The Dependence of Certain Planar Convex Hull Algorithms on Weak External Visibility *

by

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Convex Hull and Weak External Visibility

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Abstract

The widely used procedure CHS, first introduced by Sklansky for finding the extreme points of simple polygons, (also called “Graham Scan” by Shamos) was shown by Toussaint and Avis to success on weakly externally visible polygons. The more detailed investigation presented here, involving the new Mathematical construct of “Dominating Chains”, leads to a rigorous proof of the sufficiency of weak edge visibility for the success of CHS as a convex hull algorithm. A secondary outcome of the theory of dominating chains is an elegant proof of how a minor modification of CHS turns it into a convex hull algorithm for any simple polygon.

1. Introduction

A basic computation task is that of finding the extreme points of the convex hull of an arbitrary finite set of points in the plane. The principal algorithms for this purpose (see [7], Algorithms 3.2-3.3], [1] and [4]) have all incorporated the simple O(n) algorithm (henceforth referred as CHS) originally proposed by Sklansky [8] for treating this question when the points are given as the successive vertices of an arbitrary simple polygon. Although CHS has since been shown to fail in some cases [3, 9], Toussaint and Avis [9] have shown that CHS works for all weakly externally visible polygons. In a subsequent publication [2], they show how CHS can be adapted to test for weak edge visibility. Our results show that weak edge visibility is actually characterized by the simple condition (referred to henceforth as Condition X) which Avis and Toussaint check for in steps 2 of their procedure RIGHTSCAN [2]. Thus, their additional steps to determine intersections of lines or relative positions of points on lines are not necessary. Our investigation of Condition X shows also that it is in a sense the only criterion for the success of CHS as a convex hull algorithm, leading to the conclusion that all simple polygons for which CHS successfully determines a simple polygon's extreme points (when applied in both clockwise and counterclockwise directions) are in fact weakly externally visible.

In section 2 we introduce some terminology, notable the new concept of chains, along with necessary notation to help the subsequent discussion. Section 3 contains the statement of algorithm CHS and associated notation as well as some preliminary results and a description of the overall setting under consideration. Condition X and the associated concept of dominating chains, along with the major new results in two Theorems are presented in section 4. Finally we include in section 5 a few side results of interest, including the modified CHS algorithm which succeeds in finding the convex hull of all simple polygons. An working example of this algorithm in FreeBASIC is supplied as an appendix.

2. Preliminary Concepts

a) Chains:

Throughout this paper, we speak of an arbitrary sequence of points, $\langle x_1, x_2, \ldots, x_n \rangle$, (also called vertices) in the plane as a chain, a subsequence being called a subchain. If a subchain of a given chain, C, consists of consecutive points from C, then we speak of an interval, and use the usual conventions for square and round brackets to indicate whether or not the end-points are included. Thus if C = $\langle c_1, c_2, \ldots, c_n \rangle$ then C[c_i, c_j) = C[c_i, c_{j-1}] = $\langle c_i, c_{i+1}, \ldots, c_{j-1} \rangle$.

A right <left> convex chain is a chain with only one or two vertices or a chain for which the vector cross products, $(\overrightarrow{x_i x_{i+1}}) \times (\overrightarrow{x_{i+1} x_{i+2}})$ are all negative <positive> for i=1,2,...n-2. If $\langle a, b, c \rangle$ is a right convex chain, we will say that c is right of $\langle a, b \rangle$ or just $\langle a, b, c \rangle$ is right. One and two point chains are both right and left convex chains.

As we will be studying primarily subchains of the vertex chains of simple polygons, all the vertices in our chains will be distinct. The passage to subchains does not, however, in general preserve the "non-selfintersecting" property of simple polygons.

b) Intersections:

The graph of a chain is the piecewise linear curve in the plane obtained by joining successive vertices of the chain by straight line segments, called the edges of the chain. By an intersection of a chain with itself, also called a selfintersection, we mean a point in the plane which is the image of two distinct values under the obvious parametrization of the chain's graph. By an intersection of two different chains, we mean a point in the usual set-theoretic intersection of their graphs. Edges will be denoted as [x, y] and the graph of a chain C by graph(C). Thus [x_i, x_j] = graph($\langle x_{n-1}, x_n \rangle$).

