Discrete Wavelet Transforms Based on Zero-Phase Daubechies Filters

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overheads for talk available at

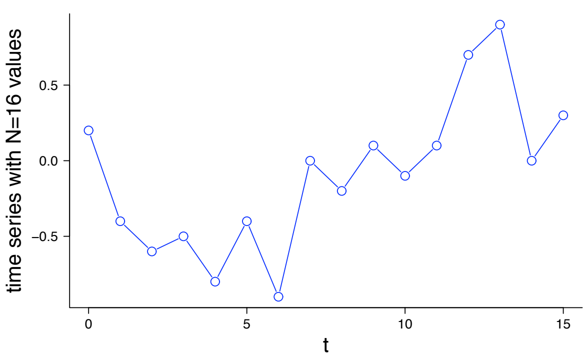
`http://faculty.washington.edu/dbp/talks.html`

Overview

- will discuss work in progress on the 'zephlet' transform, an orthonormal discrete wavelet transform (DWT) based on zero-phase filters
- will start by giving some background on the DWT as formulated in Daubechies (1992) – see, e.g., Percival & Walden (2000) or Gençay et al. (2002) for further details
- will then describe the zephlet transform and how it differs from the usual DWT, with an illustration of some of its properties

Background on DWT: I

- let $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ be a vector of N time series values (note: 'T' denotes transpose; i.e., $\mathbf{X}$ is a column vector)
- for simplicity, assume N is an even number



Background on DWT: II

- DWT is a linear transform of $\mathbf{X}$ yielding N DWT coefficients
- notation: $\mathbf{W} = \mathcal{W}\mathbf{X}$, where $\mathbf{W}$ is vector of DWT coefficients, and $\mathcal{W}$ is $N \times N$ *orthonormal* transform matrix
- orthonormality says $\mathcal{W}^T\mathcal{W} = I_N$ ($N \times N$ identity matrix)
- orthonormality is exploited heavily in, among other uses, DWT-based extraction of signals ('wavelet shrinkage')
- to focus discussion, will concentrate on so-called unit-level DWT, for which $\mathbf{W} = [\mathbf{W}_1^T, \mathbf{V}_1^T]^T$, where the two subvectors contain
  - wavelet coefficients $\mathbf{W}_1 = [W_{1,0}, W_{1,0}, \ldots, W_{1,\frac{N}{2}-1}]^T$ and
  - scaling coefficients $\mathbf{V}_1 = [V_{1,0}, V_{1,0}, \ldots, V_{1,\frac{N}{2}-1}]^T$
- higher-level DWTs use unit-level DWTs over and over again

The Wavelet Filter: I

- matrix $\mathcal{W}$ is rarely constructed explicitly, but rather is formed implicitly by use of a wavelet filter
- let $\{h_l : l = 0, \ldots, L-1\}$ be a real-valued filter of width L
- for convenience, will define $h_l = 0$ for $l < 0$ and $l \geq L$

The Wavelet Filter: II

- $\{h_l\}$ called a wavelet filter if it has these 3 properties

1. summation to zero:
$$\sum_{l=0}^{L-1} h_l = 0$$
2. unit 'energy' (i.e., squared Euclidean norm):
$$\sum_{l=0}^{L-1} h_l^2 = 1$$
3. orthogonality to even shifts: for all nonzero integers n, have
$$\sum_{l=0}^{L-1} h_l h_{l+2n} = 0$$

- 2 and 3 together are called the *orthonormality property*

The Wavelet Filter: III

- summation to zero and unit energy relatively easy to achieve
- orthogonality to even shifts is key property & hardest to satisfy (implies L must be even; common choices are $2, 4, \ldots, 20$)
- define transfer function for wavelet filter, i.e., its discrete Fourier transform (DFT), along with its squared gain function:
$$H(f) \equiv \sum_{l=0}^{L-1} h_l e^{-i2\pi fl} \quad \text{and} \quad \mathcal{H}(f) \equiv |H(f)|^2$$
- orthonormality property is equivalent to
$$\mathcal{H}(f) + \mathcal{H}(f + \tfrac{1}{2}) = 2 \quad \text{for all } f$$
(an elegant – but not obvious! – result)

