Sound propagation through an inhomogeneous and moving medium: A geometrical approach

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1 Introduction

The purpose of this note is to derive ray-equations for 2-dimensional acoustic propagation through range dependent sound speed and current fields. Solutions are sought which can be readily solved on the computer. The literature on this subject (See References herein) appears to be sorely lacking in this area.

2 The 2-Dimensional Case

Assume spatially varying but time frozen fields of sound speed and current of the form $c = c(x, z)$ and $\vec{v} = (u(x, z), 0, 0)$. In a moving medium the wave equation becomes,

$$\nabla^2\Psi = \frac{1}{c^2}(\partial_t + u\partial_x)^2\Psi \tag{1}$$

$$(\partial_{xx} + \partial_{zz})\Psi = \frac{1}{c^2}(\partial_{tt} + u^2\partial_{xx} + 2u\partial_{tx} + u\partial_x u\partial_x)\Psi \tag{2}$$

and assuming a wave of fixed frequency $\Psi = \tilde{\Psi}\exp i\omega t$, Eq. 2 takes the form,

$$[\ (1-\gamma^2)\partial_{xx} + \partial_{zz} + (2i\gamma k - u\partial_x u\partial_x) + k^2\ ]\tilde{\Psi} = 0 \tag{3}$$

where $k = \omega/c$ and $\gamma = u/c$ (the Mach number). Equation 3 is the moving medium equivalent to the Helmholtz equation.

2.1 Geometrical Acoustics

Next the geometrical acoustics limit is treated. Assume that $\tilde{\Psi}$ is a slowly modulated wave; that is

$$\tilde{\Psi} = A(x, z)\exp i\omega\phi(x, z) \tag{4}$$

where A is a slowly varying function of (x, z) and ϕ is a rapidly varying function of (x, z). Substituting Eq. 4 into Eq. 3 and taking the real part gives,

$$(1-\gamma^2)[\partial_{xx}A - \omega^2(\partial_x\phi)^2A] + \partial_{zz}A - \omega^2(\partial_z\phi)^2A + (\omega/c)^2A - u\partial_x u\partial_x A - 2\gamma(\omega^2/c)A\partial_x\phi = 0. \tag{5}$$

Since the high frequency limit is of interest and because A is a much more slowly varying function of (x, z) than ϕ only terms of order ω^2 are retained. The result is,

$$(1-\gamma^2)(\partial_x\phi)^2 + (\partial_z\phi)^2 + 2(\gamma/c)\partial_x\phi - (1/c^2) = 0. \tag{6}$$

To get the ray solutions a function $H(p_x, p_z, x, z)$ is defined as,

$$H \equiv (1-\gamma^2)p_x^2 + p_z^2 + 2(\gamma/c)p_x - (1/c^2) = 0 \tag{7}$$

where I have defined $\vec{p} = \vec{\nabla}\phi$. The function H (Eq. 7), called the Hamiltonian function, is consistent with the Hamiltonian functions defined by Keller (1954), and Uginčius (1972). The objective is to obtain a family of curves parameterized by a variable λ (i.e. $x(\lambda)$, $z(\lambda)$, etc.) such that $H = 0$ along those curves. Mathematically this is,

$$\frac{dH}{d\lambda} = \frac{\partial H}{\partial p_i}\frac{dp_i}{d\lambda} + \frac{\partial H}{\partial x_i}\frac{dx_i}{d\lambda} = 0 \tag{8}$$

where $x_i = (x, z)$ and $p_i = (p_x, p_z)$ and implicit summation is understood. The condition that $H = 0$ along these paths will hold if the following equations are satisfied.

$$\frac{dp_i}{d\lambda} = -\frac{\partial H}{\partial x_i} \tag{9}$$

$$\frac{dx_i}{d\lambda} = \frac{\partial H}{\partial p_i}. \tag{10}$$

These are Hamiltons equations.

2.1.1 Solution via Hamiltons Equations

A solution for the ray equations is now sought via Hamiltons Equations. Starting first with the spatial coordinates (x, z) we get,

$$\frac{dx}{d\lambda} = \frac{\partial H}{\partial p_x} = 2(1-\gamma^2)p_x + 2\gamma/c \tag{11}$$

$$\frac{dz}{d\lambda} = \frac{\partial H}{\partial p_z} = 2p_z. \tag{12}$$

These equations can be manipulated to obtain the ray slope, that is

$$\frac{dz}{dx} = \frac{p_z}{(1-\gamma^2)p_x + \gamma/c} \equiv \tan\theta. \tag{13}$$