A chain C is said to cross the line ℓ at x, if ℓ and an edge of C intersect at x. If an intersection, x, (of a chain with itself or another chain) lies in the relative interior of some edge, e, of the chains involved, then we speak of the intersection as a crossing of chains. (Note that an intersection of a chain with itself will always lie in the relative interior of one of its edges if all its vertices are distinct.)

c) Polygons:

A polygon or closed chain is a chain in which $x_1 = x_n$ and for which x_{n-1}, x_n and x_2 are not colinear. Only in the case of polygons (but then always) will the indexing be taken "mod n", so that x_2 "succeeds" x_n. All terms which we define for chains in general will be carried over unaltered to polygons. Thus, as would be expected, we mean by an edge of a polygon the segment joining successive vertices.

d) Simple Chains:

A simple chain is one which is non-selfintersecting and for which no three successive vertices are colinear. A simple polygon is a closed chain whose only selfintersection is at $x_1 = x_n$. Simple polygons need not generally have an orientation, but unless otherwise stated in this paper, we shall always assume that the vertices of a polygon are in clockwise order. As is standardly done, we reserve the term convex polygon for chains that are right <left> convex, closed with $\langle x_{n-1}, x_n, x_2 \rangle$ right <left> and simple.

The following more direct equivalent definition of simple chains avoids the relatively cumbersome concepts of graph and selfintersection. A simple chain is a chain in which:

(i) No three successive vertices are colinear.

(ii) No point in the plane is in more than two edges.

(iii) The only points common to exactly two edges are vertices.

Note that from these properties it is easy to see that simple chains are characterized as those chains which, except for trivial reversals and (in the case of polygons) cyclic permutations of the vertices, can be unambiguously retrieved from their graphs.

e) Simple Convex Chains and Spirals:

A spiral is a simple convex chain, C = $\langle x_1, x_2, \ldots, x_n \rangle$, with $x_1 \in$ ext(C) and will be called right or left according to whether C is respectively right or left. In what follows the term spiral will, unless otherwise stated, denote a right spiral. All concepts and conclusions developed for spirals will of course apply equally to left spirals with “left” and “right” interchanged.

Let C = $\langle x_1, x_2, \ldots, x_n \rangle$ be a spiral, ℓ be the line through $\langle x_{n-1}, x_n \rangle$, N be the number of times C[x_1, x_{n-2}] intersects ℓ and $k < n - 2$ be greatest such that x_k is right of $\langle x_{n-1}, x_n \rangle$. With C, ℓ, N and k thus we define the <u>windings</u>, W_C, and the <u>core</u>, R_C, of C as follows:

W_C = floor((1 + N) / 2)

R_C = C (if k=1) or

$R_C = \langle x_\ell, x_k, x_{k+1}, \ldots, x_n \rangle$, for x_ℓ = ℓ ∩ [x_k, x_{k+1}] (if k > 1)

Note that, except when x_n, x_1 and x_2 are colinear, R_C = C iff $\langle x_1, x_2, \ldots, x_n, x_1 \rangle$ is a convex polygon. Also, simple convex chains are characterized as the union of one right and one left spiral with a common initial vertex which can be chosen to be any extreme point of the joined chain. In fact, for every simple convex chain, its extreme points are the only vertices that can be made into the initial vertices of such right and left spiral component pairs.

f) Visibility

A simple polygon, P, is <u>weakly edge visible from the edge</u> $[u, v]$ if for every vertex, w, of P, there is a point $v' \in [u, v]$ such that the segment $[w, v']$ is entirely in the closed bounded region defined by P. We refer to the instructive article by Avis and Toussaint [2] for the basic properties of edge visible polygons.

