Modelling of Clock Behaviour

Don Percival

Applied Physics Laboratory
University of Washington
Seattle, Washington, USA

overheads and paper for talk available at

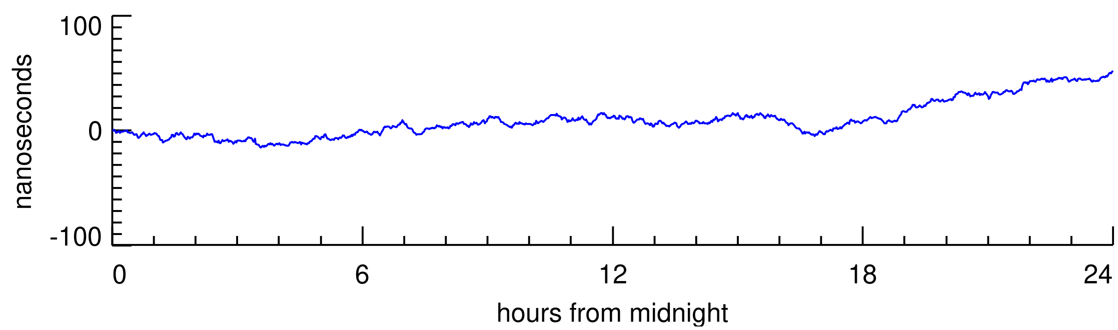
`http://faculty.washington.edu/dbp/talks.html`

Overview

- atomic clocks can keep time to unimaginable precision . . .
- . . . but some people (clock modellers!) are never satisfied and insist on focusing on unimaginable imprecisions
- will discuss
  - concepts behind clock modelling
  - ways in which current modelling practices can be improved

A Thought Experiment

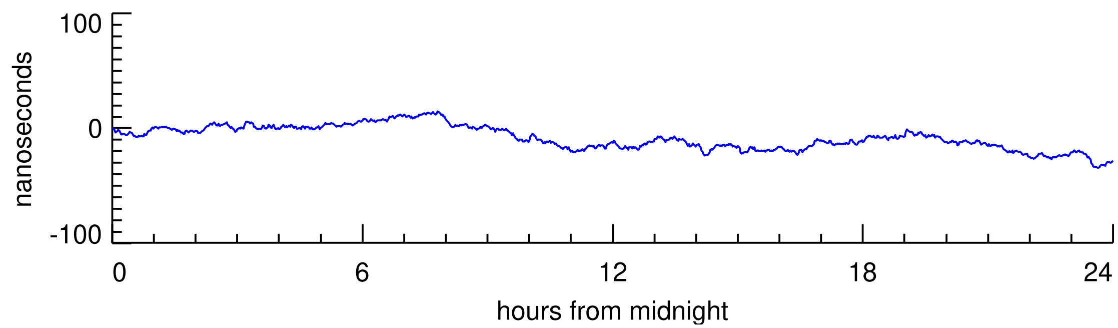
- suppose we have a clock whose performance we want to evaluate
- will assume we can compare this clock to 'perfect' time
- at midnight we set the clock to perfect time and measure how well it does over the next 24 hours:



- clocks wanders away from perfect time over 24 hour period, ending up about 50 nanoseconds ahead of perfect time
- no obvious explanation for observed time deviations