It is helpful to eliminate the variable p_x. This can be done using Eq. 7 which gives,

$$p_x = \frac{-\gamma/c + \sqrt{c^{-2} - p_z^2(1-\gamma^2)}}{(1-\gamma^2)} \tag{14}$$

where we have taken the positive solution from the quadratic formula for p_x since it reduces to the solution when $u = 0$. Therefore we obtain as a solution,

$$\frac{dz}{dx} = \frac{p_z}{\sqrt{c^{-2} - p_z^2(1-\gamma^2)}} \tag{15}$$

Next we need an equation for p_z and this equation is,

$$\frac{dp_z}{dx} = -\frac{\partial H}{\partial z}\frac{dz}{d\lambda} = \frac{p_x\gamma - c^{-1}}{\sqrt{c^{-2} - p_z^2(1-\gamma^2)}}\left(\frac{1}{c^2}\frac{\partial c}{\partial z} + p_x\frac{\partial \gamma}{\partial z}\right) \tag{16}$$

Next we go to travel time. Recall our formulation of the wave function

$$\Psi = A \exp i\omega(\phi - t) \tag{17}$$

and from stationary phase we see that travel time, T, is simply ϕ. Therefore we have,

$$\frac{dT}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}\frac{dz}{dx} = \left(p_x + p_z \frac{dz}{dx}\right) \tag{18}$$

and with a little manipulation and using Eq. 14 we get

$$\frac{dT}{dx} = \frac{1}{(1-\gamma^2)}\left(-\gamma/c + \frac{c^{-2}}{\sqrt{c^{-2} - p_z^2(1-\gamma^2)}}\right). \tag{19}$$

To first order in u/c the familiar result is obtained,

$$\frac{dT}{dx} = \left(\frac{dT}{dx}\right)_0 - \frac{u}{c^2} \tag{20}$$

where $(dT/dx)_0$ is the value with $u = 0$. A better understanding of Eq. 19 can be obtained by writing it in terms of the ray angle θ. Equation 15 is equal to the tangent of the ray angle ($\tan\theta$) therefore we can solve this equation for p_z and the result is,

$$p_z = \frac{\sin\theta}{\sqrt{c^2 - u^2 \sin^2\theta}}. \tag{21}$$

Substituting Eq. 21 into Eq. 19 gives

$$\frac{dT}{dx} = \frac{\sec\theta}{c^2 - u^2}\left(-u\cos\theta + \sqrt{c^2 - u^2\sin^2\theta}\right) = \frac{\sec\theta}{u\cos\theta + \sqrt{c^2 - u^2\sin^2\theta}}. \tag{22}$$

It can be seen now that for θ=0 we have the appropriate result $(dT/dx) = (c+u)^{-1}$. Equation 22 is consistent with the results of Thompson (1972), Uginčius (1972), and Munk (1995).

3 Putting It All Together

I will summarize by listing the equations to be solved on the computer. Re-writing simply in terms of c and u the equations are,

$$\begin{aligned}
\frac{dT}{dx} &= \frac{1}{(c^2-u^2)}\left(-u + \frac{c}{\sqrt{1 - p_z^2(c^2-u^2)}}\right) &(23)\\
\frac{dz}{dx} &= \frac{p_z c}{\sqrt{1 - p_z^2(c^2-u^2)}} &(24)\\
\frac{dp_z}{dx} &= \frac{p_x u - 1}{\sqrt{1 - p_z^2(c^2-u^2)}}\left(\frac{p_x}{c}\frac{\partial u}{\partial z} + (1 - p_x u)\frac{1}{c^2}\frac{\partial c}{\partial z}\right) &(25)
\end{aligned}$$

where

$$p_x = \frac{-u + c\sqrt{1 - p_z^2(c^2-u^2)}}{(c^2-u^2)}. \tag{26}$$

Note that these equations can be solved very efficiently on the computer because they only require the evaluation of vertical gradients of current and sound speed.

Finally the initial conditions are,

$$T(0) = 0 \quad (27)$$

$$z(0) = \text{source depth} \quad (28)$$

$$p_z(0) = \sin\theta_0/\sqrt{c_0^2 - u_0^2\sin^2\theta_0} \quad (29)$$

where c_0 and u_0 are the values of sound speed and current at the source and θ_0 is the launch angle.

3.1 First Order Effects

It is helpful to write Eqs. 23-25 to first order in u/c. We obtain,

$$\frac{dT}{dx} = \frac{1}{c^2}\left(\frac{c}{\sqrt{1-p_z^2c^2}} - u\right) \quad (30)$$

$$\frac{dz}{dx} = \frac{p_z c}{\sqrt{1-p_z^2c^2}} \quad (31)$$

$$\frac{dp_z}{dx} = -\frac{1}{c^2}\frac{\partial c}{\partial z}\frac{1}{\sqrt{1-p_z^2c^2}} + \frac{1}{c^2}\left(\frac{2u}{c}\frac{\partial c}{\partial z} - \frac{\partial u}{\partial z}\right) \quad (32)$$