In our study of CHS acting on arbitrary simple polygons, P = $\langle x_1, x_2, \ldots, x_n, x_1 \rangle$, we need only investigate what happens on the deficiency polygons of P (see [9]). In our terminology, the vertices of a deficiency polygon are given by an interval, P[x_i, x_j], for which

$\emptyset \neq$ P(x_i, x_j) ⊆ int(con(P)) while the edge [x_i, x_j] is contained in bd(con(P)). A simple polygon P is said to be <u>weakly externally visible</u> (as defined in [9]) iff each of its deficiency polygons is weakly edge visible from its unique edge in bd(con(P)).

3. Algorithm CHS

When supplied with any chain, C = $\langle x_1, x_2, \ldots, x_n \rangle$, CHS returns a right convex subchain by deleting vertices of C as follows:

```
r ¬ 1, s ¬ 2, t ¬ 3
DO WHILE t ≤ n
    IF ⟨x_r, x_s, x_t⟩ is right THEN DO        (Advance)
        BEGIN
            r ¬ s, s ¬ t, t ¬ t+1
        END
    ELSE DO                                     (Remove x_s from C)
        BEGIN
            IF r > 1 THEN DO                    (Backtrack)
                BEGIN
                    s ¬ r
                    r ¬ index of previous vertex not yet deleted
                END
            ELSE DO                             (Only 2 vertices left, so must advance)
                BEGIN
                    s ¬ t, t ¬ t+1
                END
        END
```

Notation: Throughout this paper, r, s and t are as in the description of CHS above, and D_t the subchain of C[x_1, x_t] not yet removed from C.

Toussaint and Avis [9] give a convincing proof that CHS returns the set of extreme points of a weakly externally visible polygon, P, when C consists of the clockwise vertex chain of P with $x_1 = x_n$ chosen to be an extreme point of P (Theorem 1 below). In what follows we list without proofs a few results which, although replicating some of the intermediary conclusions as well as the main theorem in [9], provide a new perspective and are sufficiently central to what we will subsequently establishing in this paper to be worth mentioning here.

Lemma 1: If C is a chain, then just before an advance, D_t is a right convex chain after at most $2t$-5 cross products.

Lemma 2: If C is a simple polygon with $x_1 = x_n \in$ ext(C), then D_n contains ext(C) when CHS terminates.

Lemma 3: If C = $\langle x_1, x_2, \ldots, x_n \rangle$ is a simple right convex chain, then, if $n > 2$ there is some i with $1 < i < n$ for which $\langle x_1, x_n, x_i \rangle$ is left.

Lemma 3 is not used in [9], but by using it together with Lemmas 1 and 2 and the fact (see [9]) that every subchain containing x_1 and x_n of the vertex chain, $\langle x_1, x_2, \ldots, x_n, x_1 \rangle$, of a polygon which is weakly edge visible from $[x_i\ x_n]$ is itself weakly edge visible from $[x_i\ x_n]$, we get a direct proof of:

Theorem 1 (Toussaint and Avis):

If C = $\langle x_1, x_2, \ldots, x_n \rangle$ is a weakly externally visible polygon and $x_1 = x_n \in$ ext(C), then D_n = ext(C) when CHS terminates.

Two further basic results, needed later, about the action of CHS on arbitrary chains are included here without proof:

Lemma 4: Just before advancing (beyond x_t), let A be the subchain of C consisting of x_s and x_t together with all those points deleted since the previous advance (to

x_t). Thus, as sets, A = $\{x_s,x_t\} \cup (D_{t-1}\backslash D_t)$, where D_{t-1} is taken just before advancing to x_t. Then A is non-selfintersecting, and no point of A is left of $\langle x_s, x_t \rangle$ or $\langle x_t, x_{t-1} \rangle$. Furthermore, if A has any points other than x_s and x_t, then no point of A is left of $\langle x_{t-1}, x_s \rangle$, so that in fact A($x_s$,$x_t$) lies entirely within the interior of the triangle formed by x_s, x_{t-1} and x_t.

