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Research Note

Cool tools: Polysemic and non-commutative Nets, subchain decompositions and cross-projecting pre-orders, object-graphs, chain-hom-sets and chain-label-hom-sets, forgetful functors, free categories of a Net, and ghosts

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Expressive tools work primarily on the problem of the clarity and style of expression of models in some underlying theory. There is a structure among mathematics, a theory of the world (e.g. music), expressive constructs with their own structure, the applied theory, and models of some real entity. Lewin's transformational networks are expressive tools of great currency in music theory. We generalize Lewin-nets to Nets, which, unlike Lewin-nets proper, are both polysemic and non-commutative. We show how all kinds of Nets work on both simple and complex data objects, and how this is all useful in expressing musical analysis. We then show some ways that Nets connect with category theory and topology. We construct chain-hom-sets and chain-label-hom-sets in Nets that form free categories on object-graphs and transformation-graphs, and show how each chain-hom-set of paths consists of all possible musical compositions between an initial and final musical object in a chain within a particular Net. We note how the temporal partial order of any piece of music plays against the structural pre-order of its Net representation, and how the various partial and pre-orders cross-project against one another. There is a family of forgetful functors that relate the various categories **Net**, **Object-graph**, **Transformation-graph**, and **Grph**. We show how to get from both proper Lewin-nets and from Nets to an underlying pre-order such that the labelling arrows of the Net can then be construed as a pre-sheaf over the pre-order. We point out the interesting ghosts that survive all the forgetful functors, that is, the particular characterizing structures of each digraph (or transformation-graph or object graph) which then, reading upwards, constrain the possibilities for labelling its arrows and nodes in any superior entity (such as a Net) built on it.

Keywords: Lewin Nets; Polysemic Nets; Non-commutative Nets; Chain-hom-sets; Free category of a Net; Forgetful functors

1. Point of view

This essay is written by a music theorist who is not a mathematician. It aims, at least, to be comprehensible to both music theorists and mathematicians, and hopes to be useful to both. It contains some new music theory, but no new mathematics.

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2. Tools and nails

> When your only tool is a hammer,
> Every problem looks like a nail
> (Or a tack, or a fender, or . . .)

This is true for any theory – if you are Schenker, you construe pieces of music to fit that theory (in this case, if they do not fit, they are not music); if you are Rudolph Reti, every piece looks like a web of melodic motives; and so on [1,2]. You are looking at the world through (colour-of-your-theory)-coloured glasses. If you are daring, you may try looking at objects that are not supposed to fit, as though they did fit, and see what happens, as Felix Salzer did with his Schenker-coloured glasses [3]. (You know, if I squint, this thing looks like a nail after all.)

In this sense, any theoretical enterprise is a search for a cool tool. You can look at any object through any glasses, and there are many, many different prescriptions for those glasses, so in using one prescription, you are at least implicitly asserting the coolness of your tool.

In scholarly discourse, competing theories are often advanced by persuading people that the tool is cool, giving your reasons for thinking so – the results are better (as in, the music is more beautiful or satisfying so construed), the machinery is simpler or more beautiful or more complex, the theory is supposed to have predictive power, no predictive power since prediction is irrelevant or pernicious, more scope, less scope and more focus, and so on – there are many possible virtues you could put forward.

Among the virtues of any tool should be listed the virtues of the nails it hits. What is the problem? What is your tool good for? Hitting its particular kind of nails – never more or less than this. If the problem does not matter, neither does the tool. From this perspective, the most important part of any theoretical enterprise is finding the right problems to address. For example, if you do not believe that you could be convinced that hearing Reti-webs of motives is very helpful to your hearing of pieces of music, you will not be very interested in Reti's theory, or any refinement of it. But if the problem is a compelling one for you, then you will probably want to look closely at any tool that purports to work for that kind of problem. So, being a really cool tool means that it hits cool nails. This is the first distinction I am making here: tool and nail.

By invoking 'cool', I mean to underline a distinction between theories of the physical world, which may be corroborated or falsified by experimentation, and theories of art. Though aesthetics may be a factor in the evaluation of physical theories, this is always subordinated to experimental evidence, coming into play only when two theories explain the same evidence in different ways. However, theories of art, like artworks themselves, lacking an experimental basis of high inter-subjectivity, have to be evaluated using aesthetic criteria. The aesthetic criteria are always applied in evaluating the results of the theory, and sometimes in evaluating the theory itself.

3. Expressive tools

The second distinction I want to make is between the tool, that is, the theory, and the way this theory is expressed, which in fact is yet another tool, an 'expressive tool'. David Lewin called any such mode of expression 'poetic', because expression is an action that makes something [4]. Any specialized language is an expressive tool. One mode of expression is mathematics generally, but formalization is really a supermode comprising many others. Benjamin Boretz has argued that explicitly fictional, explicitly metaphorical, non-formal expression can be more precise than mathematical expression, and this is certainly a view that needs to be considered, but let us assume (in this paper) that expressing musical theories and musical models formally, in mathematics, is OK. We should not, however, lose track of the poetic, fictional, metaphorical nature of theories and models that are expressed mathematically [5].

Most theories can be expressed in mathematics, but in particular ways appropriate to each theory – Schenker theory as a system of recursive functions or a transformational grammar or a formal logical system, for example. In this case, the formalization is a translation from an originally informal presentation, and things can get lost in translation, but let's say the formalization is adequate and accurate. We now have a mathematical music theory, or rather several, each expressed in a different but equally mathematical mode (recursive functions, formal grammar, and so on). If the underlying theory is well made, each of its adequate and accurate formalizations will be equivalent to the others at some level of abstraction. However, each formalization remains distinct as a mode of expression. Working in a system of recursive functions has a different feel, and will naturally lend itself to different ways of thinking, than working in a transformational grammar.

