—it equals the order of the resulting group, and as I showed, it gets very large. The idea of the model is that complicated shapes are construed cognitively in levels or stages from simpler shapes, step by step, implicitly, always. But what's simple, beside a cyclic isometry itself?

We may need to just relativize the notion of symmetry (the order of the symmetry subgroup of a figure within a given group). But it could be that the idea we want is already captured by the idea that when you need to move to a symmetry group of greater order, increasing the degree of symmetry, the figure has gotten more complex in the sense that it is not longer describable by the smaller symmetry group. A *squished square* is no longer described by any of the formulations of the symmetry group of the square, so you need to step up to a *w*-group of greater order, a *square followed by its squishing*, as represented by wreath-appending an affine group. The resulting *w*-sequence describes the figure as a square that has been squished, but this description requires more symmetry.

HERMIONE: Yes, it is sort of counterintuitive—a squished square is normally thought of as less symmetrical than a square.

XANTHUS: Yes, and the theory depends on this notion, Hermione, even though it describes the squished square by a different symmetry group, of greater order than that of the square. This requires a subtlety around the idea of symmetry which may actually be one of the strengths of the theory.

Finally, there's the Occupancy Problem. This is key; unless we can solve this, I can't see how to apply this to music in a useful way. Wreath-prepending *Z*2 in the *w*-sequence would seem to be intended to provide a switch for each element,[17] but it does not choose which are on and which, off (*God's Hand problem*). We can solve this one by inserting an agent into the model—though this just pushes the question back to a theory that would account for the agent's actions! At least, for a music-theoretical model, we don't have to account for agency and motivation, just the structure of the actions taken.

However, underlying this is a more serious God's Switch problem: does the intended switch work? Wreath-prepending *Z*2 to *Z*12, for example, *Z*2 *w* *Z*12, results in a data set of size 24 for the complete wreath group. (But the group is of order 12*(2**12).) Each group element of *Z*12 is (re)present(ed) on the compound data *both* turned on, *and* turned off (accepting the semantics of *Z*2 for this). Any action of a group element of *Z*2 *w* *Z*12 simply maps the complete data set of the wreath group into itself, of course (since we are constructing symmetries); particularly, *no part of the group action selects any subset* of the data set of *Z*2 *w* *Z*12, or of *Z*12. This is clear if one actually works out the math in detail. If we can't select subsets of the data, we can't model anything very interesting.

[silence while everyone digests all this]

HERMIONE: I couldn't really follow all that, Xanthus.

IESUS: Me, neither. The math doesn't sound that hard, but the way it's applied, the problems in the application, the difference between the intentions of the model-maker and the way the model fits the data . . .

CHLOE: I seem to remember . . . couldn't you construct a section through the data? Make a data subset such that each point of *G*2's data appears exactly once, so that each is specifically on or off in the section. That would model a subset of the *G*2 data. The group action would affect the on-ness and off-ness of each compound element, as well as affecting the things that are on or off. It's a possibility. You'd still need a metatheoretical agent, God's Hand, to do the selecting, but this structure could give you the switching effect you need, right?[18]

HERMIONE: That's a very interesting idea, Chloe.

XANTHUS: Yes, I think that would work—thanks Chloe! One more thing—the Army of Occupation. You'd need to be able to account for the particular structures of occupancy, too, if you mean to describe the structure of a particular object. The theory does not do this, or provide a place for it, and a lot of the complex specificity of the object would lie in the structures of occupancies among the structures of possibilities, so we'd have to extend or modify the theory to accommodate this. A theory of occupancy structures within the theory of shape-possibilities would have to be integrated with the theory of (at least) the structure of the agent's actions in choosing a path through the universe.

XP: What about ascesis and fit?

MEG: It seems pretty parsimonious, in that it's just one construction, *w*-sequences, and those aren't all that complicated considering the scope of its ambition, its intended applications. It does a good job of using simple things in complicated ways. In fact, I think that's the point of this model, build complex things from simple things in generative stages.

CHLOE: Is it simple enough? Or is the map larger than the territory here?

HERMIONE: By the time we plug in solutions to Xanthus's problems, it might not look simple at all. But music, and the other areas of application, are not so simple either. I think the jury is still out on ascesis.

XANTHUS: What about fit?