Lemma 5: Just before advancing, let $x_j \in D_t$ and denote by x_i and x_k those points in D_t (whenever they exist) which respectively precede and follow x_j. Then

(i) x_{j-1} is not left of $\langle x_i, x_j \rangle$

(ii) x_{j+1} is not left of $\langle x_j, x_k \rangle$

(iii) x_{j+1} is not left of $\langle x_i, x_j \rangle$

Lemma 6: Just before advancing, let $\langle a, b \rangle$ be any edge in D_t. Then the last time C[a, b] crosses the line through a and b is on the side of a opposite from b.

The proof of this is a straightforward induction on the number of vertices in C[a, b] using the fact that every interval of deleted vertices of C is a concatenation of smaller such intervals whose end-points are arranged as described in Lemma 4.

4. Dominating Chains with Condition X

Lemmas 4 and 5 place some restrictions on the possible locations of points in C deleted between successive vertices of D_t. The additional restriction imposed by Condition X is just enough to capture weak edge visibility of C. (Compare with step 2 of procedure in [2].)

Condition X: In CHS, just after an advance, $\langle x_r, x_s, x_t \rangle$ not right, or $\langle x_{s-1}, x_s, x_t \rangle$ is not left.

Just prior to an advance in CHS, we will say that "X holds in D_t" or X holds at x_t if Condition X was satisfied just after every advance so far. In this case it is clear that the four-vertex chain $\langle x_i, x_{j-1}, x_{j+1}, x_k \rangle$ is in "counterclockwise order" about x_j, whereby x_i, x_j and x_k are as in Lemma 5. (A chain $\langle x_1, x_2, \ldots, x_n \rangle$ is in counterclockwise order about a point y if $\langle y, x_k, x_{k+1} \rangle$ is left for all k < n.) We can thus generalize to arbitrary subchains, D, of simple chains, C, by saying that, in a slight abuse of language, "X holds in (an arbitrary subchain) D (of the simple chain C)" if any three consecutive vertices x_i, x_j and x_k of D are related in this way to x_{j-1}, and x_{j+1} in C.

A right convex subchain, D= $\langle x_1, x_2, \ldots, x_n \rangle$, of a simple chain, C, dominates C if i) X holds in D and ii) for any two successive vertices a and b of D, no vertex in C[a,b] is left of $\langle a, b \rangle$. Note that by this definition every interval of a subchain that dominates C will also dominate C.

Our first objective is to show that if X holds in D_t, then D_t actually dominates C. Establishing this in Lemma 7, along with all further results, is greatly simplified by

Proposition 1: Suppose that D is a subchain of the simple chain C and that C and D intersect in points *x* and *y* such that all of the following hold:

a) The portions of the graphs of C and D between *x* and *y* form a simple closed curve surrounding a region, R, whose interior is right of D.
b) x is in the relative interior of some edge, $e = \langle v_1, v_2 \rangle$ of D.
c) The vertex of *e* on the side of *x* opposite from *y* is not in R.
d) In case both *x* and *y* are in *e*, then *y* must be a vertex of D.

Then D cannot dominate C.

Proof: Assume C and D do intersect as stated. By assuming also that D dominates C, we will reach a contradiction.

Let $\langle v_1, v_2, \ldots, v_k \rangle$ be the smallest interval of D whose graph contains x and y. By symmetry, it is enough to consider the case in which *x* precedes y in D, so that $v_1 \notin$ R. We proceed by induction on k.

The least possible value for k is 2, in which case the requirements of d) must hold: i) implies that $y = v_2$. Furthermore, since the dominating character is trivially violated if C[v_1, v_2] crosses D[v_1, v_2] at x, there is a vertex w such that x is instead in the graph of C' = C[v_2, w] $\subseteq$ C[v_1, k_2]. Because R is to the right of D[v_1, v_2], w must be left of D. But then, if Condition X is to hold at v_2, the Jordan Curve Theorem shows that C[v_1, v_2] must also cross D[v_1, v_2], a contradiction. (See Figure 1.)

