Sample Midterm Questions

Math 125A
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There may of course be problems on the midterm that are not similar to any of these. The midterm will be about 4-5 pages long.

1. A particle is moving along a straight line with acceleration $a(t) = -2\sin(t)$. At time $t = 0$, its velocity is $v_0 = 1$.

(a) (5 points) Find the velocity $v(t)$ of the particle as a function of time t.

(b) (5 points) What is the *total distance* travelled by the particle from time $t = 0$ to time $t = \pi$?

2. (a) (5 points) Find the indefinite integral $\int x\, e^{x^2} \sec^2(e^{x^2})\, dx$.

(b) (5 points) Evaluate the definite integral $\int_1^{\sqrt{e}} \frac{\cos(\pi \ln x)}{x}\, dx$.

3. Consider the region in the xy-plane enclosed between the curves $y = 2\sqrt{x}$ and $y = x$.

(a) (6 points) Find the area of this region.

(b) (4 points) Express the volume of the solid of revolution obtained when this region is rotated around the vertical line $x = -1$ in terms of a definite integral with respect to x. DO NOT EVALUATE THE INTEGRAL.

4. Consider the region in the xy-plane between the lines $x = 0$ and $x = 1$, above the x-axis, and below the curve $y = \sqrt{\sin^{-1}(x)}$.

(a) (4 points) Express the volume of the solid of revolution obtained when this region is rotated around the x-axis in terms of a definite integral with respect to x.
DO NOT EVALUATE THE INTEGRAL.

(b) (6 points) Express the volume of the solid of revolution obtained when this region is rotated around the x-axis in terms of a definite integral with respect to y, and evaluate that integral.

5. A rocket is taking off, going straight up. At time t, its height $s(t)$ is given by the formula

$$s(t) = \int_0^{\pi t} \sqrt{4\sin^2 x + 1}\ dx.$$

(a) (5 points) Estimate the height $s(1/2)$ of the rocket at time $t = 1/2$ by using a right-hand Riemann sum with $n = 3$ equal subintervals.

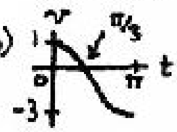
(b) (5 points) Find the velocity $v(t)$ of the rocket as a function of time t.

Math 125 C,D
Winter 2003
Midterm I Solutions

1. (a) $v(t) = \int a(t)\,dt = \int(-2\sin t)\,dt = 2\cos t + C$ (by guess & check)

$1 = v(0) = 2\cos(0) + C = 2 + C \Rightarrow C = -1 \Rightarrow \boxed{v(t) = 2\cos t - 1}$

(b)



$v(t) = 2(\cos t - \frac{1}{2}) \geq 0$ when $\cos t \geq \frac{1}{2} \Leftrightarrow 0 \leq t \leq \frac{\pi}{3}$ (for $0 \leq t \leq \pi$)

So $|v(t)| = \begin{cases} v(t) = 2\cos t - 1 & \text{for } 0 \leq t \leq \frac{\pi}{3} \\ -v(t) = 1 - 2\cos t & \text{for } \frac{\pi}{3} \leq t \leq \pi \end{cases}$

Tot. dist. $= \int_0^{\pi} |v(t)|\,dt = \int_0^{\pi/3}(2\cos t - 1)\,dt + \int_{\pi/3}^{\pi}(1 - 2\cos t)\,dt$

$= (2\sin t - t)\Big|_0^{\pi/3} + (t - 2\sin t)\Big|_{\pi/3}^{\pi} = (\sqrt{3} - \frac{\pi}{3}) - 0 + (\pi - 0) - (\frac{\pi}{3} - \sqrt{3}) = \boxed{2\sqrt{3} + \frac{\pi}{3}}$

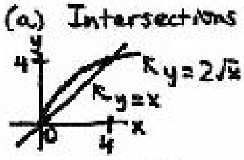
2. (a) $\int x e^{x^2} \sec^2(e^{x^2})\,dx = \int(\sec^2 u)\cdot\frac{1}{2}du = \frac{1}{2}\tan u + C = \boxed{\frac{1}{2}\tan(e^{x^2}) + C}$

$\left[\begin{array}{l} u = e^{x^2} \\ du = 2xe^{x^2}dx \end{array}\right.$

(b) $\int_1^{\sqrt{e}} \frac{\cos(\pi \ln x)}{x}\,dx = \int_0^{1/2} \cos(\pi u)\,du = \frac{1}{\pi}\sin(\pi u)\Big|_0^{1/2} = \frac{1}{\pi}(1 - 0) = \boxed{\frac{1}{\pi}}$

$\left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}\right.$ $\quad x = \sqrt{e} \Leftrightarrow u = \ln\sqrt{e} = \frac{1}{2}$, $\quad x = 1 \Leftrightarrow u = \ln 1 = 0$