XP: Fit may be a problem, too. From your description, this begins to sound like a pretty elaborate way to model a basic idea of generation in stages. But all the *w*-sequence stuff may actually be essential, too, if it works as a theory both of construction and cognition, transfer and recoverability. And we saw how complex the theory could get if we extended it to address your objections, Order and the agent's actions, and the Army of Occupation and so on.

XANTHUS: I agree—it seems to me that I'll have to work with this theory in its application to music for a while before we can decide about good fit and ascesis.

CHLOE: Hey, we're getting too serious! Talking about ascesis. Here. [passing around the food platter]

[They all sit and munch, thinking or relaxing.]

XP: You know what? Meg never did get to explain his Categories. Meg?

[Meg snores.]

CHLOE: Meg, wake up, dear. [shakes Meg.]

XP: Well, I'm off to Soc.

HERMIONE: I'll walk you part way, Xp.

IESUS: I'll walk Meg home. [helps Meg up] Come on, Meg. It's been a long day.

[Exit all, leaving Chloe standing in the doorway with her arm around Xanthus.]

Afterword

Chloe and her friends were too sleepy to follow up on some of the immediate threads after Xanthus's discussion of occupancy, and Chloe's idea (which is really Michael Leyton's idea) about using sections of the wreath-product group's data. In Leyton's general theory, occupancy is a tool which (as Xanthus points out) he assumes rather than discusses in detail. From the musician's standpoint, *Z*2 *w* *Z*12 ought to be interpretable as manipulating subsets of the aggregate of pitch classes. Following Chloe's section idea, each pitch class (element of the data of *Z*12, which is the group of pc transpositions) is equipped with a state, on or off. A subset of pcs is represented by those pcs that are turned on in the section. For example, {1 4 5} would look like {<off, 0>, <on, 1>, <off, 2>, <off, 3>, <on, 4>, <on, 5>, and all the rest off}. A group element of *Z*2 *w* *Z*12 is of form <vector | *z*> where *z* is a group element of *Z*12 (a pc transposition operation), and vector is a 12-place *n*-tuple of elements of *Z*2; each of the positions in the vector is indexed by an operation in *Z*12 and will apply to the wreath data element containing the *Z*12 data element referenced by its index (the "selective effect"; see Leyton Chapter 3). The elements of *Z*2 are best thought of as two operations, identity and swap; when applied to a binary state, identity leaves it alone and swap turns it into the other state. So the vector part of the wreath operation has either an identity or a swap operation in each of the 12 positions indexed by the operations of *Z*12.

You can also think of this section of the wreath data as a 12 digit binary number, written left to right instead of right to left as usual, so that 2**0 is in the first place, 2**1 in the second place, and so on. The pcs index the offs and ons, as the exponents of the binary number places. (As a historical sidelight, my *normal form* and *representative form* of a set or its type, from *Basic Atonal Theory*, are then simply the smallest number, in this representation.)

This representation of a pc subset has interestingly unusual features for a musician. To get from one subset to its transposition, you could just apply the wreath group element that leaves all states alone (a vector full of identities) and has as its *Z*12 group element the desired transposition, as usual. But you could also apply the wreath group element with identity (T_0) as its *Z*12 group element, and the new state-operation vector would just switch off the old pcs and switch on the new ones; or some combination of the two. So there are multiple one-step ways to get from any pc subset to any other, only one of which will be the familiar way.

We can even tie this idea from Leyton's into Mazzola's work.

The so-called "section" amounts to a "characteristic function" for its subset. Reading the ordered pairs in the above example backward ("contravariant," as it were), the function yields "on" just for those pcs that are in the subset, and "off" elsewhere. In category theory, it is this characteristic function which amounts to the *subobject classifier* for the category **Sets** (Mazzola, 1126). The definition of a *Topos* requires the existence of a subobject classifier for the category (Mazzola, 1127 infra). But neither the category of abelian groups nor the category of *R*-modules has a subobject classifier, and thus these categories have no topoi (Mazzola, 1127). Mazzola wants to work with Grothendieck topologies, which require topoi (Mazzola, 1129). Mazzola's work-around is to translate the category of modules into the category of presheaves over modules, notated **Mod@**. This is the category *Func*(**Mod**opp, **Sets**) of contravariant set-valued functors on **Mod**, which provides the required translation from modules into a category that has a subobject classifier, namely **Sets** (see Mazzola, 1126, example 97, and p. 1119, example 92).