Second Day of Thought Experiment

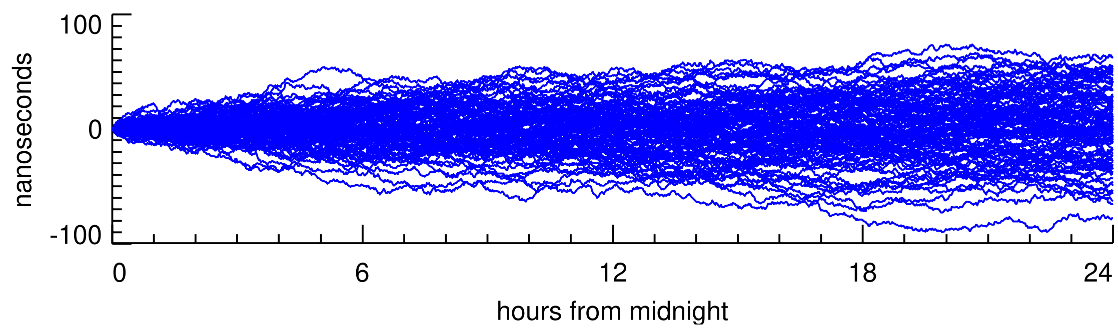
- let's do this again!
- at midnight we reset the clock to perfect time and now get this:



- after 24 hours, clock is now about 30 nanoseconds behind
- again, no obvious explanation for observed time deviations; however, deviations seem to have the same visual 'bumpiness'

Thought Experiment After 100 Days

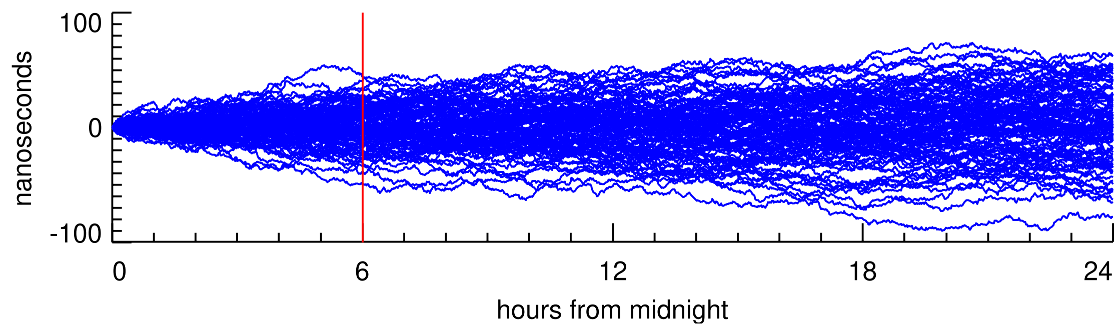
- we keep on doing this for 100 days:



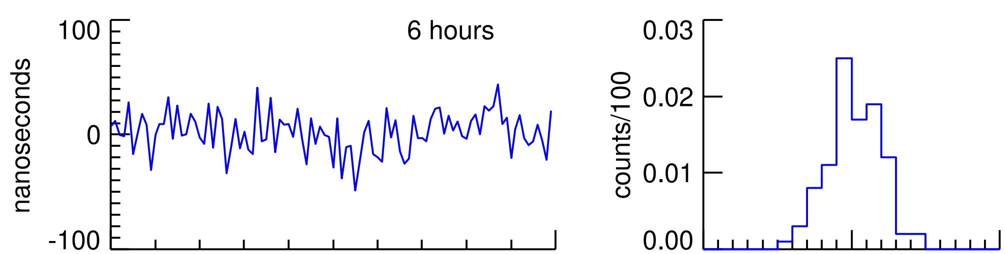
- can't predict exactly what will happen on any given day
- can make some statistical statements about the nature of the time deviations over the 24 hour period
  - average deviation after 6 hours close to 0, and 95% of curves are between about -30 and 30 nanoseconds
  - average deviation after 24 hours also close to 0, but now 95% of curves are between about -60 and 60 nanoseconds

Purpose of Clock Models

- clock models summarize statistical information in our data



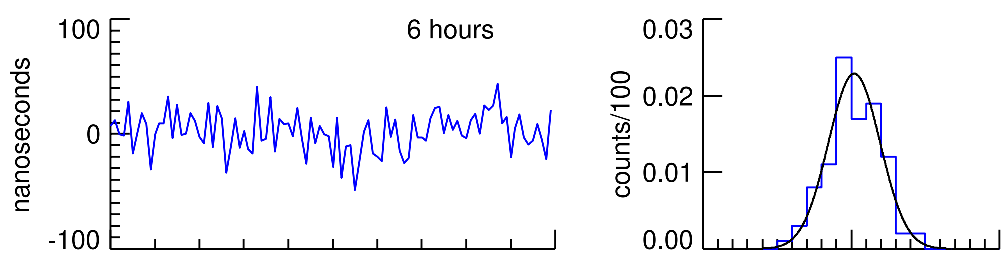
- let's focus again on what we observe each day at 6AM



- histogram (right-hand plot) offers some summary, but we can more with the help of a theoretical distribution

Gaussian Distribution to the Rescue!