David Lewin's transformational networks are an interesting case, worth exploring in some detail, because Lewin-nets are not really a theory of music at all, but rather a poetic language.* Suppose there is a theory that musical objects are relatable (viz. in a cool way) by only 'transposition' in the musical sense. We can build a model of a piece of music using this theory, and when we have the model, since the group of transpositions is one of the algebraic entities that fits into the Lewin-net framework, we have the option of expressing that model as a Lewin-net – or not.† Clearly, some kinds of things will be easier to say in some language, according to the nature of that language, and this is true of Lewin-net language. Beyond this, the special value of Lewin's poetic language might lie in its ability to express models of music made in many different theories, so long as the theories conform to the structural constraints of the Lewin-net language; or in the simplicity or beauty of the language of Lewin-nets; or in its facilitation of making clear and subtle distinctions among different models of the same music; or in the connection of such expressive networks to other mathematical objects.

Lewin-nets require a theory of music whose relations form a monoid. The net's arrows are consistently labelled in the monoid, and the net's vertices are labelled in a set of objects acted on by the monoid.‡ Lewin's own definitions in GMIT explicitly require a semi-group, but his definition there of the underlying abstract directed graph, what would be called a 'diagram scheme' in category theory, requires reflexivity (as a relation), which provides the identity that makes the labelling semi-group a monoid [7, p. 193, 9.1.1].¶

* Lewin's transformational network idea appeared initially as [6], and subsequently was extended and illustrated with many examples in his influential book [7].

† Obviously, the T_n group is only one of the many algebraic structures that would be expressible by a Lewin-net; see below.

‡ In terms of Lewin's [7] Definition 9.2.1, the monoid I am talking about would be SGP, but including the identity element implicit in the definition of node-arrow system.

¶ 'For present purposes, we shall stipulate that every node is in the arrow relation with itself'. This is in the paragraph defining node/arrow system, and is never modified by Lewin. In original article in PNM, there is no stipulation that every node is in the arrow relation with itself (see below); but since the arrows here are labelled in a group, not a semi-group, an identity is available for every node. About monoids: a semi-group is associative, a semi-group with identity is a monoid, a monoid in which every mapping has an inverse is a group. Every group is also a monoid.

Whole categories of musical theories are excluded, but many theories are included, given the pervasiveness of monoidal structures in the world. So Lewin-nets can be seen as defining a category of music theories, those whose relations are monoidal. Theories in this category are the nails that Lewin's expressive tool hits.

Or, if we take the original definition of Lewin-nets from the PNM article, the relations are those that form a group rather than a monoid. It is interesting that Lewin turned to semi-groups and monoids later. The algebraic theory of machines models machines as semi-groups or monoids rather than groups, because not every machine runs backwards – you can't assume the inverse of every operation. The theory of semi-group actions is less tamed than the theory of group actions.*

4. Lewin-nets labelled in GIS groups and percolated with group theory

Lewin's Generalized Interval System (GIS) is one important monoidal structure that can label the arrows of Lewin-nets. Its component interval group is what Lewin calls a 'simply transitive group'.† Note that Lewin-nets can be labelled in any monoid, of which a GIS interval group is only one kind. For example, the T_n/T_nI group of musical transpositions and inversions acting on the pitch classes (isomorphic to D_{24}, the dihedral group of order 24 acting on Z_{12}),‡ which labels Lewin-net arrows in applications such as Klumpenhouwer networks, is not itself a kind of group action Lewin had in mind for a GIS interval group [8, pp. 83–120].¶

Oren Kolman has shown that any Lewin-GIS can be re-written as a group. More precisely, two GIS are isomorphic iff their interval group components are isomorphic; 'every GIS is isomorphic to a canonical GIS associated with its interval group, and with

* See Yust [9] for some interesting pathological examples of Lewin-nets labelled in semi-groups that lack the identity, and in which the example graphs do have loops for each node.

† Note that a 'simply transitive group' is neither exactly a 'simple group', nor only a group with a 'transitive group action'. A 'simply transitive group' is defined thusly [7, p. 157]: a group acting on a set is simply transitive iff for any two elements in the set being acted on, there is exactly one group element that maps one into the other. Properly speaking, it is the group action that is 'simply transitive' in this sense, not the group. All this can be confusing given the entirely different other standard definitions of a 'transitive' group action (an action that has only one orbit) and 'simple' group (a group whose only normal subgroups are the trivial ones, the identity and the entire group itself). A simply transitive action does not imply that the group is simple. However, a simply transitive group action is also a transitive group action. D_{24}, the dihedral group of order 24 acting on Z12, is isomorphic to the T_n/T_nI group of musical transpositions and inversions on the pcs. Clearly, the action of D_{24} on Z_{12} is not 'simply transitive' in the sense that would qualify it as part of a GIS, since in D_{24} acting on Z_{12} there are exactly two (not one) elements of D24 mapping any given element of Z_{12} into some other given element of Z_{12}. Translated directly into Lewin's language from [7] 7.1.1, this would read, 'there exist two members OP1 and OP2 of STRANS such that OP1(s) = t and OP2(s) = t'. It explicitly contradicts Lewin's condition from 7.1.1.

‡ For a primer on music theory dealing with the T_n/T_nI group, see Rahn [10]. For a basic algebra reference, see Dummit and Foote [11]. In music theory, the notation T_n/T_nI refers to the group of musical transpositions and inversions, not some quotient group.

