An Omnibus Test for Red Noise, with Applications to Climatology Time Series

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Collaborative effort with You-Gan Wang

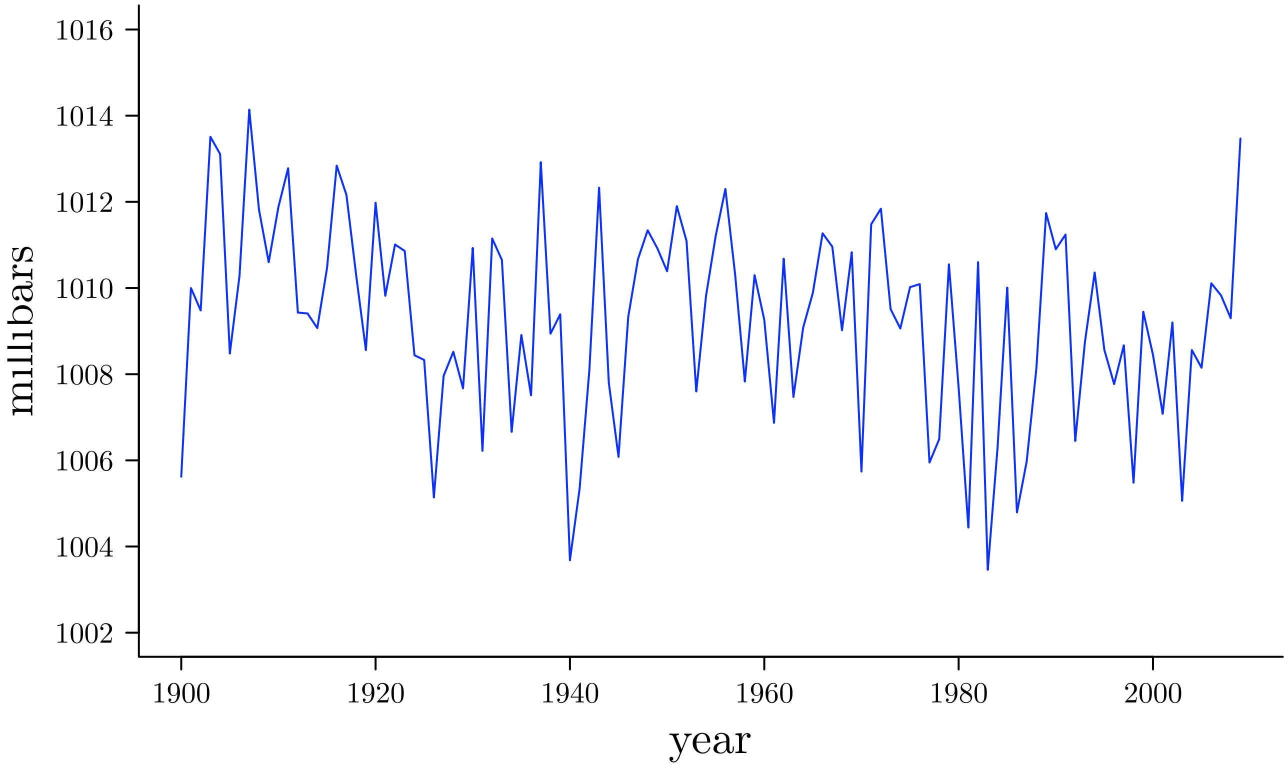
Overview

- analysis of time series related to climate often rely on 'red noise' as a simple model for correlation in the series
- after giving background on
  - red noise,
  - partial autocorrelation sequences (PACSs) and
  - portmanteau tests for white noise,

  will describe an omnibus test – and a variation thereof – designed to point out when a model other than red noise is needed
- will discuss adapation of tests to handle time series with missing values (a common occurrence in climatology)
- will demonstrate use of tests on two climatology time series

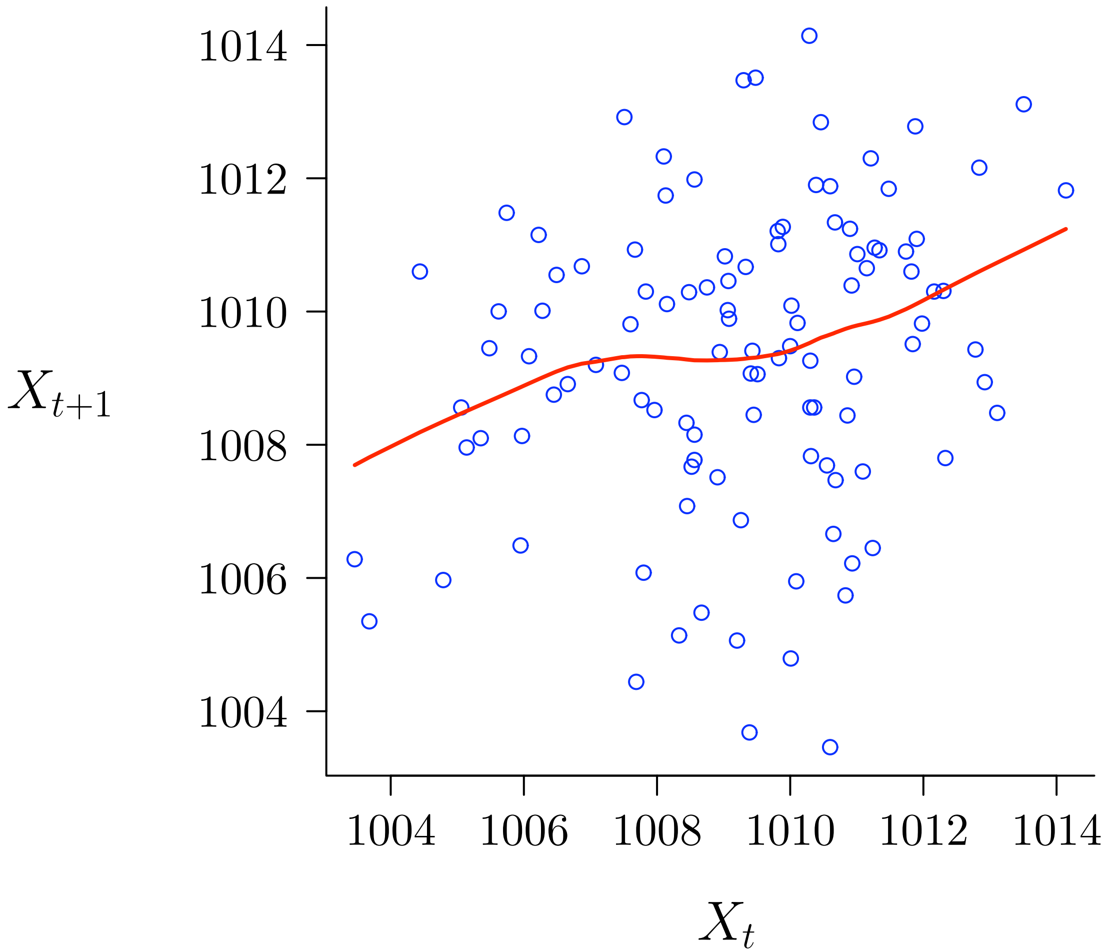
Example of a Climatology Time Series (NPI): I