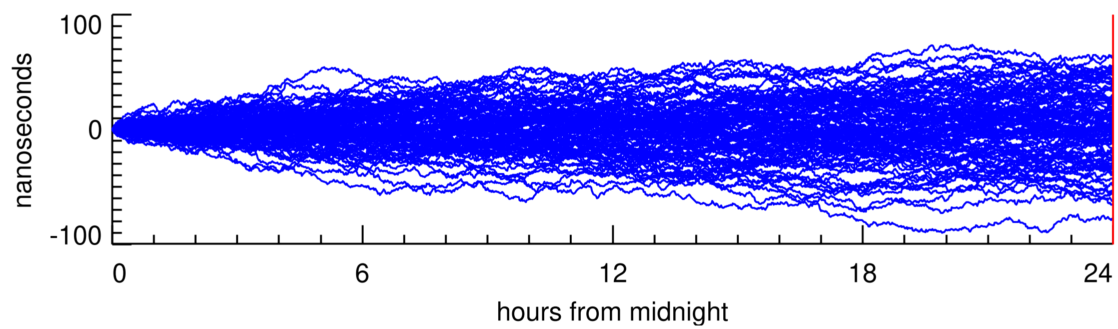
- popular theoretical distribution is Gaussian (normal)
- bell-shaped curve determined by two parameters
  - mean, which is set by average (≈ 0) of 100 6AM deviations
  - variance, which quantifies the fact that most values occur between -30 and 30 nanoseconds
- bell-shaped curve shows Gaussian approximation:



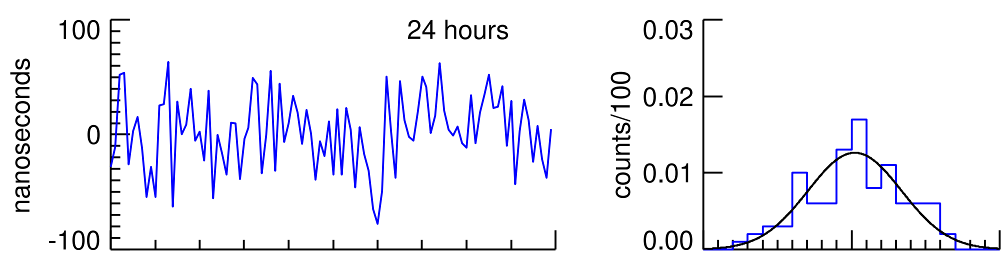
- our simple clock model has summarized statistical information about 100 6AM measurements using just 2 parameters