Now assume $k > 2$. Condition X means $v_1 \notin$ R and C[v_1, v_2] not crossing $\langle v_1, v_2 \rangle$ means (Jordan Curve Theorem) that C[v_1, v_2] crosses D[v_2, v_k]. Choose the smallest j such that C[v_1, v_2] crosses D' = D[v_2, v_j]. Then D' intersects C in a manner that satisfies the proposition's assumptions. But then we have a contradiction, since D' has fewer than k vertices and therefore by induction it cannot dominate C and from which it follows that D[v_1, v_k] cannot either. (See Figure 2.) □

Corollary: Let D = $\langle v_1, v_2, \ldots, v_k \rangle$ be a right convex subchain of a simple chain C with $v_1 \in$ ext(D) and a single selfintersection on the edges $\langle v_1, v_2 \rangle$ and $\langle v_{k-1}, v_k \rangle$. Then D cannot dominate C.

Proof: Because x_1 is an extreme point of D, and D[v_1, v_{k-1}] is a simple right convex chain, v_2 through v_{k-1} must be right of $\langle v_1, v_2 \rangle$. In particular, v_{k-1} is right of $\langle v_1, v_2 \rangle$ and thus $\langle v_{k-1}, x, v_2 \rangle$ is also right, showing that P = $\langle x, v_2, v_3, \ldots, v_{k-2}, v_{k-1} x \rangle$ is a convex polygon, and so also is $\langle v_2, v_3, \ldots, v_{k-2}, v_{k-1} \rangle$.

Assume D dominates C. Let y be the vertex in C following v_{k-1}. Then, because of the dominating nature of D at v_{k-1}, [v_{k-1},y] passes through the interior of P (see Figure 3). But then the Jordan Curve Theorem and the fact that C[v_{k-1},v_k] cannot have any vertices left of $\langle v_{k-1}, v_k \rangle$ forces C[v_{k-1},v_k] to cross D[v_1,v_{k-1}]. Therefore C and D[v_1,v_{k-1}] satisfy the conditions of Proposition 1, from which it follows that D[v_1,v_{k-1}] cannot dominate C, contradicting the assumption that D dominates C. □

From this it is easy to conclude that:

Proposition 2: <u>If</u> D = $\langle x_1, x_2, \ldots, x_n \rangle$ <u>is a dominating right convex subchain of a simple chain</u> C with $x_1 \in$ D <u>and</u> $x_1 \in$ ext(C) (<u>and thus</u> $x_1 \in$ ext(D)), <u>then</u> D <u>is simple</u>.

With this result it is now straightforward to show that

Lemma 7: <u>If</u> X <u>holds in</u> D_t, <u>then</u> D_t <u>dominates</u> C.

Proof: We need to show that none of the vertices of C between any two consecutive vertices a and b of D_t are left of $\langle a, b \rangle$. Assume that t is least such that the conclusion fails. Then prior to advancing to x_t, D_{t-1} satisfied all the necessary characteristics of a dominating subchain and was therefore, by Proposition 2, non-selfintersecting. But then, using Lemma 6, we see that the only way a vertex, z, in C[x_s, x_t] could be left of $\langle x_s, x_t \rangle$ without C[x_s, z] intersecting D_{t-1} (in violation of Proposition 1) would be for C[x_s, z] to cross the supporting line at x_1. (See Figure 4, and note how Lemma 6 prevents C[x_s, z] from first crossing the line through x_s and x_t on the same side of x_s as x_t.) □

Equipped now to thoroughly analyze Condition X, we assume throughout the rest of this paper that the chain, C, supplied to CHS is the vertex chain, $\langle x_1, x_2, \ldots, x_n \rangle$, such that $\langle x_1, x_2, \ldots, x_n, x_1 \rangle$ is a counterclockwise polygon with edge [x_i x_n] $\subseteq$ bd(con(C)). The apparent inconsistency here with our prior "clockwise" assumption for simple polygons, P, in the preceding section is a consequence of only needing to restrict the study of CHS to P's deficiency polygons whose vertex chains are inherited from P in reverse orientation. Occasionally we will speak of a clockwise traversal of CHS, in which case we mean that the indexing of the vertices of C has been reversed and that all occurrences of “right” and “left” have been interchanged in CHS as well as any results pertaining to it.