- consider North Pacific Index (NPI): area-weighted sea level pressure over 30° N to 65° N & 160° E to 140° W and over November to March for each year from 1900 to 2009



Example of a Climatology Time Series (NPI): II

- unit lag scatter plot & locally weighted regression fit ($\hat{\rho}_1 \doteq 0.21$)



Modelling Correlation in Time Series as Red Noise

- cannot regard NPI and most other climatology time series X_t as realizations of independent random variables (RVs)
- widely-used simple model for correlated time series is 'red noise'
- red noise is the same as a first-order autoregressive (AR(1)) stationary Gaussian process with a positive correlation at unit lag (see, e.g., von Storch and Zwiers, 1999)
- assuming $E\{X_t\} = 0$ for convenience, such a process satisfies

$$X_t = \phi X_{t-1} + \epsilon_t,$$

where $|\phi| < 1$, and ϵ_t's are IID Gaussian with mean 0 and variance σ_ϵ^2 (i.e., Gaussian white noise)

Properties of AR(1) Processes

- can argue that

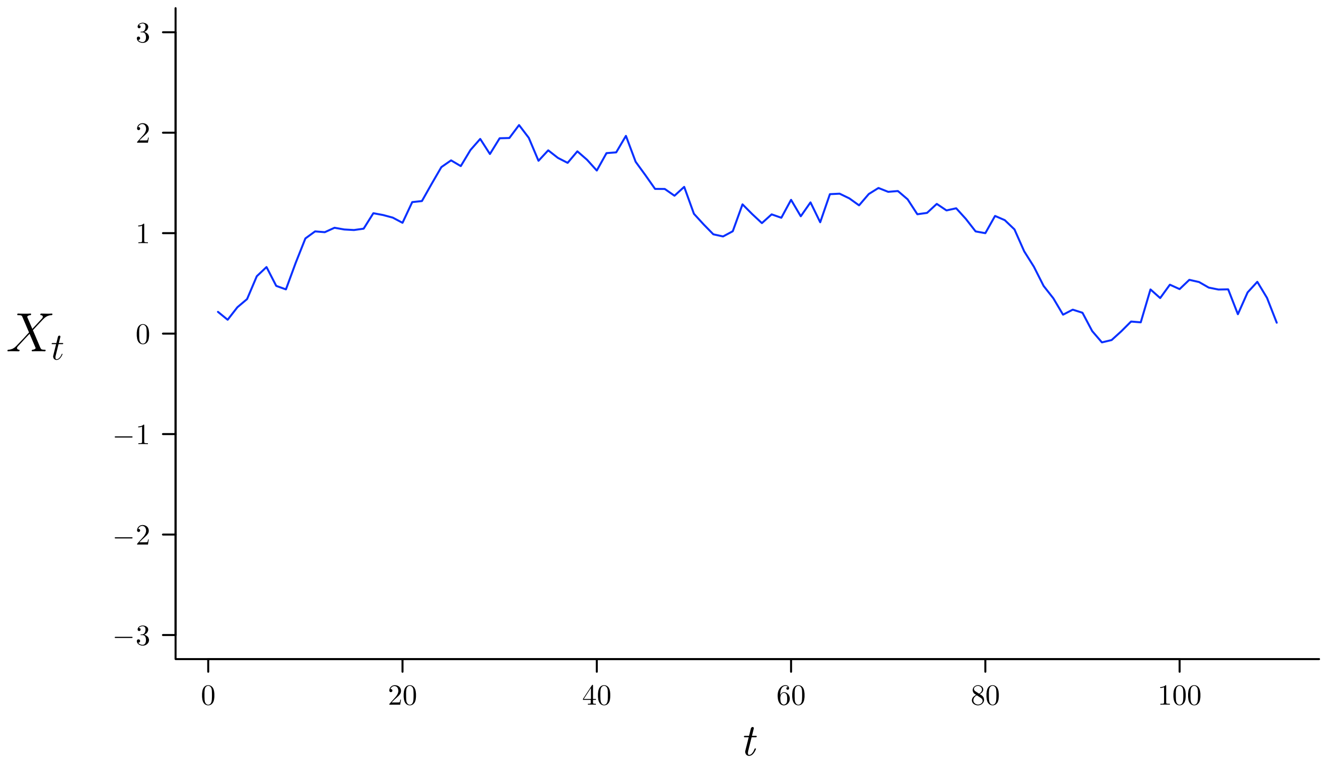
$$\sigma_X^2 \equiv \operatorname{var}\{X_t\} = \frac{\sigma_\epsilon^2}{1-\phi^2} \text{ and } \operatorname{cov}\{X_{t+k}, X_t\} = \phi^{|k|}\sigma_X^2,$$