The Wavelet Filter: IV

- simplest wavelet filter is Haar ($L = 2$): $h_0 = \frac{1}{\sqrt{2}}$ & $h_1 = -\frac{1}{\sqrt{2}}$
- note that $h_0 + h_1 = 0$ and $h_0^2 + h_1^2 = 1$, as required
- orthogonality to even shifts also readily apparent
- squared gain function is
$$\mathcal{H}(f) = 2\sin^2(\pi f),$$
for which
$$\begin{aligned}\mathcal{H}(f) + \mathcal{H}(f + \tfrac{1}{2}) &= 2\sin^2(\pi f) + 2\sin^2(\pi[f + \tfrac{1}{2}]) \\ &= 2\sin^2(\pi f) + 2\cos^2(\pi f) \\ &= 2,\end{aligned}$$
as required

Construction of Wavelet Coefficients: I

- given wavelet filter $\{h_l\}$ of width L & time series of even length, obtain wavelet coefficients as follows
- *circularly* filter $\mathbf{X}$ with wavelet filter to yield output

$$\sum_{l=0}^{L-1} h_l X_{t-l} = \sum_{l=0}^{L-1} h_l X_{t-l \bmod N}, \quad t = 0, \ldots, N-1;$$

  i.e., if $t-l$ does not satisfy $0 \le t-l \le N-1$, interpret X_{t-l} as $X_{t-l \bmod N}$; for example, $X_{-1} = X_{N-1}$ and $X_{-2} = X_{N-2}$
- take every other value of filter output to define

$$W_{1,t} \equiv \sum_{l=0}^{L-1} h_l X_{2t+1-l \bmod N}, \quad t = 0, \ldots, \tfrac{N}{2} - 1;$$

  $\mathbf{W}_1$ formed by *downsampling* filter output by a factor of 2

Construction of Wavelet Coefficients: II

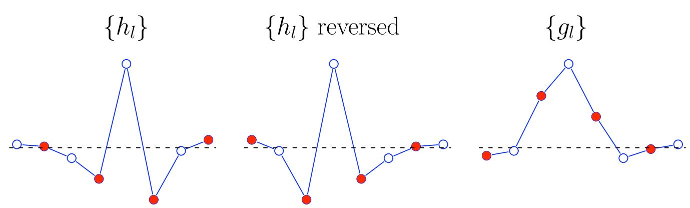
- can write $\mathbf{W}_1 = \mathcal{W}_1 \mathbf{X}$, where, when $N \ge 10$ for example,

$$\mathcal{W}_1 \equiv \begin{bmatrix} h_1^\circ & h_0^\circ & h_{N-1}^\circ & h_{N-2}^\circ & h_{N-3}^\circ & \cdots & h_5^\circ & h_4^\circ & h_3^\circ & h_2^\circ \\ h_3^\circ & h_2^\circ & h_1^\circ & h_0^\circ & h_{N-1}^\circ & \cdots & h_7^\circ & h_6^\circ & h_5^\circ & h_4^\circ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ h_{N-1}^\circ & h_{N-2}^\circ & h_{N-3}^\circ & h_{N-4}^\circ & h_{N-5}^\circ & \cdots & h_3^\circ & h_2^\circ & h_1^\circ & h_0^\circ \end{bmatrix}$$

- here $h_l^\circ = h_l$ when $L \le N$, but takes different form if $L > N$; for example, if $N = 10$ and $L = 20$, $h_l^\circ = h_l + h_{l+10}$
- can argue that $\mathcal{W}_1 \mathcal{W}_1^T = I_{N/2}$ for all L and N
- $\mathcal{W}_1$ is the top *half* of orthonormal transform matrix $\mathcal{W}$

The Scaling Filter: I

- create scaling filter $\{g_l\}$ by reversing $\{h_l\}$ and then changing sign of coefficients with even indices



- precise definition is $g_l \equiv (-1)^{l+1} h_{L-1-l}$

The Scaling Filter: II

- properties 2 and 3 (orthonormality) of $\{h_l\}$ are shared by $\{g_l\}$:

  2. unit energy:

$$\sum_{l=0}^{L-1} g_l^2 = 1$$

  3. orthogonality to even shifts: for all nonzero integers n, have

$$\sum_{l=0}^{L-1} g_l g_{l+2n} = 0$$

- squared gain function $\mathcal{G}(\cdot)$ for scaling filter satisfies

$$\mathcal{G}(f) = \mathcal{H}(f + \tfrac{1}{2}) \text{ and hence } \mathcal{H}(f) + \mathcal{G}(f) = 2$$

  is equivalent way of stating orthonormality property

Construction of Scaling Coefficients: I

- orthonormality property of $\{h_l\}$ is all that is needed to prove $\mathcal{W}_1$ is half of an orthonormal transform (never used $\sum_l h_l = 0$)
- going back and replacing h_l with g_l everywhere yields another half of an orthonormal transform
- circularly filter $\mathbf{X}$ using $\{g_l\}$ and downsample to define scaling coefficients:

$$V_{1,t} \equiv \sum_{l=0}^{L-1} g_l X_{2t+1-l \bmod N}, \quad t = 0, \ldots, \frac{N}{2} - 1$$

Construction of Scaling Coefficients: II

- have $\mathbf{V}_1 = \mathcal{V}_1 \mathbf{X}$, where $\mathcal{V}_1$ is analogous to $\mathcal{W}_1$:

$$\mathcal{V}_1 = \begin{bmatrix} g_1^\circ & g_0^\circ & g_{N-1}^\circ & g_{N-2}^\circ & g_{N-3}^\circ & \cdots & g_5^\circ & g_4^\circ & g_3^\circ & g_2^\circ \\ g_3^\circ & g_2^\circ & g_1^\circ & g_0^\circ & g_{N-1}^\circ & \cdots & g_7^\circ & g_6^\circ & g_5^\circ & g_4^\circ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ g_{N-1}^\circ & g_{N-2}^\circ & g_{N-3}^\circ & g_{N-4}^\circ & g_{N-5}^\circ & \cdots & g_3^\circ & g_2^\circ & g_1^\circ & g_0^\circ \end{bmatrix}$$

- as before, can argue that $\mathcal{V}_1 \mathcal{V}_1^T = I_{N/2}$
- in addition, each row in $\mathcal{W}_1$ is orthogonal to each row in $\mathcal{V}_1$ and hence

$$\mathcal{W} \equiv \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{V}_1 \end{bmatrix} \quad \text{is an orthonormal transform}$$

Daubechies Scaling Filters

- Daubechies (1992) constructs a family of scaling filters $\{g_l\}$ with squared gain functions given by

$$\mathcal{G}_{(\mathrm{D})}(f) \equiv 2\cos^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2} - 1 + l}{l} \sin^{2l}(\pi f)$$

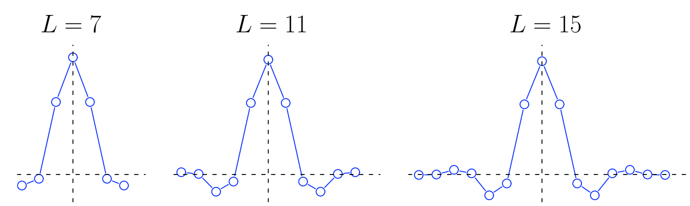
(corresponding wavelet filter given by $h_l = (-1)^l g_{L-1-l}$)

- for given L, there are multiple filters with the same $\mathcal{G}_{(\mathrm{D})}(\cdot)$, with these filters being distinguished by their phase functions $\theta(\cdot)$; i.e., their transfer functions can be written as

$$G(f) \equiv \sum_{l=0}^{L-1} g_l e^{-i2\pi f l} = \mathcal{G}_{(\mathrm{D})}^{1/2}(f) e^{i\theta(f)}$$

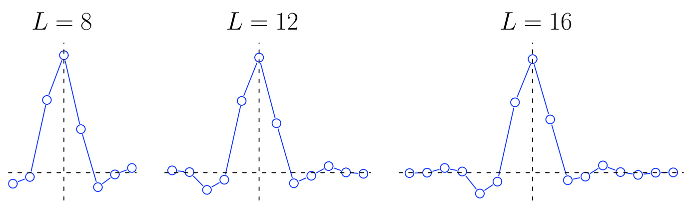
Zero-Phase Filters

- Oppenheim and Lim (1981) note that filters with zero phase (i.e., $\theta(f) = 0$ for all f) are important for eliminating distortions in filtered signals (particularly in images)
- zero-phase filters also facilitate aligning filter output with input
- conventional zero-phase filters $\{a_l\}$ must be of *odd* length, say $L = 2M + 1$, and take the form $a_{-l} = a_l$ for $l = -M, \ldots, M$
- three examples of zero-phase filters