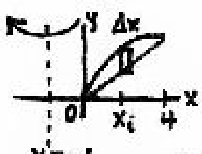
3. (a) Intersections: $x = 2\sqrt{x} \Leftrightarrow x^2 = 4x \Leftrightarrow x^2 - 4x = 0 \Leftrightarrow x(x-4) = 0 \Leftrightarrow x = 0, 4$



$A = \int_a^b (f_2(x) - f_1(x))\,dx = \int_0^4 (2\sqrt{x} - x)\,dx$

$= \left(\frac{4}{3}x^{3/2} - \frac{1}{2}x^2\right)\Big|_0^4 = \left(\frac{4}{3}\cdot 8 - \frac{1}{2}\cdot 16\right) - (0) = \boxed{\frac{8}{3}}$

(b)



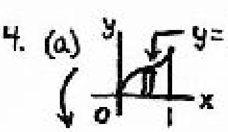
Shells. Volume of shell near $x_i \approx 2\pi r h\,\Delta r \approx 2\pi(x_i + 1)(2\sqrt{x_i} - x_i)\Delta x$

$r = x_i + 1$
$h = 2\sqrt{x_i} - x_i$
$\Delta r = \Delta x$

$$\text{Volume} \approx \sum_{0 \leq x_i \leq 4} 2\pi(x_i + 1)(2\sqrt{x_i} - x_i)\,\Delta x$$

$$\boxed{\text{Volume} = \int_0^4 2\pi(x+1)(2\sqrt{x} - x)\,dx}$$

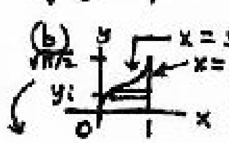
4. (a)



Slices (Disks)

$V = \int_a^b \pi(f(x))^2\,dx = \int_0^1 \pi\left(\sqrt{\sin^{-1}x}\right)^2 dx = \boxed{\int_0^1 \pi \sin^{-1}x\,dx}$

(b)



Shells

Solve for x: $y^2 = \sin^{-1}x$, $x = \sin(y^2)$. Upper limit: $y = \sqrt{\sin^{-1}(1)} = \sqrt{\frac{\pi}{2}}$

$V = \int_c^d 2\pi y\,(g_2(y) - g_1(y))\,dy = \int_0^{\sqrt{\pi/2}} 2\pi y\,(1 - \sin(y^2))\,dy$

$\left[\begin{array}{l} u = y^2 \\ du = 2y\,dy \end{array}\right.$ $\quad y = \sqrt{\pi/2} \Leftrightarrow u = \pi/2$, $\quad y = 0 \Leftrightarrow u = 0$

$= \int_0^{\pi/2} \pi(1 - \sin u)\,du$

$= \pi(u + \cos u)\Big|_0^{\pi/2} = \pi\left[\left(\frac{\pi}{2} + 0\right) - (0 + 1)\right] = \boxed{\frac{\pi^2}{2} - \pi}$

5. (a) $s(\frac{1}{2}) = \int_0^{\pi/2} \sqrt{4\sin^2 x + 1}\,dx$. Let $a = 0$, $b = \frac{\pi}{2}$, $n = 3$, $\Delta x = \frac{b-a}{n} = \frac{\pi}{6}$; $x_0 = 0, x_1 = \frac{\pi}{6}, x_2 = \frac{\pi}{3}, x_3 = \frac{\pi}{2}$

$R_3 = \Delta x\,(f(x_1) + f(x_2) + f(x_3)) = \frac{\pi}{6}\left(\sqrt{4\sin^2\frac{\pi}{6} + 1} + \sqrt{4\sin^2\frac{\pi}{3} + 1} + \sqrt{4\sin^2\frac{\pi}{2} + 1}\right)$

$= \boxed{\frac{\pi}{6}(\sqrt{2} + 2 + \sqrt{5})} \approx 2.95848$

(b) Let $G(u) = \int_0^u \sqrt{4\sin^2 x + 1}\,dx$. Then $G'(u) = \sqrt{4\sin^2 u + 1}$ (by F.T.C. Part 1)

Since $s(t) = G(\pi t)$, $v(t) = \frac{ds}{dt} = G'(\pi t)\cdot\pi = \boxed{\pi\sqrt{4\sin^2(\pi t) + 1}}$

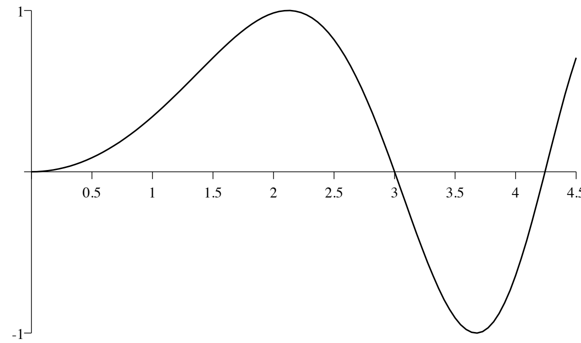
1 (12 points) Compute the following integrals. Give your answers in exact form.