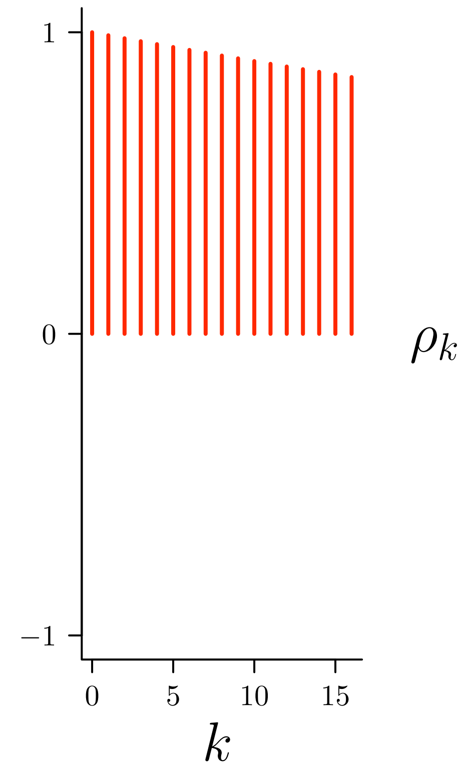
  so $\rho_k \equiv \operatorname{corr}\{X_{t+k}, X_t\} = \phi^{|k|}$ – the autocorrelation sequence (ACS) – dies down exponentially as $k \to \infty$ ('short-range' dependence)

- when $\phi = 0$, AR(1) process reduced to white noise
- AR(1) process is related to a first-order stochastic differential equation with 'correlation time' dictated by ϕ
- AR(1) model is simple enough to offer analytic tractablity for calculations (e.g., getting an expression for $\operatorname{var}\{\overline{X}\}$, where $\overline{X}$ is the mean of $X_0, X_1, \ldots, X_{N-1}$)

Examples of Realizations of AR(1) Processes

- easy to generate realizations of Gaussian AR(1) processes (see Kay, 1981, for details)
- with $\sigma_X^2 = 1$, here is a realization when $\phi = 0.99$





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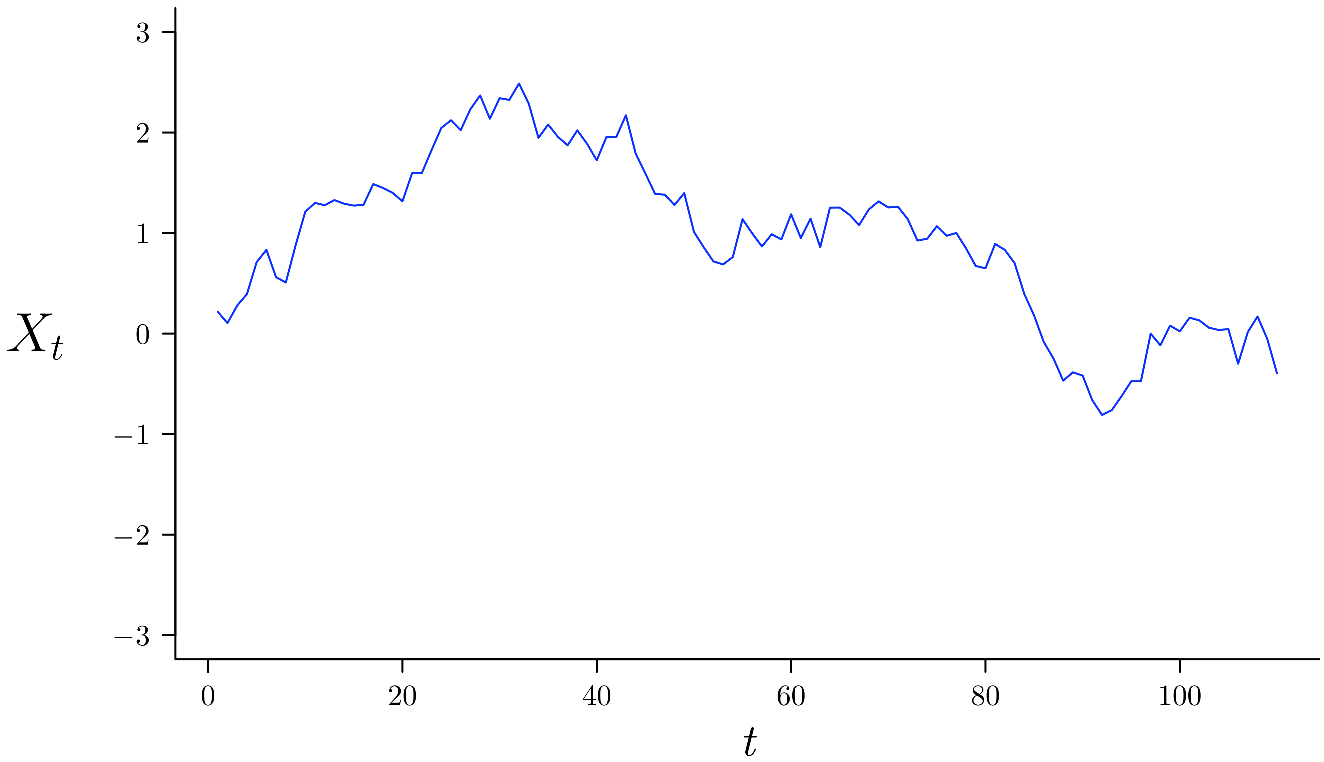