‘Least Asymmetric’ Scaling Filters (Symlets)

- in recognition of importance of zero-phase filters, Daubechies (1992) uses spectral factorization to obtain filters of widths $L = 8, 10, 12, \ldots$ closest to having zero phase (after a reindexing)
- three members of her class of ‘least asymmetic’ scaling filters



- cannot achieve filters with *exact* zero phase under her scheme because L must be even

Zero-Phase Wavelet (Zephlet) Transform: I

- possible to construct orthonormal DWT based on filters whose squared gain functions are consistent with those of Daubechies, but with *exact* zero phase, as following theorem states
- let $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ be squared gain functions satisfying

$$\mathcal{G}(\tfrac{k}{N}) + \mathcal{G}(\tfrac{k}{N} + \tfrac{1}{2}) = 2 \text{ and } \mathcal{H}(\tfrac{k}{N}) + \mathcal{G}(\tfrac{k}{N}) = 2 \text{ for all } \tfrac{k}{N}$$

- let $\{\bar{g}_l\}$ & $\{\bar{h}_l\}$ be inverse DFTs of the sequences $\{\mathcal{G}^{1/2}(\frac{k}{N})\}$ & $\{\mathcal{H}^{1/2}(\frac{k}{N})\}$:

$$\bar{g}_l \equiv \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{G}^{1/2}(\tfrac{k}{N}) e^{i2\pi kl/N}, \quad l = 0, 1, \ldots, N-1,$$

with an analogous expression for $\bar{h}_l$

Zero-Phase Wavelet (Zephlet) Transform: II

- define the $\frac{N}{2} \times N$ matrices

$$\mathcal{D}_1 = \begin{bmatrix} \bar{h}_1 & \bar{h}_0 & \bar{h}_{N-1} & \bar{h}_{N-2} & \bar{h}_{N-3} & \cdots & \bar{h}_5 & \bar{h}_4 & \bar{h}_3 & \bar{h}_2 \\ \bar{h}_3 & \bar{h}_2 & \bar{h}_1 & \bar{h}_0 & \bar{h}_{N-1} & \cdots & \bar{h}_7 & \bar{h}_6 & \bar{h}_5 & \bar{h}_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{h}_{N-1} & \bar{h}_{N-2} & \bar{h}_{N-3} & \bar{h}_{N-4} & \bar{h}_{N-5} & \cdots & \bar{h}_3 & \bar{h}_2 & \bar{h}_1 & \bar{h}_0 \end{bmatrix}$$

and

$$\mathcal{C}_1 = \begin{bmatrix} \bar{g}_0 & \bar{g}_{N-1} & \bar{g}_{N-2} & \bar{g}_{N-3} & \bar{g}_{N-4} & \cdots & \bar{g}_4 & \bar{g}_3 & \bar{g}_2 & \bar{g}_1 \\ \bar{g}_2 & \bar{g}_1 & \bar{g}_0 & \bar{g}_{N-1} & \bar{g}_{N-2} & \cdots & \bar{g}_6 & \bar{g}_5 & \bar{g}_4 & \bar{g}_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{g}_{N-2} & \bar{g}_{N-3} & \bar{g}_{N-4} & \bar{g}_{N-5} & \bar{g}_{N-6} & \cdots & \bar{g}_2 & \bar{g}_1 & \bar{g}_0 & \bar{g}_{N-1} \end{bmatrix}$$

(note that, while $\mathcal{D}_1$ has a form analogous to $\mathcal{W}_1$ & $\mathcal{V}_1$, rows of $\mathcal{C}_1$ are circularly shifted to the left by one)

Zero-Phase Wavelet (Zephlet) Transform: III

- then the $N \times N$ matrix formed by stacking $\mathcal{D}_1$ on top of $\mathcal{C}_1$ is a real-valued orthonormal matrix; i.e,