¶ See Lewin [8, p. 84]: 'Any network that uses T and/or I operations to interpret interrelations among pcs will be called a *Klumpenhouwer Network*.' However, one may note that Lewin's original article on Lewin-nets in PNM uses this D_{24} group for labelling networks among pcs; and in [8], Lewin is mostly concerned with constructing a general theory of isography among such networks, and the same isographies between such networks as contents of nodes of larger networks and the larger networks that contain them recursively, that eventually is extended to larger groups of operations including the multiplicative ones M_5 and M_7. So, K-nets are not restricted to pcs as node contents and not restricted to D_{24} for labelling the arrows, while Lewin-nets labelling in D_{24} with pc node-content showed up much earlier in Lewin's work. For motivation behind the terminology, see Lewin [8, pp. 114–116]. It was Klumpenhouwer who initiated the investigation into isomorphisms among Lewin-nets of pcs with arrows labelled in T_n/T_nI.

every group one can associate a GIS in a canonical way', so that the categories **Grp** and **GIS** are equivalent, and all of group theory applies to any GIS. Kolman draws far-reaching consequences from this, involving mathematical logic and topological groups, whose results include (among others) that the class of all commutative GIS has a decidable theory, and that the question of whether, in general, any two musical motives (represented as transformational networks labelled in the group of a GIS) are isographic in Lewin's sense, is formally undecidable [12].

Kolman's paper shows how the GIS, and therefore to some extent the Lewin-net expressive construct, itself relies on underlying 'mathematical theory' which can be used to leverage further revelations about the expressive construct, and therefore about any model of music in any 'musical theory' expressed in this expressive construct. This is an unusual and appealing twist on the usual relationship between mathematical theory and an applied theory via a formal expressive construct that mediates between the two.

In Example 1 below, the arrows may be read as 'constrains'. Another way to read these arrows is tool → nail. In Example 1, the models do not constrain the problems, so the actual, final nails in the chain – the terminal nails – are left out of the diagram.

Example 1

Mathematics → theory of intervals → expressive construct → applied theory
→ models of the world in the applied theory.

However, a hammer can be no better than the nails it hits. Example 2, reading the arrows in the other direction, as 'conditions', and this time for the general case rather than the special one discussed above, shows that the value of the whole chain depends on the value of its nails. Mathematics itself is unconditioned, though the problem may pick out one aspect of it.

Example 2

Problem = nail → model/applied theory → expressive construct if any
→ auxiliary theories if any → selection of appropriate kinds of mathematics.

By removing, or 'transferring', the concept of the GIS to another realm without forgetting any of its structure, Kolman has been able to expand the context for the GIS in a way that leads to greater knowledge not only about the constraints on the GIS, but about new possibilities for the GIS idea. Group theory transfers back down into GIS theory. The transfers up and down amount to a circulatory percolation.

In a similar way, as we shall see below, it is possible to use category theory and topology to explore Lewin networks: category theory transfers into Lewin network theory, while topology informs the underlying digraphs of both.*

* For related discussions see two earlier papers by this author: Rahn [13] is a popularized introduction in the form of a Socratic dialogue which eventually gets into some technicalities about Michael Leyton's model of structure using wreath-product sequences and touches on Mazzola's work; Rahn [14] surveys music theory, including some of Lewin's contributions, and urges more dynamic mathematical models for musical change and flow.

5. Polysemic Lewin-nets

A digraph with arrows labelled consistently in a monoid, and vertices labelled in objects acted on by the monoid – a Lewin-net – looks a lot like a commutative diagram in category theory.* But under Lewin's own definitions, it can't be, for several reasons which we will take up separately. The first reason is because Lewin defines an arrow in the node-arrow system as an ordered pair of nodes [7, p. 193]. So in Lewin-nets proper, there is at most one arrow between any two objects, while in a diagram in a category, there may be multiple arrows (in the same direction) between a pair of objects. Perhaps a Lewin-net arrow can have another label also, amounting to a second morphism between its tail and its head? No, because Lewin accomplishes the labelling by a function (which he calls TRANSIT) from the family of arrows into the monoid.

However, there is no good musical reason for this restriction. In perhaps the most common use of Lewin-nets, Klumpenhouwer-network analytical models label the arrows in the T_n/T_nI group acting on the pitch classes. Clearly, there are two elements of T_n/T_nI that will get from any pitch-class object s to a given object t. Lewin's definitions artificially force a choice between them, via the TRANSIT function, so that only one of them in any given Lewin-net labels the unique arrow between s and t. Yet in many a music-analytical situation, one would want the simultaneous multivalence, or polysemy, that would assert both musical relations. For example, if the objects are pitch classes 1 and 4, $T_3(1) = 4$ and $T_5I(1) = 4$. These are distinct but not incompatible musical perceptions of the way in which pitch class 1 moves to 4.

Lewin's restriction to monosemy would make sense if the labelling entity were always a simply transitive group such as is required for a GIS. Lewin's 'simply transitive' simply means there is exactly one possible arrow from s to t for any s and t [7, p. 157]. However, most useful labelling entities are semi-groups or groups for which this is not the case, including the ubiquitous T_n/T_nI acting on the pitch classes.

Of course, a better way to allow polysemy would be simply to define Lewin-nets consistently with the way categories are defined. There may be any number of arrows connecting a pair of objects, with tail and head functions on the family of arrows giving exactly one tail and one head for each arrow [15, p. 10].

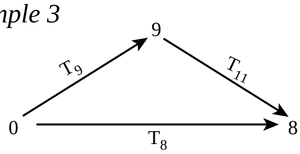
Example 3 shows a little Lewin-net (of the Klumpenhouwer-net kind), in which arrows are unique and labelled consistently in T_n/T_nI. You could think of this as a model of the first three melodic pitch-classes of Schoenberg's Opus 11, no. 1, <B G# G>, with B = 0. Example 4 shows a supernet of Example 3, in which there is an arrow from 8 to 9 as well as an arrow from 9 to 8. This is allowed under Lewin's definitions, which define an arrow as an ordered pair of vertices. In general, two such inverse arrows will be labelled by inverse transformations in the labelling monoid.† If our Lewin-net

* See Mac Lane [15]. Category is defined on p. 10 but see also metacategory on pp. 1–2. There are objects and arrows between them such that there is an identity arrow for each object, and each composable pair of arrows composes to another arrow in an associative way. A hom set hom(b, c) is the set of all arrows from b to c. 'We will later observe that a category is a monoid for the [composition] product ... ' (p. 10). The set of hom(a, a) of all arrows from a to a is a monoid (p. 11); and so on.