Proposition 1 makes it clear that for any two successive vertices *a* and *b* of a dominating subchain, D, of a simple chain, C, the region enclosed by the graphs of D[*a*,*b*] and C[*a*,*b*] cannot contain any vertex of D in its interior. Lemma 7 proves that D_t in CHS dominates C, is therefore simple by Proposition 2 and thus a right spiral. Equipped with these facts, Theorems 2 and 3 are fairly straightforward to establish. Keep in mind that the two Theorems assume that the simple chain, C, supplied to CHS are just the counterclockwise unclosed deficiency polygons of clockwise simple polygons. In other words $x_1 \neq x_n$, are both in ext(C) and for CHS to “succeed” means it ends with all but x_1 and x_n deleted ($D_n = \langle x_1, x_n \rangle$).

Theorem 2: Condition X fails for some *t* iff D_n is selfintersecting when CHS terminates.

Proof: We have seen that if X holds in D_n then D_n dominates C (by Lemma 7) and is therefore simple (Proposition 2). To show the converse, assume X fails for the first time just after advancing from $x_{t-1} = x_s$ to x_t, so that $D_s = D_t[x_1,x_s]$ dominates C. Let the points of D = D_s be given by $\langle z_1, z_2, \ldots, z_k \rangle$. As remarked above, D is then a right spiral with $z_1 = x_1$ and $z_k = x_s$.

Now if D_n is non-selfintersecting when CHS terminates, then because x_1 are extreme points of C, no points in C are left of $\langle x_1, x_n \rangle$ and so by Lemma 3, $D_n = \langle x_1, x_n \rangle$, i.e. all the points in $D(z_1,z_k]$ have been deleted. This means there must exist a subchain E = $\langle y_1, y_3, \ldots, y_\ell \rangle$ of points in $C[x_s,x_n]$ containing all the points causing the deletion of every vertex in

D(z_1,z_k] and with the additional property that y_i causes the deletion of y_{i-1}, for all i = 2, 3, . . , ℓ. In finding the possible locations for points in E, we will show that all of D(z_1,z_k] cannot be deleted unless some vertices of E are on the far side of the supporting line at x_1.

Some clarification is needed of the figures used throughout the remainder of this paper: Heavily traced polygonal lines represent D_t at the point of reference in the text, and the light wavy curves are portions of C. The remarks just preceding this Theorem give these figures their generality, since the fact that D_t is a spiral and dominates C as long as condition X holds makes the general case, apart from vertex counts and spiral windings, topologically equivalent to those illustrated.

First we show that y_1 cannot lie in the region denoted by A in Figure 5. Assume then that $y_1 \in A$. $x_s = z_k$ cannot immediately precede y_1 in C because the failure of condition X at x_t requires $\langle x_r, x_s x_t \rangle$ to be right. Therefore there is at least one vertex other than x_s which is deleted by y_1 and which by Lemma 4 lies to the right of $\langle x_r, x_s \rangle$ on the simple right convex subchain H = $\langle h_1, h_2, \ldots, h_j \rangle$ of C consisting of all the points deleted by y_1. But this situation cannot arise, for it is clear that, in order for C not to selfintersect and - in case the spiral of D does not wind around far enough - in order for C[x_s, h_2] (which contains x_t) not to cross the supporting line at x_1, it is necessary for C[x_s, h_2] to cross the line through x_s and h_2 for the last time on the same side of x_s as h_2, contradicting Lemma 6.

Thus y_1 must lie at least one winding of the spiral D_s ahead of x_s as illustrated in Figure 6. (Note that if D does not wind around far enough, then the proof is complete, since our argument would have shown that x_s cannot be deleted.)

