Solution to Homework 5

Problem 1.a

If $\hat{O}$ is a Hermitian Operator, it should satisfy the condition $\left\langle f|\hat{O}g\right\rangle = \left\langle \hat{O}f|g\right\rangle$ for all $f(x)$ and all $g(x)$ in Hilbert space. Let's check this for momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$,

$$\langle f|\hat{p}g\rangle = \int_{-\infty}^{+\infty} f^{*}(-i\hbar)\frac{dg}{dx}dx = (-i\hbar)f^{*}g|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\left(-i\hbar\frac{df}{dx}\right)^{*} g dx = \langle\hat{p}f|g\rangle$$

where I have used integration by parts and ignored the boundary terms. Hence $\hat{p}$ is a Hermitian operator.

Problem 1.b

The Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$, then

$$\left\langle f|\hat{H}g\right\rangle = \int_{-\infty}^{+\infty} f^{*}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right) g dx = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} f^{*}\frac{d^2 g}{dx^2}dx + \int_{-\infty}^{+\infty} f^{*}V g dx$$

Integration by parts twice for the first term and ignore the boundary terms, we have

$$\left\langle f|\hat{H}g\right\rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\frac{d^2 f^{*}}{dx^2} g dx + \int_{-\infty}^{+\infty} f^{*}V g dx = \int_{-\infty}^{+\infty}\left(-\frac{\hbar^2}{2m}\frac{d^2 f}{dx^2} + V f\right)^{*} g dx = \left\langle \hat{H}f|g\right\rangle$$

Hence $\hat{H}$ is a Hermitian Operator.

Problem 1.c

Suppose $f(x)$ is an eigenfunction of a Hermitian operator $\hat{A}$ with eigenvalue a, namely $\hat{A}f = af$, and $\left\langle f|\hat{A}f\right\rangle = \left\langle \hat{A}f|f\right\rangle$. Then

$$a\langle f|f\rangle = a^{*}\langle f|f\rangle$$

$\langle f|f\rangle$ is nonzero, otherwise $f(x) \equiv 0$, so $a = a^{*}$, the eigenvalue a must be real.

Problem 2.a

The coordinate space wavefunction $\Psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} e^{-iE_0 t/\hbar} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} e^{-i\omega t/2}$, Fourier transform to momentum space,

$$\Phi_0(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi_0(x,t)\,dx \tag{1}$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2}\int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-\frac{m\omega}{2\hbar}x^2}\,dx \tag{2}$$

The integral in (2) has the form $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx$ (a is real and $a > 0$), and can be evaluated as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ax^2+bx}dx &= e^{b^2/4a}\int_{-\infty}^{\infty} e^{-a(x-b/2a)^2 dx} \\ &= e^{b^2/4a}\int_{-\infty}^{\infty} e^{-ay^2 dy} \\ &= \sqrt{\frac{\pi}{a}} e^{b^2/4a}\end{aligned}$$

where I have changed the variable $y = x - b/2a$. Thus,

$$\Phi_0(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2}\sqrt{\frac{2\pi\hbar}{m\omega}} e^{-p^2/2m\omega\hbar} = \frac{1}{(\pi m\omega\hbar)^{1/4}} e^{-p^2/2m\omega\hbar} e^{-i\omega t/2}$$

Problem 2.b

Express the creation operator in momentum space with the substitution $x = i\hbar\frac{d}{dp}$,

$$\begin{aligned} a^\dagger &= \frac{1}{\sqrt{2m\omega\hbar}}(-ip + m\omega x) \\ &= \frac{1}{\sqrt{2m\omega\hbar}}\left(-ip + im\omega\hbar\frac{d}{dp}\right) \end{aligned}$$

Then we can generate the first excited state by acting the creation operator on ground state,

$$\begin{aligned} \Phi_1(p) &= a^\dagger \Phi_0(p) \\ &= \frac{1}{\sqrt{2m\omega\hbar}}\frac{1}{(\pi m\omega\hbar)^{1/4}}\left(-ip + im\omega\hbar\frac{d}{dp}\right) e^{-p^2/2m\omega\hbar} \\ &= -i\left(\frac{4}{\pi m^3\omega^3\hbar^3}\right)^{1/4} p e^{-p^2/2m\omega\hbar} \end{aligned}$$

Its time dependence is $e^{-iE_1 t/\hbar} = e^{-i3\omega t/2}$, hence

$$\Phi_1(p,t) = -i\left(\frac{4}{\pi m^3\omega^3\hbar^3}\right)^{1/4} p e^{-p^2/2m\omega\hbar} e^{-i3\omega t/2}$$

Problem 3.a

$$LHS = ABC - CAB$$

$$\begin{aligned} RHS &= A(BC - CB) + (AC - CA)B \\ &= ABC - ACB + ACB - CAB \\ &= ABC - CAB \end{aligned}$$

Therefore, $[AB, C] = A[B, C] + [A, C]B$.

Problem 3.b

$$\begin{aligned} [p^n, x] &= p[p^{n-1}, x] + [p, x]p^{n-1} \\ &= pp[p^{n-2}, x] + p[p, x]p^{n-2} + (-i\hbar)p^{n-1} \\ &= p^2[p^{n-2}, x] + 2(-i\hbar)p^{n-1} \\ &= \cdots \\ &= p^{n-1}[p, x] + (n-1)(-i\hbar)p^{n-1} \\ &= -i\hbar n p^{n-1} \end{aligned}$$

Where I have used the fundamental commutator $[p,x]=-i\hbar$.
In general, commutator $[f(p),x]=-i\hbar\frac{df(p)}{dp}$ for any function $f(p)$

Problem 4.a

$$\begin{aligned}\frac{d}{dt}\langle x\rangle &= \frac{i}{\hbar}\langle [H,x]\rangle + \left\langle \frac{\partial x}{\partial t}\right\rangle \\ &= \frac{i}{\hbar}\langle [H,x]\rangle\end{aligned}$$

For x does not explicitly depend on time.
The commutator $[H,x]$ is,

$$\begin{aligned}[H,x] &= [\frac{p^2}{2m}+V(x),x] \\ &= \frac{1}{2m}[p^2,x] \\ &= \frac{1}{2m}(-i\hbar)\frac{dp^2}{dp} \\ &= (-i\hbar)\frac{p}{m}\end{aligned}$$

Thus, $\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\left\langle (-i\hbar)\frac{p}{m}\right\rangle = \frac{\langle p\rangle}{m}$, consistent with the Erenfest's theorem.

Problem 4.b

$$\begin{aligned}\frac{d}{dt}\langle p\rangle &= \frac{i}{\hbar}\langle [H,p]\rangle + \left\langle \frac{\partial p}{\partial t}\right\rangle \\ &= \frac{i}{\hbar}\langle [H,p]\rangle\end{aligned}$$

For p does not explicitly depend on time.
The commutator $[H,p]$ is,

$$\begin{aligned}[H,p] &= [\frac{p^2}{2m}+V(x),p] \\ &= [V(x),p] \\ &= i\hbar\frac{dV}{dx}\end{aligned}$$

where we have used the fact that for any function $f(x)$, the commutator $[f(x),p]=i\hbar\frac{df(x)}{dx}$.
Thus, $\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\left\langle i\hbar\frac{\partial V}{\partial x}\right\rangle = -\left\langle\frac{\partial V}{\partial x}\right\rangle$, consistent with Erenfest's theorem.