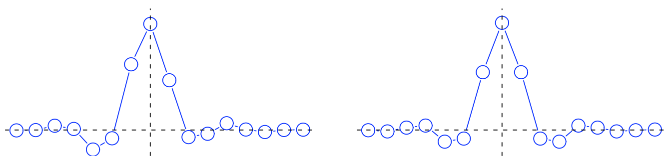
$$\mathcal{D} \equiv \begin{bmatrix} \mathcal{D}_1 \\ \mathcal{C}_1 \end{bmatrix} \text{ is such that } \mathcal{D}^T \mathcal{D} = I_N$$

- moreover, the zero-phase circular filters $\{\bar{h}_l\}$ and $\{\bar{g}_l\}$ are related by $\bar{g}_l = (-1)^l \bar{h}_l$ (note that this is in contrast to what holds for DWT filters, namely, $g_l = (-1)^{l+1} h_{L-1-l}$)
- proof of above theorem is similar in spirit to proof that $\mathcal{W}$ is orthonormal, but details differ
- algorithms for computing DWT and zephlet transform are, respectively, $\mathcal{O}(N)$ and $\mathcal{O}(N \cdot \log_2(N))$

Zero-Phase Wavelet (Zephlet) Transform: IV

- for case $N = L = 16$, let's compare values in rows of $\mathcal{V}_1$ based on Daubechies' least asymmetric filter and corresponding $\mathcal{C}_1$ (after alignments for easier comparison)

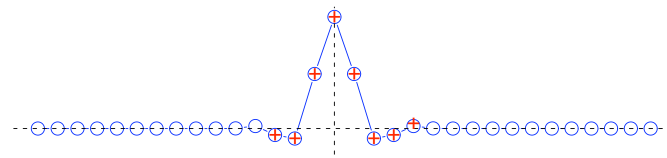
DWT filter $g_l^\circ = g_l$ zephlet transform filter $\bar{g}_l$



- for any N and L, squared magnitudes of DFTs of $\{g_l^\circ\}$ & $\{\bar{g}_l\}$ at $f_k = k/N$ are exactly the same, but phase functions differ, with that for $\{\bar{g}_l\}$ given by $\theta(f_k) = 0$

Zero-Phase Wavelet (Zephlet) Transform: V

- for fixed $L \geq 8$, values in rows of zephlet transform change as N increases (DWT rows just add more 0's for all $N \geq L$)
- consider zephlet transform based on least asymmetric filter for $L = 8$ and cases $N = 8$ (pluses) and $N = 32$ (circles)



Zero-Phase Wavelet (Zephlet) Transform: VI

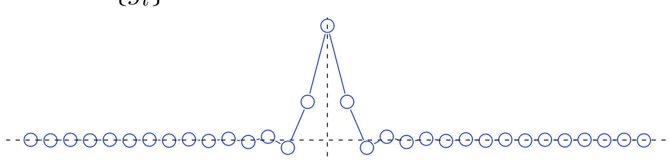
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1 - 4l^2)}$$

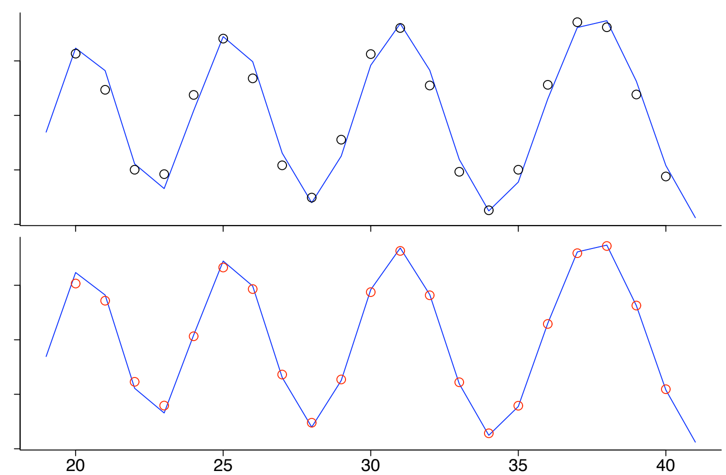
for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 32$:



Comparison of Outputs from LA(8) & Zephlet Scaling Filters (Input is Doppler Signal)



Concluding Remarks

- more work needed to elicit advantages/disadvantages of zephlet transform over usual DWT (in particular, for economic applications)
- can also formulate 'maximal overlap' version of zephlet transform (details in Percival, 2010)
- thanks to Ramo Gençay & conference organizers for opportunity to talk!

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