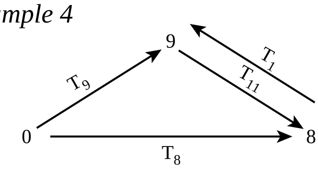
† See section 6 in this paper.

model of a piece contains many of these reverse-arrow situations, so many that in fact every label must have an inverse, the labelling monoid must be a group.

Example 3



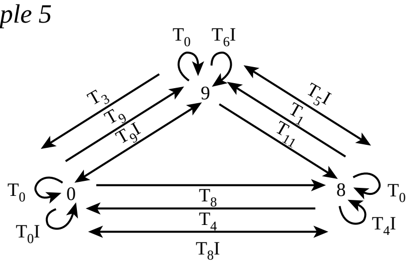
Example 4



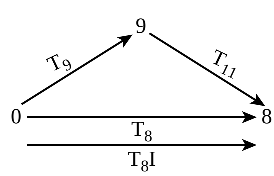
Mathematically, the addition of some reverse arrows is perhaps not very interesting. It may have implications about the further structure of the labelling monoid, but if we know already that the monoid is also a group, as we do for T_n/T_nI, we know that every labelling element has its unique inverse whether we show it on the diagram or not, so the diagram with reverse arrows tells us nothing new about the structure of the labelling monoid. However, remember that this diagram is a poetic expression of a musical analysis. Adding the arrow from 8 to 9 adds a musical assertion to the model: 'I hear 9 moving to 8, but also 8 moving to 9, in the ways described by the arrows and their labels'. A reason to assert this might be a musical observation elsewhere, for example, that the T_1 from 8 to 9 re-appears as a relation between the Bb and B in the right-hand thumb in bar 4, and as part of a larger melodic complex <C Bb B> which is a transposed retrograde (in fact a T_3, the inverse of the T_9 between 0 and 9) of <G# G A>. The point I am trying to make is simply that an assertion that is mathematically trivial can be musically significant, so that Example 4 is meaningfully different from Example 3 as a music-analytical assertion.

Example 5 shows a maximally polysemic Lewin-net, in which the three objects are connected by all relations provided for them by T_n/T_nI. The multiple arrows between two events (in the same order) are not provided for or allowed under Lewin's definitions. In practice, a given analysis expressed as a Lewin-net, or as such a polysemic generalization of a Lewin-net, is always (well, almost always) a pruning of the maximal network. In a sense, the maximal network does not assert much. The analyst adds meaning by choosing to exhibit only those relations which she has good musical reason for considering significant in this context. Example 6 shows a pruned version of Example 5, which still is polysemic rather than the original monosemic Lewin-net. Example 6 has only one more arrow than Example 3, the extra arrow from 0 to 8 labelled T_8I. Why might one want to assert both the T_8 and the T_8I? Many reasons; for example, the T_8 could also label the arrow between the A and F which follow directly in the melody, but the T_8I could also label the relation between the G = 8 in the diagram and the right-hand thumb 0 = B which appears harmonically beneath it in bar 2. In fact, this initial three-note *Grundgestalt* for the Opus 11 no. 1 might be, exceptionally, best represented analytically as the maximal polysemic network, since its relations are milked of every drop of implication by the piece.

Example 5



Example 6



In summary: I am asserting that there are good musical reasons for moving from Lewin's original monosemic formulation to polysemic Lewin-nets. We can do this independently of any other considerations.

Lewin does require that his transformational networks, what I am calling Lewin-nets, be commutative in the usual sense: the arrow-labels on any two chains from one object to another must compose to the same element of the labelling entity for the arrows, viz. the same monoid element [7, p. 195, 9.2.1].* The astute reader will have noticed that Examples 5 and 6 – the polysemic Lewin-nets – are not, strictly speaking, Lewin-nets at all because they are not commutative. This is a problem for any properties of Lewin-nets that depend on commutativity, but is not a problem in using such nets as expressive tools for an analysis in ways that do not so depend. We will have more to say about commutativity in section 6.

We will see in section 9 that polysemic and non-commutative nets do fit into category theory, so these qualities need not disturb us greatly, though we should keep track of the distinction between nets that are and those that are not commutative. This more general kind of construction should no longer be called Lewin-nets, since the ones that are non-commutative and/or polysemic do not conform to his definition.

What remains is (Definition) *a directed graph, the graph defined as in category theory, with arrows labelled in a monoid and nodes labelled in a set of objects acted on by that monoid, in such a way that the label on each arrow takes its tail into its head under the action*. This last is known as the commutation condition, and is not equivalent to commutativity (see section 9). We will call this construction simply a *Net*.

6. Polysemic and monosemic Nets on simple and complex objects; commutativity

Earlier I said: 'In general, two such inverse arrows will be labelled by inverse transformations in the labelling monoid': in general, because, depending on the objects that correspond to the vertices, a particular pair of objects may have a reverse arrow that is not the group inverse of the forward arrow. This will arise when the objects are complex and symmetrical. Take two vertices, A = {0 1 4 5 8 9} and B = {2 3 6 7 10 11}. A maps into itself under T_0, T_4, T_8, T_1I, T_9I, and T_5I. A maps into B under T_2, T_6, T_{10}, T_3I, T_7I, and $T_{11}I$. A 'monosemic' Net containing only one instance each of objects A and B could show a T_6 from A to B and a $T_{11}I$ from B to A, yet T_6 and $T_{11}I$ are not inverse operations in T_n/T_nI. A 'polysemic' Net could show all six ways of getting from A to B and all six ways of getting from B to A, and while they can pair up as group inverses (e.g. $T_{11}I$ is the inverse of $T_{11}I$), only five pairings are pairs of inverses ({T_2/T_{10}}, {T_6/T_6}, {T_3I/T_3I}, {T_7I/T_7I}, {$T_{11}I/T_{11}I$}).