Modelling Deviations at 24 Hours

- let's now look at observations after 24 hours of elapsed time



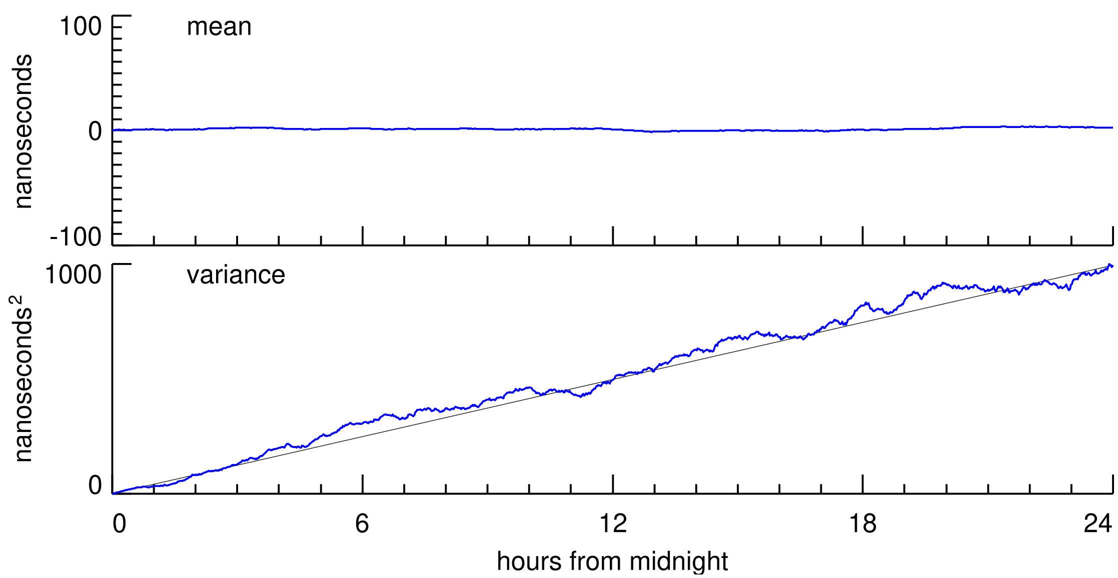
- here are the 100 time deviations, histogram and Gaussian fit:



- mean ≈ 0 still, but variance is larger, reflecting fact that 95% of the curves are now roughly between -60 and 60 nanoseconds

Variation of Gaussian Parameters over Elapsed Time

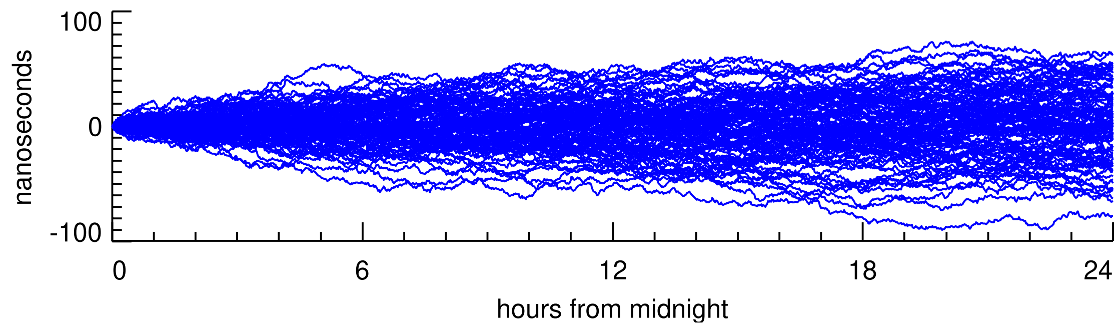
- can repeat the above procedure for all elapsed times from 0 hours to 24 hours
- let's plot the two Gaussian parameters versus elapsed time:



- mean is always close to 0, while variance increases approximately linearly

Summary Offered by Clock Model

- with help of Gaussian distribution, can summarize univariate statistic properties observed in this plot

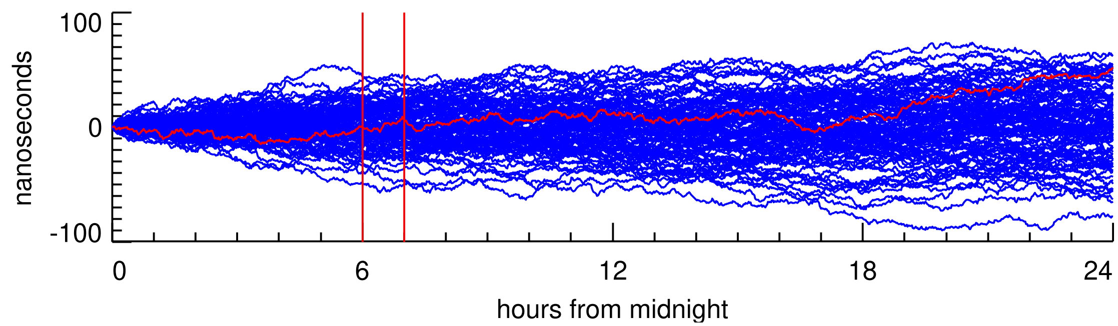


using just *one* variable (a level parameter C determining rate of linear increase in variance)!!!