In a similar manner we can now invoke Lemma 4 to show that, unless some vertices of E are located on the far side of the supporting line at x_1, each point $x_{m_i} = y_i$ remains at least one winding of D_s ahead of z_{k_i}, the point immediately preceding y_i in D_{m_i} just prior to advance beyond x_{m_i}. We just showed this for $i = 1$. Now we note that for each $i = 2, 3, \ldots, \ell$, the region in which y_i must lie is, according to Lemma 4, the set of points left of $\langle z_{k_i}, x_{m_{i-1}} \rangle$ from which line segments extend to the right of $\langle z_{k_i}, x_{m_{i-1}} \rangle$ without intersecting C or the ray from $x_{m_{i-1}}$ through z_{k_i}. Thus, for $i = 2$ in Figure 6, y_2 cannot lie in the inner spiral region denoted by A. Applying this argument inductively it is clear that y_{k-1} cannot delete z_2 since there will still be at least one winding of D_s between y_ℓ and the vertex it deletes, so z_2, being the second vertex on that winding cannot be it. □

Theorem 3: C <u>is weakly edge visible from the edge</u> $[x_1, x_n]$ <u>iff condition</u> X <u>holds throughout both counterclockwise and clockwise traversal of</u> CHS.

Proof: That failure of X implies lack of edge visibility was shown by Avis and Toussaint (See [2], Lemma 5.) Assume on the other hand that X holds throughout both counterclockwise (forming right convex chains, D_t), and clockwise (forming left convex chains, S_t) traversal of CHS. If at each vertex, x_t, D_t and S_t (just before advancing beyond x_t), do not intersect (other than at x_t) then it is clear that C is edge visible from the edge $[x_1, x_n]$. (See Figure 7.)

To conclude the proof, we show that D_t and S_t cannot intersect anywhere other than x_t without X failing during one of the traversals of CHS. Assume then that D_t and S_t intersect and X holds in both.

Let $S_t = \langle s_1, s_2, \ldots, s_k \rangle$ with $s_1 = x_n$ and $s_k = x_t$

$D_t = \langle d_1, d_2, \ldots, d_\ell \rangle$ with $d_1 = x_1$ and $d_\ell = x_t$.

Choose the least i such that $S_t[s_{i+1}, s_k]$ does not meet D_t other than at x_t, and j largest such that $[s_i, s_{i+1}]$ and $[d_j, d_{j+1}]$ intersect. Although $S_t[s_{i+1}, s_k]$

and $D_t[d_{j+1}, d_\ell]$ meet only at x_t, there is no reason why $S_t[s_1, s_i]$ should not meet $D_t[d_{j+1}, d_\ell]$ in some points other than x_t. We can, however. Assume that this does not happen, for in so doing we allow the most freedom in locating vertices of C between d_j and s_i, and if X can be shown to fail under this assumption, then X will have to fail in the more restrictive setting.

The Jordan Curve Theorem shows that exactly one of the following must happen: Either the simple closed curve defined by $[d_j, d_{j+1}]$ and the graph of $C[d_j, d_{j+1}]$ encloses s_{i+1}, or $[s_i, s_{i+1}]$ and the graph of $C[s_i, s_{i+1}]$ enclose d_{j+1}. Without loss of generality, we assume the former and also that s_{i+1} is the only vertex of S_t so enclosed.