Would the monosemic $T_6/T_{11}I$ Net fulfil Lewin's condition of commutativity? As he puts it in [7, p. 195, 9.2.1], but rephrased here, for any two different chains of arrows from nodes A to B in the Lewin-net, the semi-group composition of the labels on the arrows must be the same semi-group element for both chains. In this example, there is a chain of two arrows from A to A consisting of T_6 followed by $T_{11}I$; these compose to T_5I. This is the only chain in the Lewin-net, so Lewin's condition holds. So for this particular simple case on complex, symmetrical node-objects with only one arrow in each direction, though labelled with group elements that are not group inverses,

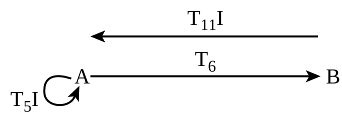
* Lewin does not use the word 'commutative', but his definition does require commutativity.

commutativity holds. It will hold any time there is only one chain between each pair of objects, because there is no second chain to compare.

However, this simple, monosemic Lewin-net only works for node-contents such that $T_{11}I(A) = T_6(A)$ – the arrow from A to B equals this particular reverse arrow from B to A, not in general, but for this particular pair of node objects. This in turn tells us something about possible node-content for this Lewin-graph – it has implications for the internal structure of the node-objects. In particular, knowing nothing else about the node contents, this Lewin-net tells us that object A must map into itself under T_5I, the composition of $T_{11}I$ and T_6.

What if we were to add an arrow directly from A to A in this little Lewin-net, a chain of one arrow from A to A, or a reflexive loop. We can label this reflexive arrow T_5I. The resulting Net is still monosemic. Does commutativity still hold? There are now four different arrow-chains from A to A, under the assumption (not generally warranted) that the reflexive loop is traversed at most once in each chain: <A A>, <A B A>, <A A B A>, and <A B A A>. The compositions work out as follows: <A A> T_5I; <A B A> $T_{11}I\ T_6 = T_5I$; <A B A A> $T_5I\ T_{11}I\ T_6 = T_0$; <A A B A> $T_{11}I\ T_6\ T_5 = T_0$. This Net is not commutative (Example 7).

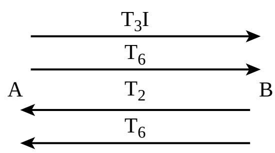
Example 7



In fact, as we have seen, there are six ways of labelling a reflexive loop from A to A, due to the symmetry of A. These six different reflexive loops themselves are sufficient to break commutativity in a Net that has any two of them. Even a graph consisting of just one reflexive arrow is not commutative, unless the monoid element labelling the arrow is idempotent. For example, a reflexive arrow labelled T_5I is not commutative, because $T_5I\ T_5I$ does not equal T_5I.

For the polysemic case on complex, symmetrical node-objects, take for simplification a Net consisting of one instance each of A and B with arrows from A to B labelled by T_6 and T_3I, and arrows from B to A labelled by T_2 and T_6. There are four chains from A to A: T_6 followed by T_2, T_6 followed by T_6, T_3I followed by T_2, and T_3I followed by T_6. They compose in the reverse order: $T_2T_6 = T_8$, $T_6T_6 = T_0$, $T_2T_3I = T_5I$, and $T_6T_3I = T_9I$. The four chains compose to four different group elements, so commutativity does not hold (Example 8).

Example 8



For the polysemic case on simple, non-symmetrical node-objects, the only monoid element that will map *s* into *s* for every *s* is the identity element in the monoid, so that 'in general' any chain from *s* to *s* must compose to this identity element. If the monoid is D_{24}, the identity is T_0. If the monoid is the Klein four-group $\{M_1, M_5, M_7, M_{11}\}$, the identity is M_1, and so on. If there is no identity element so that the labelling entity is not a monoid, but a semi-group, we cannot in general have both commutativity and chains from an object back to itself.

The exceptions to the 'in general' here have to do with the stabilizer of each element of the set acted on by the monoid or group. The stabilizer of *s* in G is the set of all elements of G that leave *s* alone. For example, pc 1 maps into itself under T_0 and T_2I: the point stabilizer of 1 in T_n/T_nI acting on the pitch classes is $\{T_0, T_2I\}$. So polysemic or monosemic chains from 1 to 1 (for example) might compose to either (but just one of) T_0 or T_2I without breaking commutativity. The label on an additional reflexive loop from 1 to 1 in the polysemic Net would, if the Net were to be commutative, constrain the monoid-element composition of the chains from 1 to 1.

7. Homotopy classes and chain-hom-sets; chains and subchains

We have just seen that chain-paths in a Net from a node to itself can tell us some interesting things, whether the paths are reflexive loops or longer chains out into the Net and back.

We can make a distinction between a 'chain' and a 'path'.* A chain takes the direction of the arrows into account, while a path does not. The structure of the set of such paths has a standard description in algebraic topology, given below as a close paraphrase of the discussion in Mac Lane [15, p. 20].