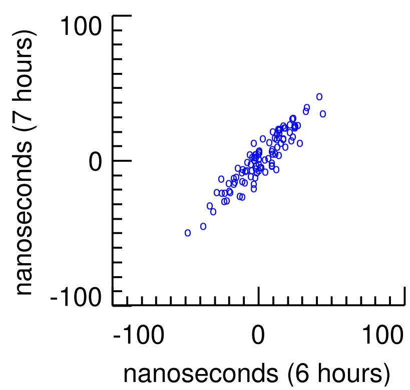
- impressive simplification, but only of univariate properties

Bivariate Statistical Properties

- also interested in relationship between deviations at two distinct elapsed times, e.g., 6 hours and 7 hours after midnight:



- deviations at 7 hours and 6 hours are positively correlated:



Clock Models Incorporating Multivariate Properties

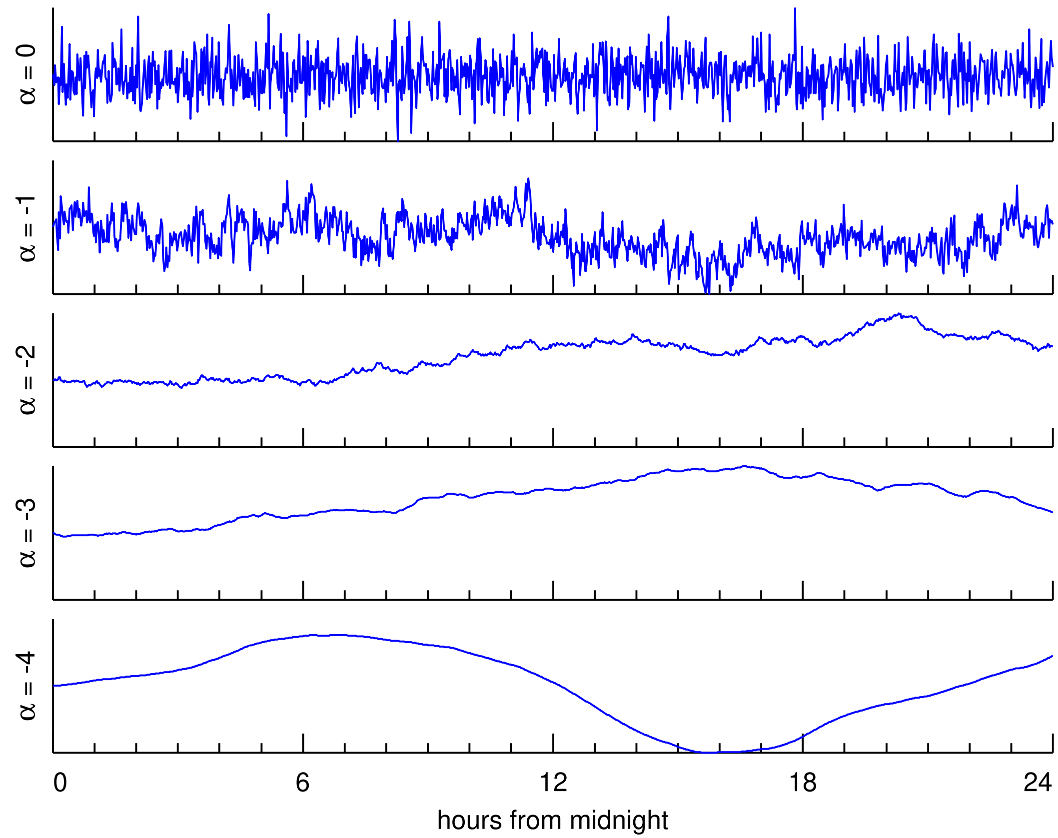
- in general, interested in relationship amongst deviations at multiple elapsed times, i.e., multivariate statistical properties
- interestingly enough, with help of multivariate Gaussian distribution, can summarize information with *one* more parameter beyond what we need to summarize univariate properties!!!
- additional parameter α essentially selects a particular pattern for the increase in variance (this was linear in our experiment)
- class of models parameterized by C and α known as 'power law' models