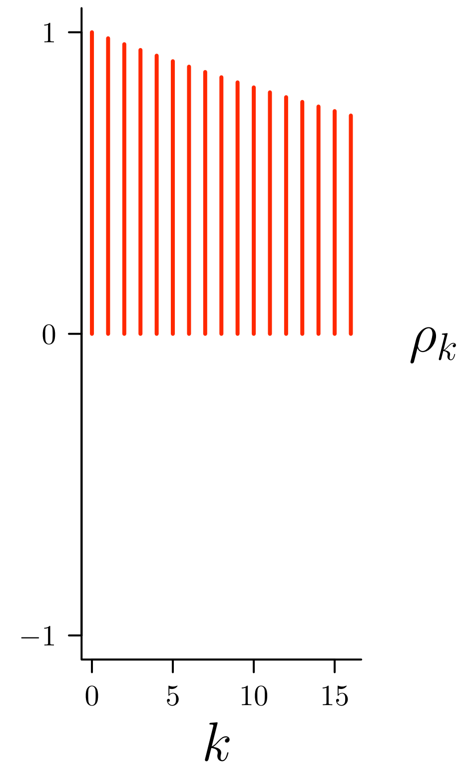

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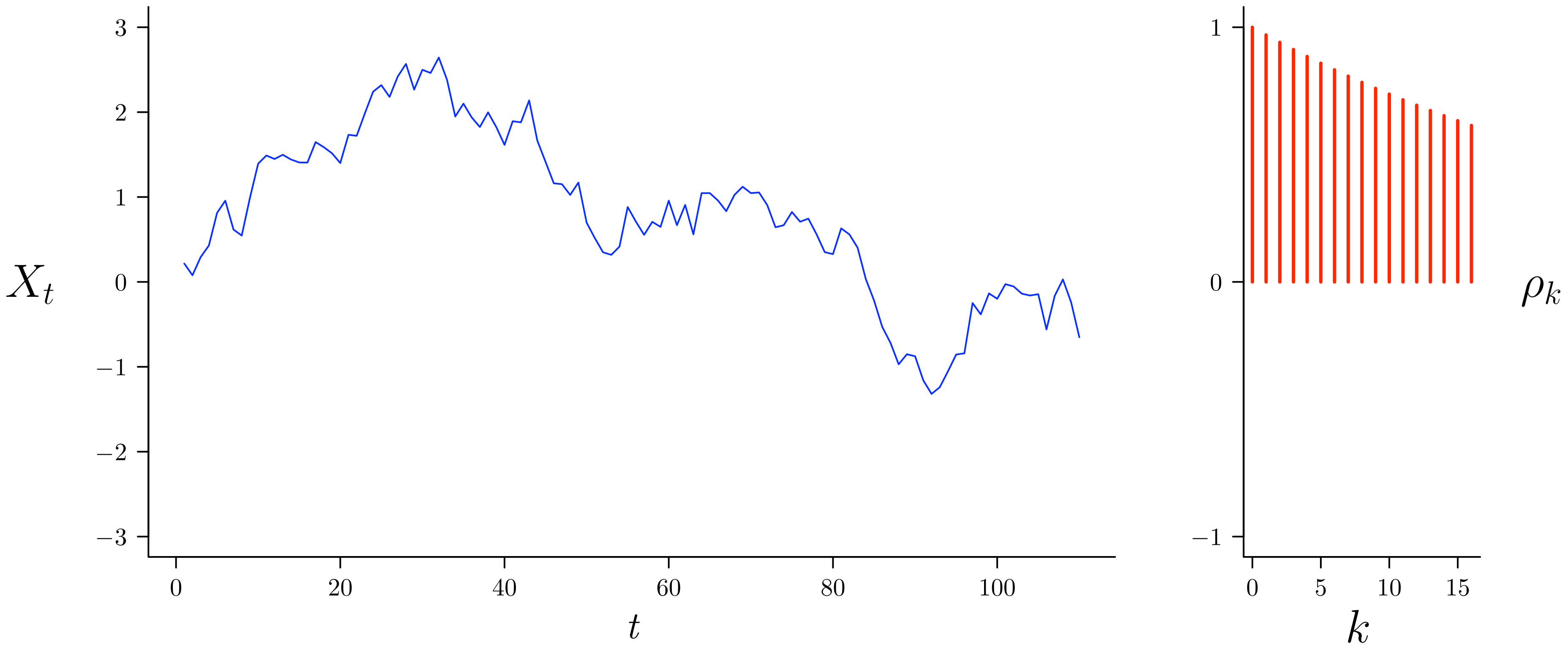

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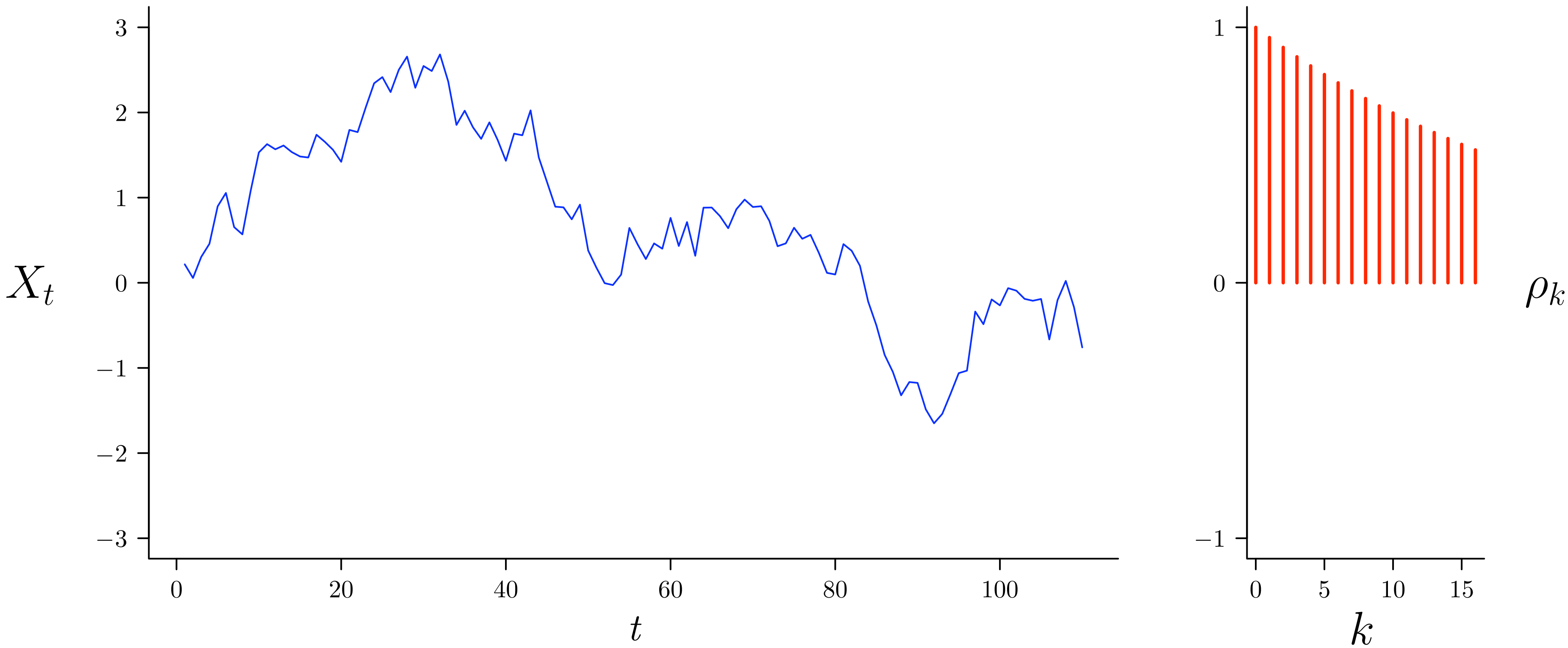