A 'groupoid' is a category in which every arrow is invertible; recall from topology the fundamental groupoid p(X) of a topological space X. An object of the fundamental groupoid is a point of X, and an arrow of the fundamental groupoid is a 'homotopy class' of paths from one point to another. A homotopy class is an equivalence class of all paths that are homotopic to one another. Two paths are homotopic iff they have the same initial and final points, and are continuously deformable into each other. The composite of two paths is first path followed by the second, so that the composite path begins with the tail of the first arrow and ends with the head of the second arrow. The inverse of any path is the same path traced in the opposite direction. Composition and inverse also apply to homotopy classes of paths, which makes the fundamental groupoid a category and a groupoid. Since each arrow is invertible, each point *x* in X determines a group *homg*(*x*, *x*) consisting of all paths from *x* to *x*. This is the 'fundamental group' of the topological space X. If there is a path from one point to another, the group of loops to one point is isomorphic under conjugation ($g \to fgf^{-1}$) to the group of loops to the other point. A connected groupoid – the fundamental groupoid of a path-connected space – is determined up to isomorphism by any of its isomorphic fundamental groups and the set of its points.

These definitions depend on the availability of continuous deformation in the space. They will be useful for music-theoretical ideas which take place in such a space. What about a finite, discrete (non-continuous) space such as a Net? What we want to preserve is the idea of constructing equivalence classes of paths or chains based on their having the same origin and destination.

Construct equivalence classes of all chains shown in a particular Net from a musical object *s* to a musical object *t*. These will partition the set of all chains shown in the Net, and so do form equivalence classes, as do homotopy classes. Call these the chain-hom-sets_N with respect to a particular Net N (referring to the different but related notion of hom-sets in category theory). A chain-hom-set of chains from *s* to *t* in a Net N is

* Lewin does this in [7, 9.1.2 and 9.1.3].

notated chain-hom-set$_N(s, t)$. The chain-hom-set$_N(s, s)$ contains all loops (reflexive or otherwise) from s to s in net N.

The chain-hom-sets do not care whether the net is commutative or not, or monosemic or polysemic. The only criteria for two chains being in the same chain-hom-set is that both chains begin with the same object, end with the same object, and both are shown in that particular Net.

A chain-hom-set in a commutative Net will be called a commutative chain-hom-set. In this case, the composition of the arrow-labels of each chain in the chain-hom-set must compose to the same element of the monoid labelling the arrows. This is true iff the Net itself is commutative.

Define a chain-label-hom-set as the set of strings of arrow labels in the chains of a chain-hom-set. These are the transformation-graphs underlying each chain in the Net, sorted by origin and destination like chain-hom-sets.

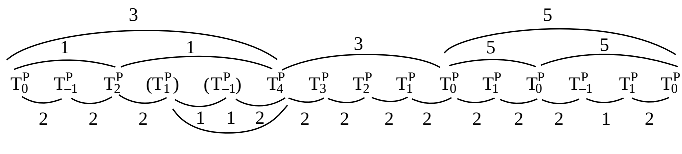
The partitioning of the set of all chains of the Net induced by equality of chain-label strings plays against the partitioning induced by the chain-hom-sets. That is, the same string of labels can appear in more than one chain-label-hom-set, and two chains in the same chain-hom-set can be different strings of labels.

We will see in sections 10 and 11 some other ways of characterizing the structure of the underlying unlabelled digraph of a Net or diagram. In particular, we will see how the chain-hom-sets and chain-label-hom-sets form free categories on the object-graphs and transformation-graphs.

First, however, consider why we might care about the chain-hom-sets of a Net. Musically, a chain between two objects is a compositional way of getting from the one to the other. Within the standard T_n/T_nI group acting on the pitch classes, there are an infinite number of ways of getting from one musical object to another. This is obvious even from the case of the identity loop labelled by T_0: an infinite number of different chains models traversing the loop some number of times, as in the car-alarm background to Laurie Anderson's ‘O Superman’, or any musical repetition. A chain composed of various group elements is a composition; ‘... any simple operation ... can be *expressed* as more than one compound operation of any given length .... Compositionally, this means that, if you have decided that you want to go from some set (or line or row or matrix, etc.) to some transformation of [it], ... you can “prolong” this transition in many different ways by substituting an equivalent compound operation of any length you want’ [10, p. 52]. A Net as a musical compositional scheme for a new piece, or a Net-analysis of an existing piece, will choose to exhibit only some from the set of all possible musical objects (of whatever type), and some from among the infinite number of such compound monoid-transformations connecting every pair of musical objects in the Net. The chain-hom-sets gather together into equivalence classes the various ways shown in this net of getting from one musical object to another. The subchains are coloured by their membership in chain-hom-set in the same way that pitches are coloured by their membership in pitch classes.

In *Basic Atonal Theory*, there is an analysis of Schoenberg's *Farben* as a composed chain of arrows labelled in T_n on pitches (see Example 9) [10, p. 67, Example 13]. There is a compositional substructure to the chain inherent in its analytical decomposition into subchain segments, or even (though not in this analysis), subchain partitioning extractions from the original chain. For example, partition a chain into an interlocking comb or hocket pattern such that each comb has the same initial and final object with respect to the ordering imposed by the comb – two extracted chains in the same chain-hom-set as the original, with respect to some Net that shows all these chains.* So each chain-hom-set of paths consists of all possible musical, and monoidal, compositions between an initial and final musical object *in a particular Net*. Some of the compositions may merge together in various ways to make larger or fuller compositions.

Example 9



8. Projected temporal partial order vs structural pre-order; object-graphs

We have been talking about musical compositions as modelled by chains, but Nets model compositions as labelled digraphs. However, there is always a temporal partial order that can be superimposed on (or if the node-objects contain temporal information, projected from) a Net-model of music. Of course, any partial order can be analysed, or decomposed, into linearly ordered chains, which brings into play as more fundamental the analysis in the previous section.