(a) (4 points) $\displaystyle\int_1^8 \frac{2x+5}{\sqrt[3]{x^2}}\,dx$

(b) (4 points) $\displaystyle\int_0^\pi \frac{\sin t}{1+\cos^2 t}\,dt$

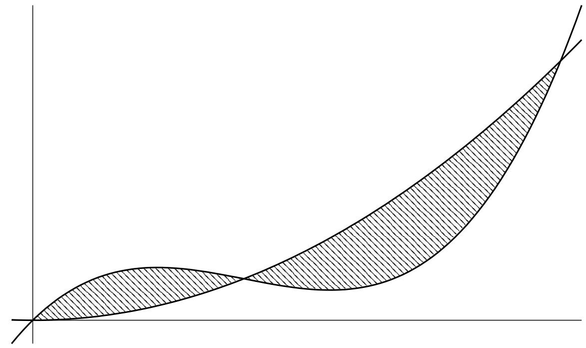
(c) (4 points) $\displaystyle\int y^3\sqrt{y^2-7}\,dy$

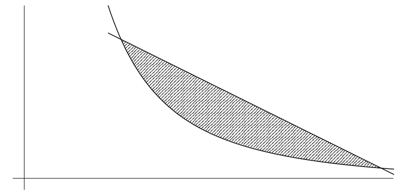
2 (10 points) A model car travels along a straight track. Its velocity is given by the function $v(t) = \sin\left(\frac{\pi t^2}{9}\right)$, where t is in seconds and v is in feet per second. Use the Midpoint Rule and $n = 6$ to estimate the **total** distance traveled by the car between $t = 1$ and $t = 4$ seconds.



3 (6 points) Let $f(x) = \int_0^{x^2-4x} e^{\sqrt{t}}\, dt$. Find the interval on which $y = f(x)$ is increasing.

4 (10 points) Compute the total area bounded by the curves $y = x^2$ and $y = x^3 - 6x^2 + 10x$.





5 (12 points) Let R be the region in the first quadrant bounded by $y = \dfrac{9}{x^2}$ and $y = 13 - 4x$. Set up the following integrals.

DO NOT EVALUATE.

(a) (6 points) Set up an integral that computes the volume of the solid generated by rotating R around the x-axis using the method of washers.

(b) (6 points) Set up an integral that computes the volume of the solid generated by rotating R around the line $x = -2$ using the method of shells.

1.(a) $\displaystyle\int_1^8 \frac{2x+5}{\sqrt[3]{x^2}}\,dx = \int_1^8 2x^{1/3} + 5x^{-2/3}\,dx = \frac{3}{2}x^{4/3} + 15x^{1/3}\Big|_1^8 = \frac{75}{2}$

(b) Let $u = \cos t$ so $du = -\sin t\,dt$. Then $\displaystyle\int_0^\pi \frac{\sin t}{1+\cos^2 t}\,dt = -\int_1^{-1} \frac{1}{1+u^2}\,du = -\tan^{-1} t\Big|_1^{-1} = \frac{\pi}{2}$

(c) Let $v = y^2 - 7$ so that $dv = 2y\,dy$ and $y^2 = v + 7$. Then $\displaystyle\int y^3\sqrt{y^2-7}\,dy = \frac{1}{2}\int (v+7)\sqrt{v}\,dv =$ $\displaystyle\frac{1}{2}\int v^{3/2} + 7v^{1/2}\,dv = \frac{1}{5}v^{5/2} + \frac{7}{3}v^{3/2} + C = \frac{1}{5}(y^2-7)^{5/2} + \frac{7}{3}(y^2-7)^{3/2} + C$

2. $\Delta t = \frac{1}{2}$ and the t-coordinates of the midpoints are $\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}$. The function $v(t)$ is positive at the first 4 values and negative at the last 2. Thus the total distance is

$$\frac{1}{2}\left[v\left(\frac{5}{4}\right) + v\left(\frac{7}{4}\right) + v\left(\frac{9}{4}\right) + v\left(\frac{11}{4}\right) - v\left(\frac{13}{4}\right) - v\left(\frac{15}{4}\right)\right] \approx 2.1784 \text{ feet.}$$

3. By the Fundamental Theorem of Calculus, $f'(x) = (2x-4)e^{\sqrt{x^2-4x}}$. This is defined when $x^2 - 4x \geq 0$, that is $x \geq 4$ or $x \leq 0$. Since $e^u > 0$ for any u, the derivative is positive if it is defined and if $2x - 4 > 0$. Thus the function is increasing when $x \geq 4$.