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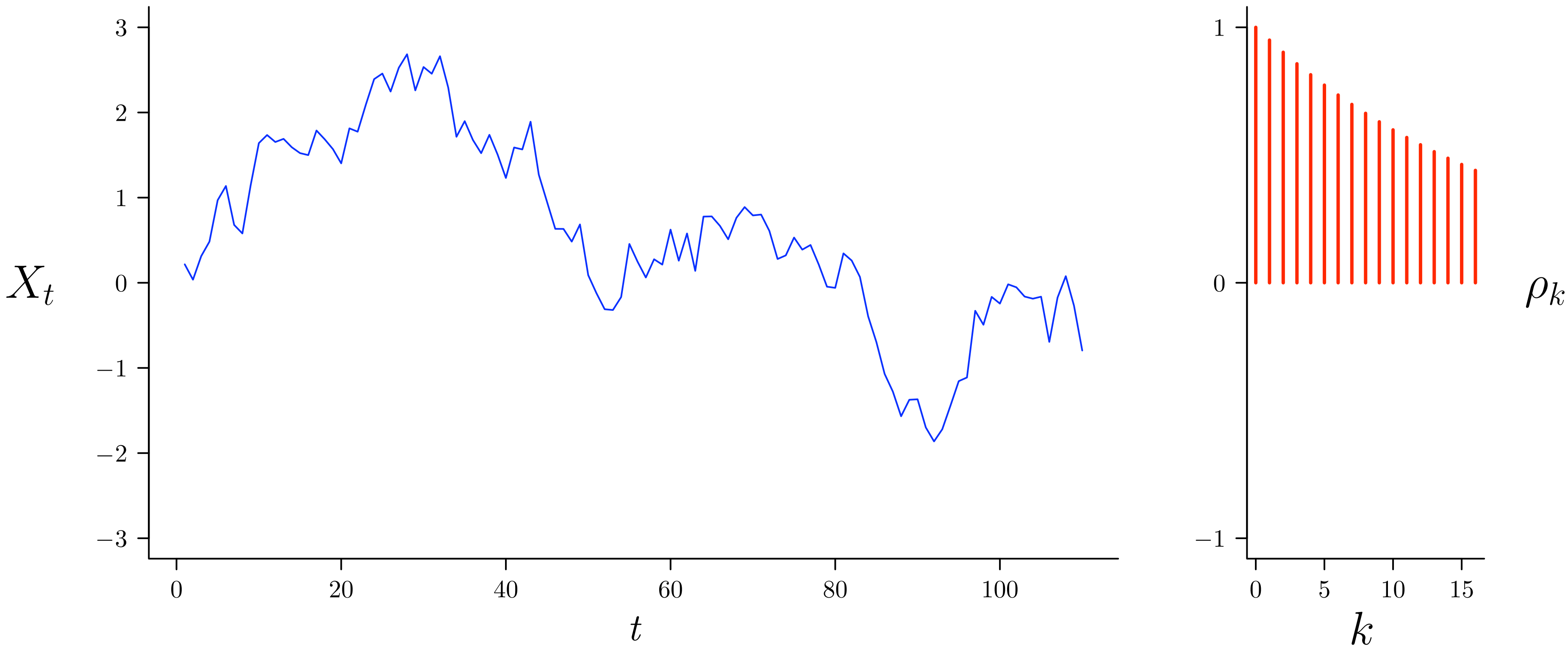

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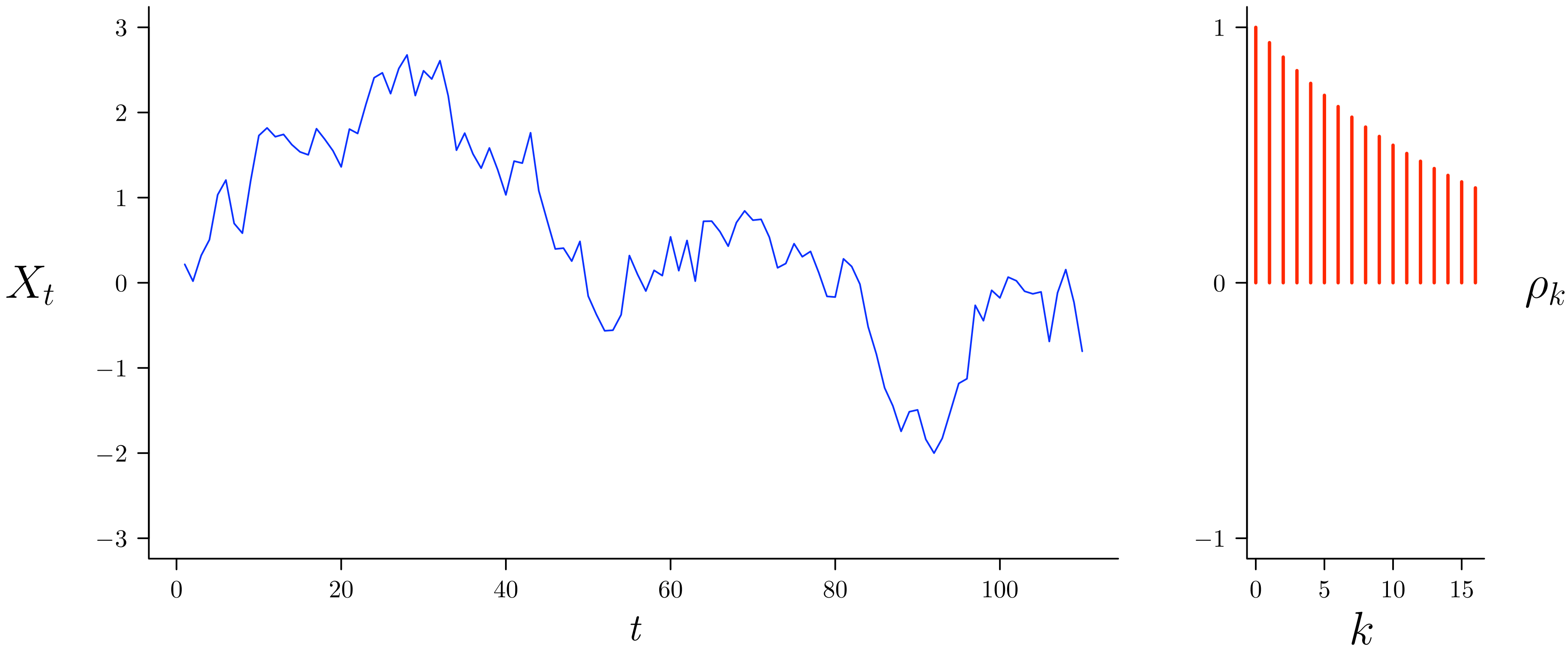