The temporal partial order plays against the structural pre-order of the Net, as Lewin noted [7, pp. 209–219]. The various partial and pre-orders cross-project against one another. Every instance of temporally non-ordered nodes in the overall temporal partial order, as in elements of a chord in a chain of chords, may be split into 'simultaneous' subchains each of which is temporally linearly ordered. A judicious analysis may decompose the whole into extracted (but not necessarily partitioning) subchains each of which is heard as a voice or line. Indeed, this is the mechanism behind musical counterpoint in general. The individual structure of the subchains and the structure of their weaving together can be explored using these notions of segmental and extractional subchains.

In a sense, the labels on the arrows also are a colouring of the more basic structures of the digraph and of the nodes. If the piece begins and ends with some object A, and moves through a certain partial order of other objects temporally between the two events, the analyst may choose to notice or assert certain relations between the nodes which form some arrow-labelling entity such as a group of transpositions, or a different one such as the T_n/T_nI group or the $T_n/T_nI/T_nM$ group, or some neo-Riemannian group.

In a bottom-up analysis of an unfamiliar kind of piece, for which you do not already think you know what the relations among the musical objects might be, or even what the musical objects are, that is, what granularity to use in the analysis, it seems to me that as you feel your way forward, the first thing to emerge structurally might well be a network of musical objects and certain unqualified connections among them. You might feel that 'this' object is connected to 'that' one, somehow, without knowing exactly how.

There is no Lewinesque name for a Net whose nodes are labelled but not its arrows. We will call them 'object-graphs'. This echoes Lewin's 'transformation graphs', and conforms to the mathematical notion of O-graph [15, p. 48].

* The logic of such counterpoint is explored in Rahn [16].

9. Nets percolated with category theory

Earlier, we postponed a discussion of exactly how Nets fit into category theory. First let us review what a category is. Here is the definition, paraphrased closely from Mac Lane: A 'directed graph' (also called a 'diagram scheme') is a set O of objects, a set A of arrows, and two functions from A into O. These are the head and tail functions, whose values for a given arrow are the co-domain and domain, respectively, of the individual arrow. The tail is the domain and the head of the arrow is the co-domain of that arrow. Within this construct, there is a set of composable pairs of arrows such that the head of the first is the tail of the second. A category is a graph such that composition of arrows is associative, with an identity arrow for each object that behaves as an identity for composition. Hence its monoidal quality: a category can be construed as a sort of monoid, but on object-graphs rather than on sets. An arrow is a morphism between objects in a category, a functor is a morphism of categories, and a natural transformation is a morphism of functors [15, pp. 10, 16, 49].

Note that in a category, an arrow points from its entire domain to its entire co-domain. In Nets, the objects for the tails and heads are elements of the domain and co-domain of the mappings labelling the arrows. This causes no problem for the construction of polysemic Nets here, but it means that individual Nets are not equivalent to categories, though Nets as a whole may constitute a category. Mazzola and Andreatta have recently shown one way of construing Lewin-nets within category theory: an individual Net is equivalent to an element of the limit of the diagram in a category [17].* As such, a Net need not be commutative, and as we have seen for both the monosemic and polysemic cases on simple and complex objects, it often is not commutative. By the way, it will be commutative whenever the labelling monoid's action on its objects is 'simply transitive' in Lewin's sense, so that the arrow-labels are equivalent to musical transpositions and therefore, by Lewin's insightful construction, to intervals. But it is also possible for it to be commutative under other conditions.

10. Forgetful functors; far isography; structural pre-orders

The objects labelling the nodes of Nets can be single pitches, single pitch classes, sets of pitches or sets of pc, sets or orderings of pc or notes or sounds with many parameters, more abstract objects such as Riemannian scale-degree-function-triads, or in fact whole swatches of musical score or musical performance, or entire musical pieces on each node – anything that can be related by some monoid labelling the arrows. We have seen that the internal structure of the node-objects is part of the constraint system represented by the Net. Removing the labels on the nodes of a Net is an abstraction that takes you back to the underlying (quasi)-Lewin-transformation-graph. In category theory terms, this is a forgetful functor from categories **Net** to **Transformation-graph**. It forgets structure. The structure that remains in the transformation-graph has been

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* I am afraid there is no really easy, brief, musician-friendly way to characterize the notion of categorical limit, nor that of the Grothendieck Topologies Mazzola is led rather inexorably to use for the most general and complete possible model of music.

preconditioned by the internal structure of the now-removed node-objects of the Net, but that information has otherwise been lost.

Lewin models this abstraction explicitly in one yet more abstract way in [7, pp. 199–200], as 'isography': two Nets are isographic iff their underlying transformation-graphs (without node-objects) are isomorphic. However, in practice Lewin immediately [7, p. 200] recognizes a distinction between having isomorphic transformation-graphs and having the same transformation graph (his figure 9.5a vs 9.5b). This latter relation comes to be known as strong isography, and is less abstract than 'isography' as Lewin defines it.

We can also note when two Nets have the same underlying digraph. For example, the five Lewin-nets in [7, figure 9.5], all have the same underlying digraph or node-arrow system, which is two nodes connected by two inverse arrows. Lewin conducts a stimulating discussion of homomorphic images of transformation graphs and product networks. In the accompanying figure 9.8, of the six Lewin-nets all depicting the same music in different ways, only two pairs have the same digraph, figures 9.8b and 9.8f, and 9.8e and 9.8g, as Lewin points out [7, pp. 204–206]. These particular pairs also have the same transformation graph, but if the labels on the arrows were changed, they would still have the same underlying digraph. I do not know of a name for 'having the same underlying digraph' in the music theory literature, though the concept is basic and certainly not new. Speaking in music theory, we will call this relation among Nets 'far isography', since it is three further moves of abstraction from strong isography, two moves from isography, and one move from the underlying transformation graph of a Net.