Five Canonical Power Law Models

- in practice, parameter α is usually set to one of five values
  - $\alpha = 0$ yields model known as white phase noise
  - $\alpha = -1$ yields flicker phase noise
  - $\alpha = -2$ yields random walk phase noise (our experiment!)
  - $\alpha = -3$ yields flicker frequency noise
  - $\alpha = -4$ yields random walk frequency noise
- variance versus elapsed time
  - is constant for $\alpha = 0$
  - increases for $\alpha = -1$ (but hard to describe exactly!)
  - increases linearly for $\alpha = -2$
  - increases for $\alpha = -3$ (again, hard to describe!)
  - increases quadratically for $\alpha = -4$

Deviations Generated by Canonical Models

- here is one set of time deviations from each model:



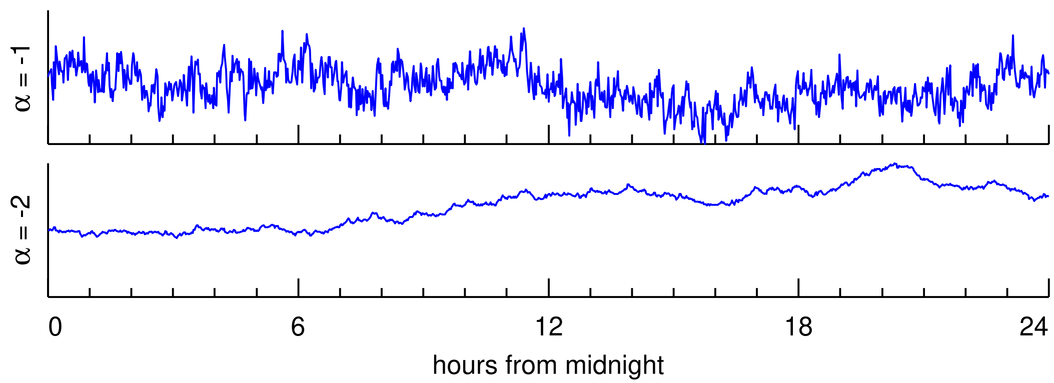
- note: level parameter C merely sets vertical scaling

Use of Canonical Models in Theory

- when applicable, canonical models along with the multivariate Gaussian distribution offer a *very* simple summary of the statistical properties of time deviations (only need C and α)
- useful for comparing statistical properties of different clocks
- can use models to form an ensemble time that is better than any individual clock

Limitations of Canonical Models in Practice

- picture painted so far of clock modelling is too simplistic
- real-world complications make it harder to reap benefits
  - need different canonical models for different elapsed times
  - spacing of α is too coarse (models exist for all α)



  - lack of data for decent parameter estimation
  - problem of trend and its estimation

Opportunities for Improvements

- regard α as parameter to be estimated rather than identified (typically preselection of α not accounted for, but this can be an important source of sampling variability)
- sampling variability in trend estimates usually not propagated properly
- lack of attention to sampling variability in α/trend estimates might explain failure of 'optimal' procedures for forming ensemble time scales
- much has been done, but still more to do!
- use of proper statistical procedures will hopefully lead to better performance of systems depending on clocks, but work must be done to see if this is so

Final Word

- thanks to organizers for making this presentation possible!!!