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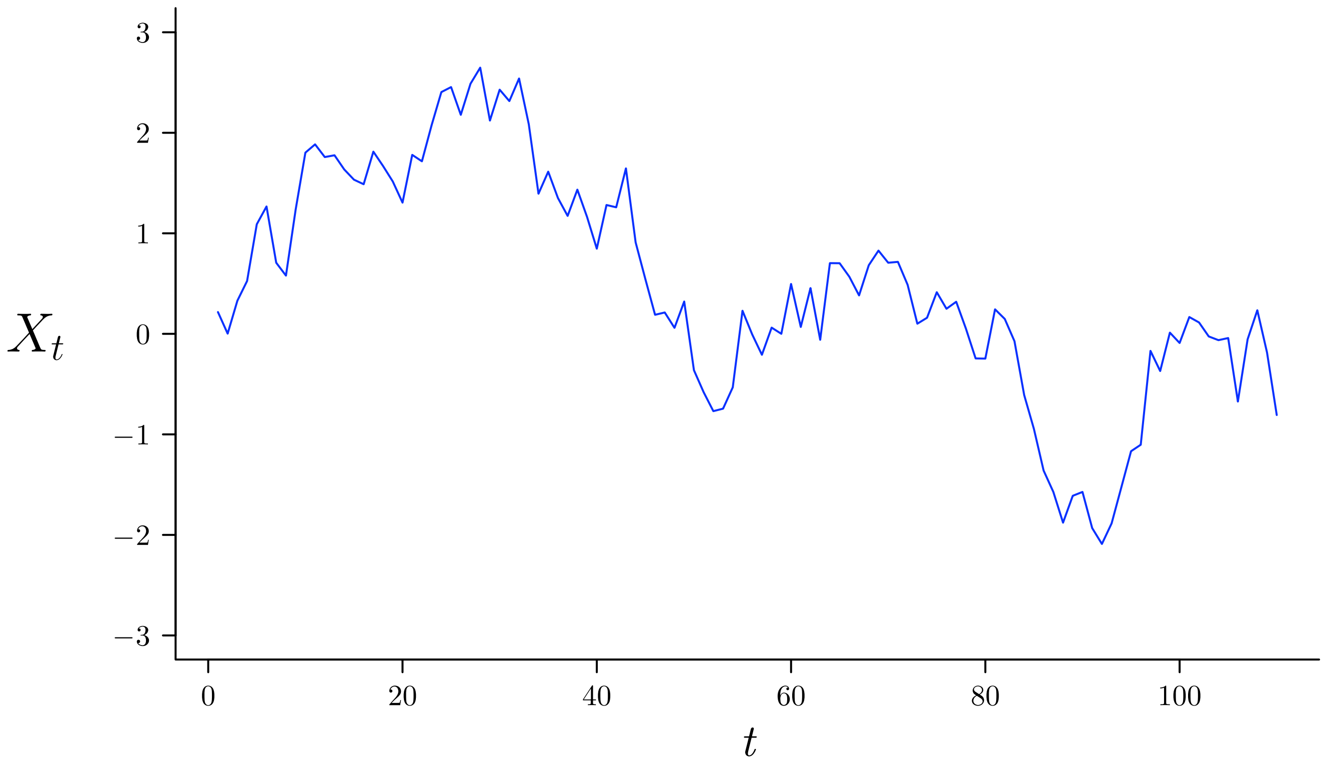