We can define further abstractions, other forgetful functors to the underlying digraph category **Grph** from categories **Net**, **Object-graph**, and **Transformation-graph**, removing arrow-labels and/or object labels. There is also a forgetful functor from **Net** to **Object-graph**. However, an object-graph and a transformation-graph are not relatable by any forgetful functor. An arbitrary object-graph *ob* and transformation graph *tr* need not share the same underlying digraph, or if they do, the node-objects may be incompatible with the arrow labels so that the action does not work, as in neo-Riemannian-group arrows and pitch class node-objects. In the special case in which *ob* and *tr* each 'descend' from the same Net by their forgetful functors, they not only share the same underlying digraph but are complementary, parallel constructions in the diagram of these related categories, and can be merged to recreate the forgotten Net from which they individually descend.

Perhaps we need to take an inventory of all these equivalence relations. There are a total of six from Nets: two Nets may have the same transformation graph, the same object-graph, or the same digraph. That is three; replacing 'the same' with 'isomorphic' adds three more. Also, two transformation graphs, or two object-graphs, may have the same (or isomorphic) digraphs. That is four or eight more, though these are really subsumed under the relations from Nets according to the diagram of the relations among these categories. It is debatable how much value using the isomorphic equivalence lends to any of these; the most plausible is from Nets to transformation-graphs, which is Lewin's 'isography'. Our constructions all rely on equivalence of 'the same'.

Finally, we note that the underlying digraph and object-graph of any proper Lewin-net (one that does conform to Lewin's definitions) can be understood as a restriction of a pre-order to the particular objects and labelled arrows shown in the Lewin-net. But this is not true for Nets. A pre-order (or quasi-order) is a relation that is reflexive and transitive. A pre-order, **P**, is a category in which there is at most one arrow from any object s to any object t [15, pp. 11]. Pre-orders include partial orders and linear orders. To make sense of the structural order of Nets, which may have multiple arrows between s and t, one would define a forgetful functor from **Net** to **P** that includes a forgetful functor from **Grph** to **P**, collapsing multiple parallel arrows in the Net's underlying digraph to one arrow. (There is a unique arrow in *P* from s to t whenever there is at least one arrow in *Grph* from s to t.) Alternatively, construct the underlying digraph of the chain-hom-sets of **Net**. These form a pre-order induced by the structure of the Net. The labelling arrows of the Net can then be construed as a pre-sheaf over the pre-order, leading further into topology and into category theory [18, pp. 744ff].*

11. Free categories and object-graphs; ghosts

Returning to topological aspects within category theory, the 'free monoid' generated by a set consists of all finite strings of elements of the set, with composition given as juxtaposition and the empty string as the identity. Construct a digraph category **Grph** whose morphisms (arrows from graph to graph within the category) are pairs of functions that work together to send the images of the domains of graph arrows into the domains of the images of the arrows, and the same for co-domains. There is then, of course, a forgetful functor from **Cat** to **Grph**, from the category of small categories to the category of digraphs [15, pp. 48–51]. Graphs are categories that, rather existentially, have forgotten how to compose arrows and lack an identity. However, it is easy to see that any digraph has its chains of arrows, so we can define the composable arrows as those juxtaposed in a chain in the digraph. The 'free category' generated by an object-graph has arrows (morphisms) from b to a whenever the digraph has a chain from b to a. For example, if the graph is just one reflexive loop, the free category consists of looping any number of times, which connects us to our earlier remarks on topology and to Laurie Anderson's car alarm.

This free category for a musical object-graph is formed by the chain-hom-sets of the object-graph as defined previously, and the free category for the musical transformation-graph is formed by the chain-label-hom-sets. Together, they work to characterize the categorical structure of a Net. The free category is an algebraic structure inherent in the underlying object-graph and transformation-graph of a Net. The forgetful functors from **Net** to **Object-graph**, **Transformation-graph**, and **Grph** only forget the explicit algebraic structures provided by the labels on the arrows and nodes, that is, the monoidal action of the labels on the labels. However, the ghosts of the internal structures of the lost node-objects and arrow-labels remain to haunt the digraph and its own structures. For example, a digraph consisting of six reflexive loops on the same node might be the relic of a Net whose node-object was {0 2 4 6 8 10} and whose arrow-labels formed the group of transpositions. Or, that same digraph could be the relic of a Net whose node-object was {0 1 4 5 8 9} and whose arrow-labels formed the T_n/T_nI group. Some vague residual memory survives the forgetful functors.

The classes of Nets equivalent under the relation of far isography also consist of all Nets that leave the same ghosts behind them. This is perhaps less interesting than the

* This path also leads to the Grothendieck topologies employed by Guerino Mazzola in [11].

ghosts themselves, that is, the particular characterizing structures of each digraph which then constrain the possibilities for labelling its arrows and nodes.

12. Nails again

So, all these are pretty cool tools, a toolbox of tools – polysemic and noncommutative Nets, chain-hom-sets and chain-label-hom-sets colouring compositional paths as pitch classes colour pitches, subchain decompositions and cross-projecting partial orders, object-graphs, forgetful functors, structural pre-orders, free categories, ghosts. The nails they all hit are music-analytical and expressive. Ultimately, the nails are aesthetic nails. The formally presented structures so expressed poetically characterize, or ought to, the intrinsic musical interest and beauty of pieces of music. It is an intrinsic beauty that is personally constructed as such by the analyst.

However, the ideas expressed in this paper have pretty clearly outgrown this rather Heideggerian toolishness. At some point, a theory takes off into its own realm. While it is still useful in various ways for various purposes, a more mature theory, such as quantum mechanics, and one may hope, eventually, music theory, ties together so many instrumentalities in so many complex ways that we no longer have the handyman looking for his hammer, performing *bricolage*.

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