Mazzola's entire book, *The Topos of Music*, is based on this construction from its earliest formulations. What I have done above is to excavate from various places in the Appendices the information needed to read even the earliest formal definitions in Chapter 6 (from p. 63). This will give some idea of the mathematical overhead involved in even approaching this substantial work of mathematical music theory, which I am by no means prepared to discuss in the kind of detail evident in the discussion of Leyton's work. It is worth mentioning as an early warning to music theorists.

Leyton's work and Mazzola's work are quite different and independent of each other.

Notes

1. For a good popular treatment of cheerio theory, see Brian Greene, *The Elegant Universe: Superstrings, Hidden Dimensions, and the Quest for the Ultimate Theory* (New York: W.W. Norton, 1999). The "cheerio" and "just add dimensions" part of the ensuing discussion in this article is based only on such popularizations and does not pretend any authority. Readers fully conversant with such physical theory should try not to take it seriously here.

   My idea was to link an updated version of Pythagoreanism (historically itself linked with music theory) to the more modern music-mathematical issues raised further along in the article. The kind of algebraic geometry used in superstring theory is not unrelated to the constructions in Mazzola, see below.

2. Iesus underestimates the depth of this work; see for example Norman Carey and David Clampitt, "Self-Similar Pitch Structures, Their Duals, and Rhythmic Analogues," *Perspectives of New Music* 34, no. 2 (Summer 1996): 62–87.

3. Longchen Rabjam, *The Precious Treasury of the Way of Abiding and the Exposition of the Quintessential Meaning of the Three Categories* (Junction City, California: Padma Publishing, 1998), 15, 23.

4. See for example David Griffiths, *Introduction to Quantum Mechanics* (Englewood Cliffs, New Jersey: Prentice Hall, 1995).

5. The standard text is Saunders Mac Lane, *Categories for the Working Mathematician* (New York: Springer, 1971).

6. Meg continues to fluff this question, declining to talk about category theory and algebraic geometry as a basis for music theory—perhaps he had not finished reading Mazzola. However, there is no connection with Platonic Forms, etc.; if anything, categories are anti-essential.

7. See Example 92, p. 1119, Appendix G, in Guerino Mazzola, *The Topos of Music: Geometric Logic of Concepts, Theory and Performance* (Basel: Birkhäuser, 2002).

8. Of course this refers to Guerino Mazzola, who lives in Zurich.

9. The idea is from a conversation with Thomas Noll in Baton Rouge, March 2003.

10. Michael Leyton, *A Generative Theory of Shape* (Berlin: Springer, 2001).

11. See any good algebra text, for example, *Abstract Algebra* by David Steven Dummit and Richard M. Foote (Englewood Cliffs, New Jersey: Prentice Hall, 1991): direct product, 153ff.; semidirect product, 176ff.; wreath product, 189 (exercise 23). Also in Leyton, Appendix A.

12. A better answer is, not in general. The "standard wreath product" in not associative in general, but the permutational wreath product is associative. For proofs see J.D.P. Meldrum, *Wreath Products of Groups and Semigroups* (Pitman Monographs and Surveys in Pure and Applied Mathematics) (Harlow, England: Longman, 1995), 9–10. The *w*-sequences in Leyton are not associative.

13. Leyton, 12. Leyton defines an "iso-regular group" as one which is decomposable as a *w*-sequence whose component levels are groups that are either cyclic (finite) or 1-parameter Lie groups (continuous), and each level is represented as an isometry group. (Leyton, 12.) Xanthus is more concerned with the finite case, for music theory.

    On pp. 129 and 130, Leyton restates the iso-regularity condition in a series of definitions:

    > A group is *c*-cyclic if it is either cyclic or a connected 1-parameter Lie group (there are only two of these);
    >
    > A group is *c*-polycyclic if it has the structure of a series of group extensions whose components are *c*-cyclic;
    >
    > A group is wreath *c*-polycyclic if it has a *w*-sequence structure where the components are *c*-cyclic;
    >
    > A group is wreath-isometric if it has a *w*-sequence structure whose components are isometry groups;
    >
    > A group is iso-regular if it is a *w*-sequence structure which is wreath-polycyclic and wreath-isometric.

14. For example, Leyton, 67ff.

15. But see Leyton, 131ff.

16. Leyton, 41–44, 69–71.

17. Leyton 8, 47, et passim; subsequent *w*-sequence material refers to this book.

18. Michael Leyton, in private conversation, Seattle, April 2003.