where we see that the ray slope is unaffected to first order in u/c, and the travel time correction is simply a current head/tail wind. Equation 32 is interesting. Lets try and evaluate the relative size of the 3 terms. First, the factor $1/\sqrt{1-p_z^2c^2}$ is of order $\sec\theta = \mathrm{O}(1)$. Therefore, the factors of interests are $\frac{\partial c}{\partial z}$ and $\frac{\partial u}{\partial z}$. The second term in the equation can be neglected because it is less than the first term by order u/c. Typical sound speed gradients in the main thermocline are roughly 0.06 s^{-1} and typical Garrett-Munk internal wave sound speed gradiants are 0.01 s^{-1}. Current shears observed from the eastern NPAL mooring are shown in Fig. 1 over a 5 day period. Isotherm depths are also displayed in Fig. 1 to show the advection of the shear along the isotherms. The current shear has magnitudes of roughly 6-10 cph or 0.01 to 0.02 s^{-1}. This is comparable to the GM sound speed gradient effect, but with a different spatial structure.

The conclusion is that current shear in the upper ocean could have a significant effect on acoustic scattering.

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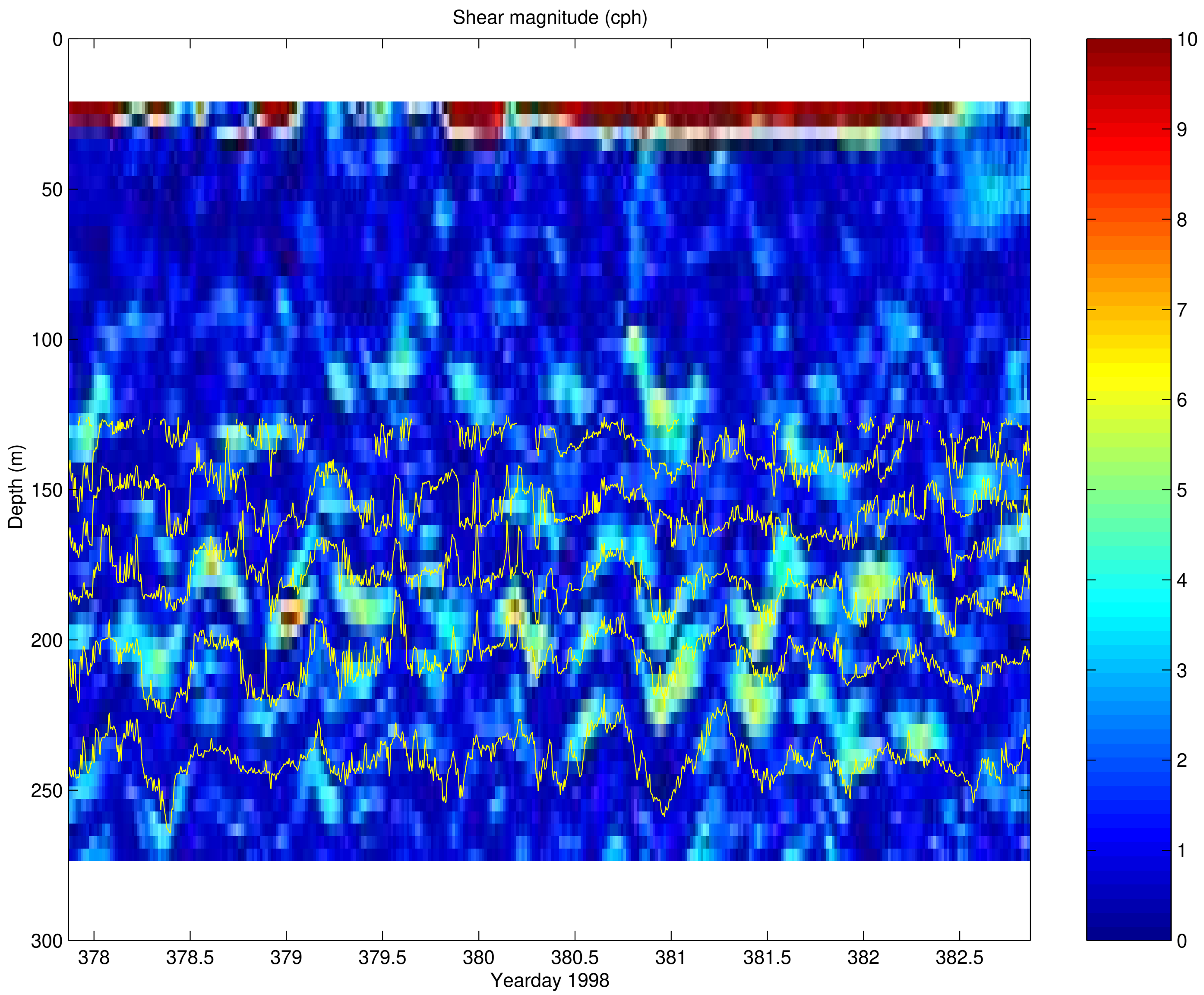

Figure 1: Shear (color panel) and isotherm depth (yellow lines) from the NPAL eastern mooring.