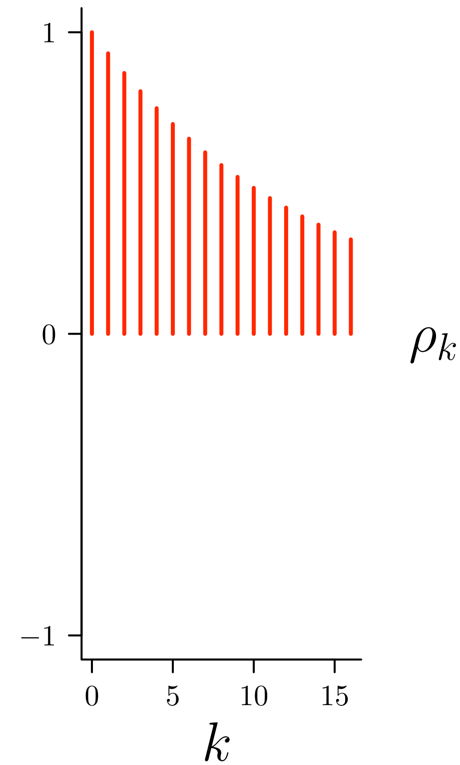

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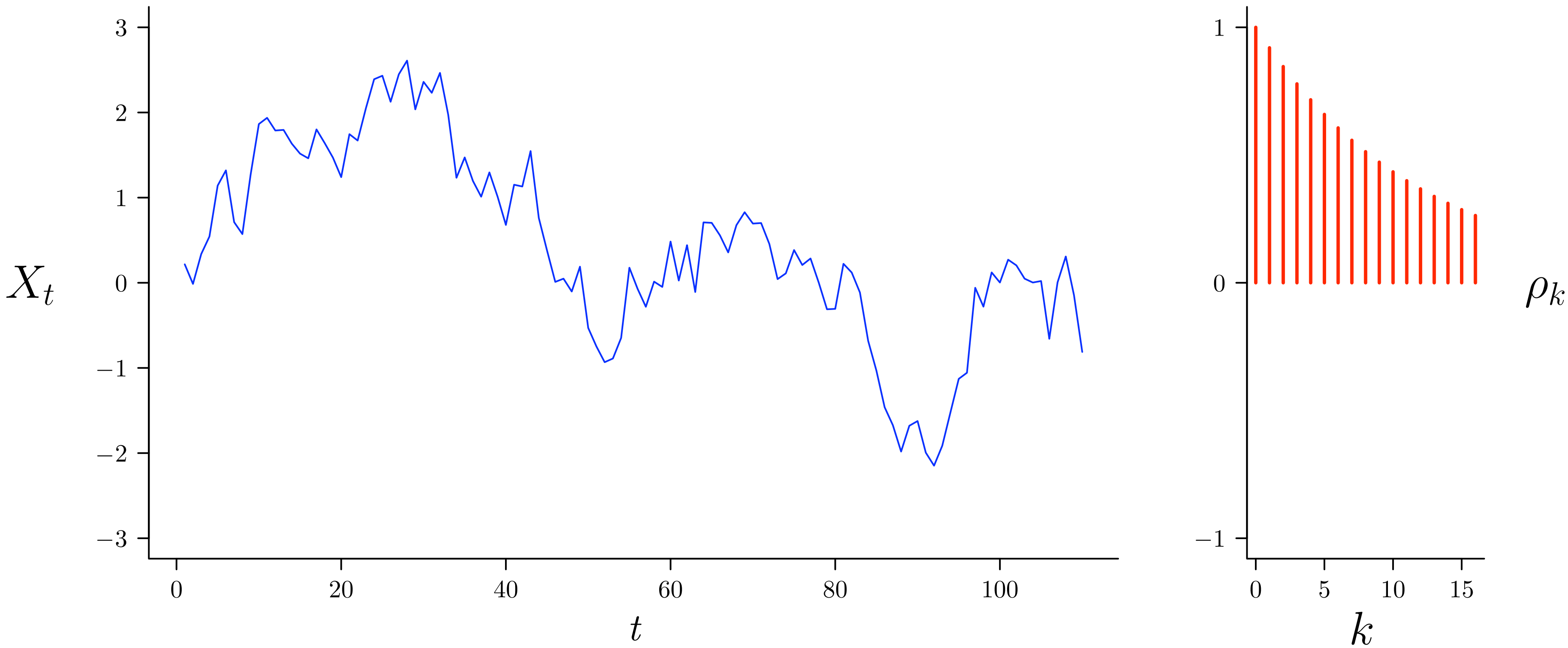

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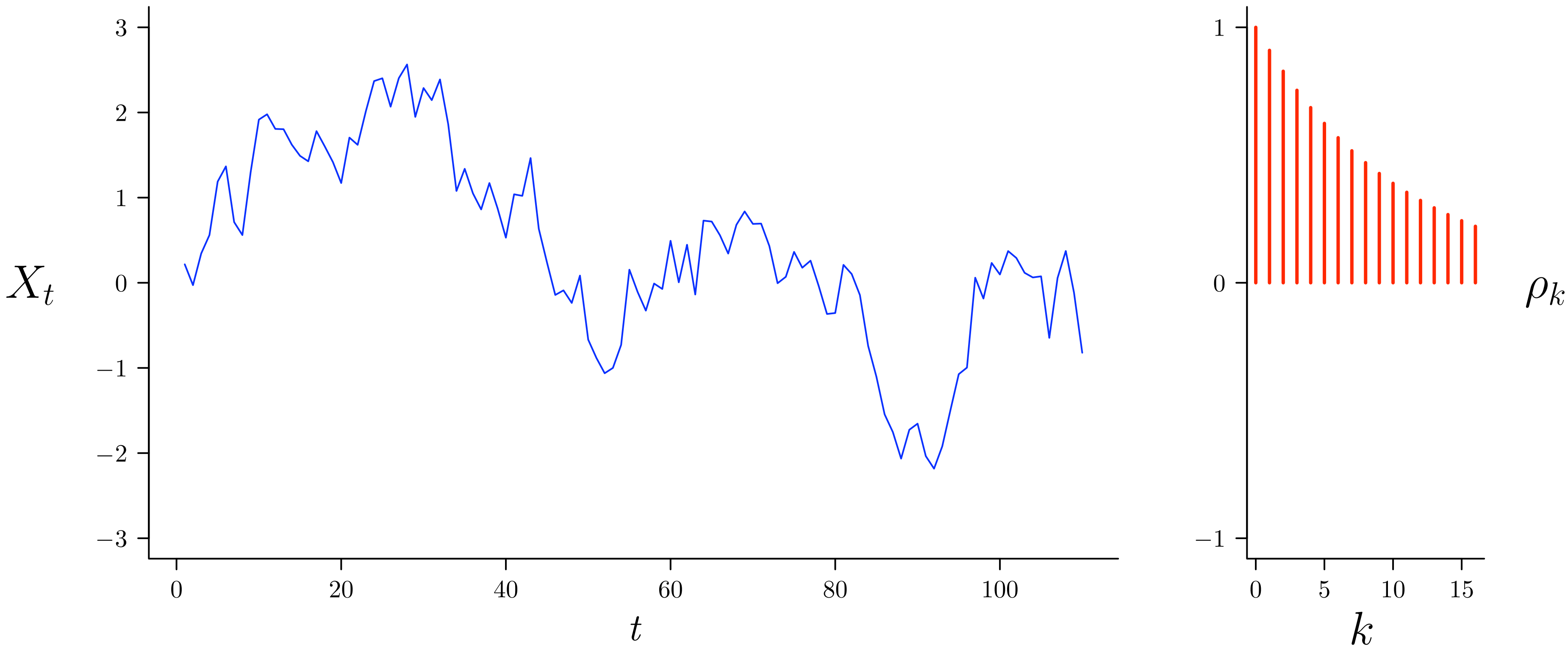