First note that when $\ell = j + 1$ ($d_{j+1} = x_t$) or $k = i + 1$ ($s_{k+1} = x_t$) S_t and D_t will respectively not dominate C. In the cases when both $\ell > j + 1$ and $k > i + 1$ (i.e. x_t is neither d_{j+1} nor s_{k+1}), the dominating character of both D_t and S_t along with the Jordan Curve Theorem can be used to show that with $s_{i_1} = s_{i+1}$, $s_{i_u} = x_t$, $d_{j_1} = d_j$ and either $d_{j_u} = x_t$ or $d_{j_u+1} = x_t$ the graph of $C[d_j, s_i]$ must run as follows for certain subchains $\langle d_{j_1}, d_{j_2}, \ldots, d_{j_u} \rangle$ and $\langle s_{i_1}, s_{i_2}, \ldots, s_{i_u} \rangle$ of D_t and S_t respectively:

For $m = 1$ through u-*1*,

the graphs of $D_t[d_{j_m}, d_{j_m+1}]$ and $C[d_{j_m}, d_{j_m+1}]$ enclose $S_t[s_{i_m}, s_{i_{m+1}-1}]$ and

$$\textit{for } m = 2 \text{ through } m = \begin{cases} u\text{-}1, \text{ if } d_{j_u} = x_t \ (\text{or } d_{j_{u-1}+1} = x_t = d_{j_u}) \\ u, \text{ if } d_{j_u+1} = x_t \ (\text{or } d_{j_{u-1}+1} \neq x_t) \end{cases}$$

the graphs of $S_t[s_{i_m-1}, s_{i_m}]$ and $C[s_{i_m}, s_{i_m-1}]$ enclose $D_t[d_{j_{m-1}+1}, d_{j_m}]$.

An example of the case where $d_{j_u} = x_t$ is shown in Figure 8, from which it is clear that X will fail right after advancing past some vertex in $C[x_t, s_{k-1})$ during counterclockwise traversal. It is easy to see that the case of $d_{j_u+1} = x_t$ is analogous and results in failure of X in $C(d_{\ell-1}, x_t]$ during clockwise traversal. □

5. Related Side Results

An immediate consequence of Theorems 2 and 3 is a stronger converse to the result mentioned in [9], that every subchain of the vertex chain, C = $\langle x_1, x_2, \ldots, x_n \rangle$, of a polygon which is weakly edge visible from [x_1, x_n] is itself weakly edge visible from [x_1, x_n]. For we have shown that if C is not edge visible from [x_1, x_n], then it has in fact a selfintersecting subchain, although this and of course its converse can be proved directly avoiding the machinery of CHS. But since the only simple right convex subchain of C (assuming counterclockwise vertex order) is $\langle x_1, x_n \rangle$, we have in fact proven that C is weakly edge visible from [x_1, x_n] iff CHS succeeds (i.e. return only x_1 and x_n) in both traversals. Lemma 7, the key to Theorems 2 and 3, has the following equivalent for arbitrary subchains, D of C: If x_1 and x_n are both in D and ext(C) and X holds in D, then D dominates C. We should also mention an extension of Lemma 7, which has a certain intuitive appeal, and is easily seen to be the strongest restriction that condition X places on the possible locations of points of C between successive vertices of D_t:

Proposition 3: Given any two vertices, x_i and x_j, of D_t just before advancing, let LH[x_i,x_j] denote those vertices of C[x_i, x_j] which are not right of $\langle x_i, x_j \rangle$. Then, as long as condition X holds,

con(LH([x_i, x_j]) $\subseteq$ con(D_t[x_i, x_j]).

Finally, our discussion makes it clear that the following slight modifications of CHS turns it into an O(n) convex hull algorithm for all simple polygons, and is a decided improvement over the algorithm in [6]: When X fails just after advancing to x_t, delete x_t and all subsequent vertices until the first vertex, x_j, is reached which is left of $\langle x_r, x_s \rangle$, at which point set $t \neg\ j$ and return to the stage in CHS at which x_s is deleted. All we need to note is that the Jordan Curve Theorem ensures the existence of x_j, and that just prior to the next advance beyond x_j, D_j dominates C once again, so that this revision of CHS ensures D_t dominates C just prior to the every advance. In particular, when CHS terminates, D_n will dominate C and if C is a simple polygon, D_n = ext(C). A slight variation of this algorithm is described in more detail in [5], though our results show that some of the step there are not needed. A FreeBASIC implementation this revision to CHS is provided as an appendix.

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