4. Solve $x^2 = x^3 - 6x^2 + 10x$ to get $x = 0, 2, 5$. Then compute

$$\int_0^2 \left(x^3 - 6x^2 + 10x\right) - \left(x^2\right)\,dx - \int_2^5 \left(x^3 - 6x^2 + 10x\right) - \left(x^2\right)\,dx = \frac{253}{12} \approx 21.083.$$

5. For both parts you need to solve $\dfrac{9}{x^2} = 13 - 4x$. This gives $4x^3 - 13x^2 + 9 = 0$ which is hard to solve by elementary methods. Guessing and checking, you find that $x = 1$ is a solution. Using long division, you get $4x^3 - 13x^2 + 9 = (x-1)(4x^2 - 9x - 9)$. The quadratic term has roots $x = 3, -\frac{3}{4}$. So the limits of integration are $x = 1$ to 3.

(a) $\displaystyle\pi\int_1^3 \left(13 - 4x\right)^2 - \left(\frac{9}{x^2}\right)^2\,dx$

(b) $\displaystyle 2\pi\int_1^3 \left(x+2\right)\left(13 - 4x - \frac{9}{x^2}\right)\,dx$

1. Find the average value of the function $f(x) = \sin^{-1}(x)$ on the interval $0 \leq x \leq 1$.

Math 125 B

Spring 2002

Midterm II Solutions

1. $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \int_0^1 \sin^{-1}x\,dx = x\sin^{-1}x\Big|_0^1 - \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$

integrate by parts: $u = \sin^{-1}x$, $dv = dx$, $du = \frac{dx}{\sqrt{1-x^2}}$, $v = x$

$= \frac{\pi}{2} + \left(\sqrt{1-x^2}\right)\Big|_0^1 = \boxed{\frac{\pi}{2} - 1}$

[Remark: Strictly speaking, the integration by parts gives an improper integral:

$\int_0^1 \sin^{-1}x\,dx = \lim_{b\to 1^-}\int_0^b \sin^{-1}x\,dx = \lim_{b\to 1^-}\left(x\sin^{-1}x\Big|_0^b - \int_0^b \frac{x\,dx}{\sqrt{1-x^2}}\right) = \lim_{b\to 1^-}\left(b\sin^{-1}b + (\sqrt{1-x^2})\Big|_0^b\right)$

$= \lim_{b\to 1^-}\left(b\sin^{-1}b + \sqrt{1-b^2} - 1\right) = 1\cdot\sin^{-1}1 + 0 - 1 = \frac{\pi}{2} - 1$]

5. The region in the xy-plane between the lines $x = 1$ and $x = 3$, above the x-axis and below the graph of $y = \ln x$, is rotated around the y-axis. (*Note: around the y-axis*)

(a) (3 points) Express the volume of the solid of revolution as a definite integral with respect to x. IN THIS PART, DO NOT EVALUATE THE INTEGRAL YET.

(b) (7 points) Evaluate the integral in part (a) to find the volume of the solid of revolution.

5. a) Shells $V = \int_a^b 2\pi x f(x)\,dx = \boxed{\int_1^3 2\pi x \ln x\,dx} = 2\pi\left[\int_1^3 x \ln x\,dx\right]$

b) Integrate by parts $u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$$= 2\pi\left[\frac{x^2}{2}\ln x\Big|_1^3 - \int_1^3 \underbrace{\frac{x^2}{2}\cdot\frac{1}{x}}_{= x/2}\,dx\right]$$

$$= 2\pi\left[\frac{9}{2}\ln 3 - \frac{x^2}{4}\Big|_1^3\right]$$

$$= 2\pi\left[\frac{9}{2}\ln 3 - 2\right] = \boxed{\pi(9\ln 3 - 4)}$$

3 (7 points) A bag of sand originally weighs 160 lbs. It is lifted at a constant rate of 4 ft/min. The sand leaks out of the bag at a constant rate so that when it has been lifted 20 ft only half the sand is left. How much work is done lifting the bag 20 ft?

3. Ignore the weight of the rope. Let y be the height of the bag. Then $y = 4t$ feet after t minutes. Let $F(t)$ be the weight of the bag after t minutes. We have $F(t) = 80$ when $y = 20$. This is when $20 = 4t$ or $t = 5$ minutes. Since $F(0) = 160$, the bag is losing $80/5 = 16$ lbs/min. Thus $F(t) = 160 - 16t$ and $F(y) = 160 - 16(y/4) = 160 - 4y$. Therefore the work is given by $\int_0^{20} 160 - 4y \, dy = 2400$ ft-lbs.