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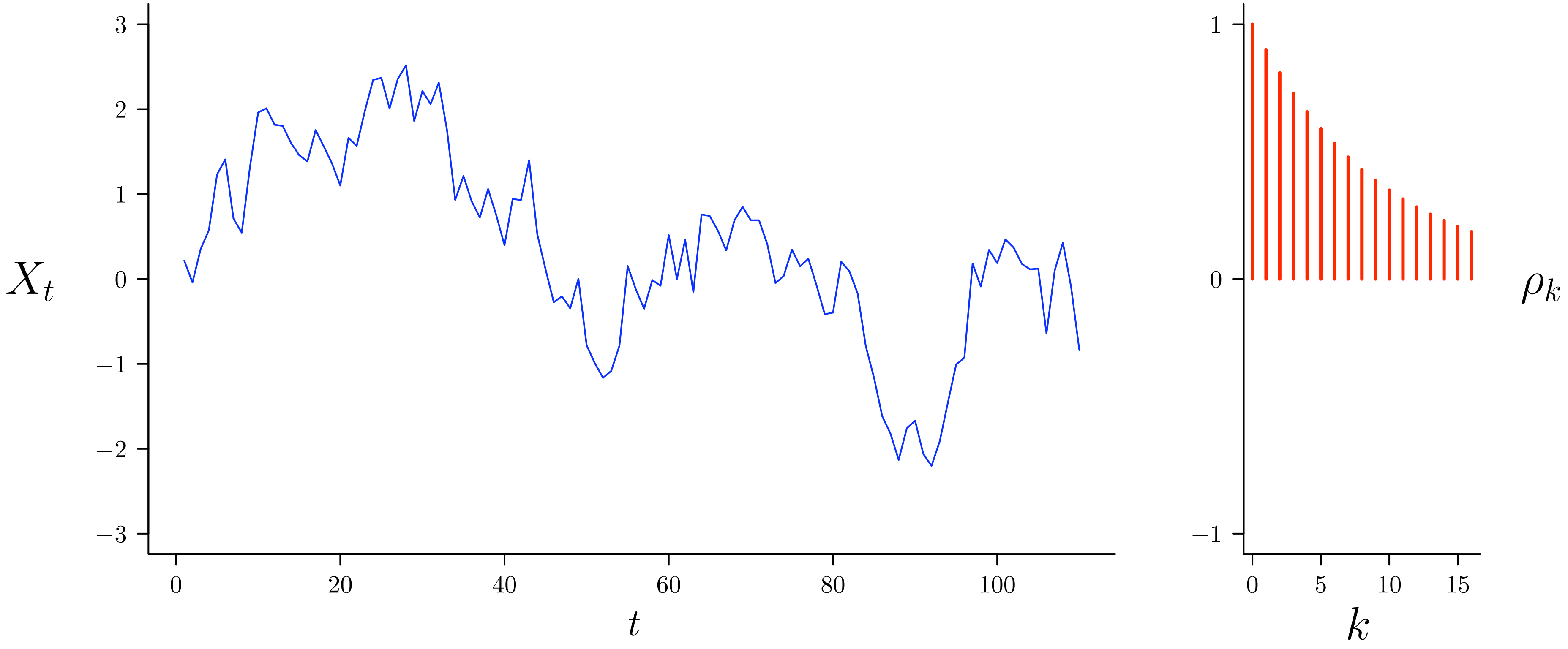

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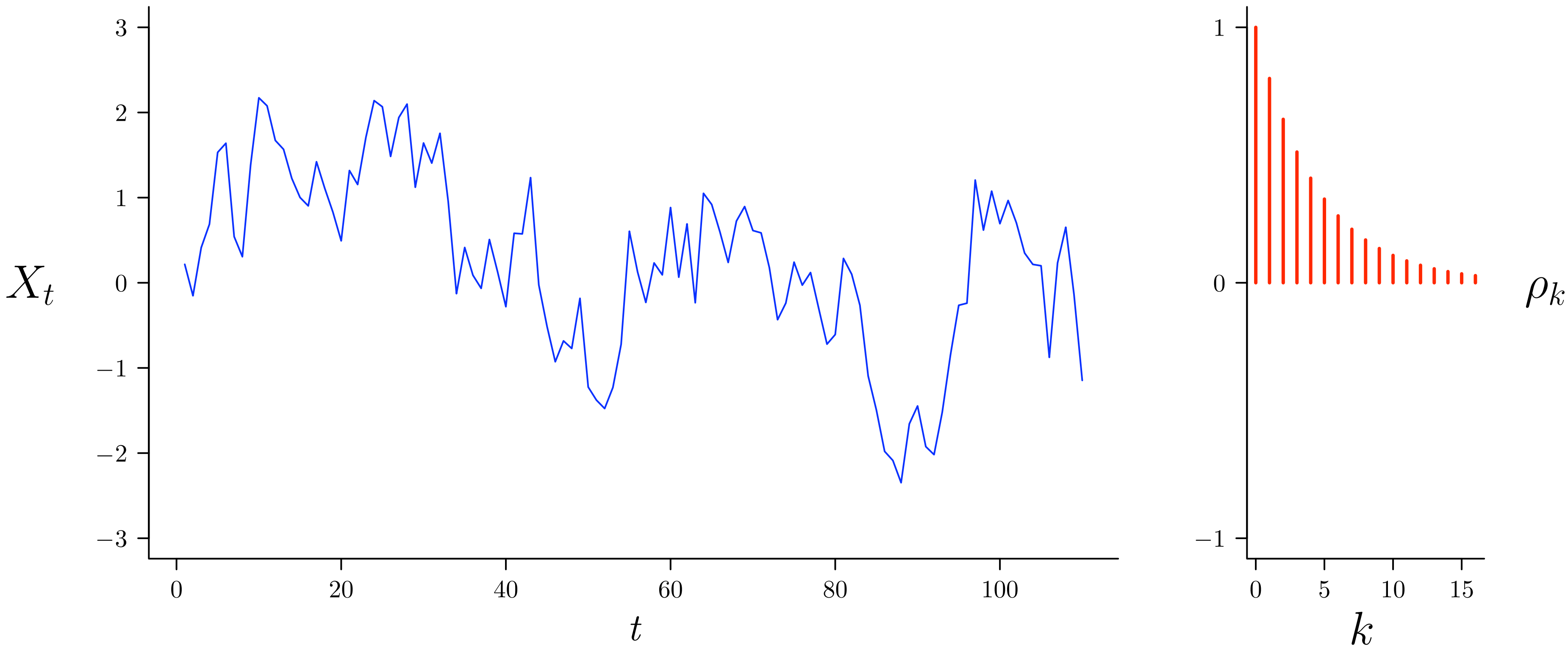

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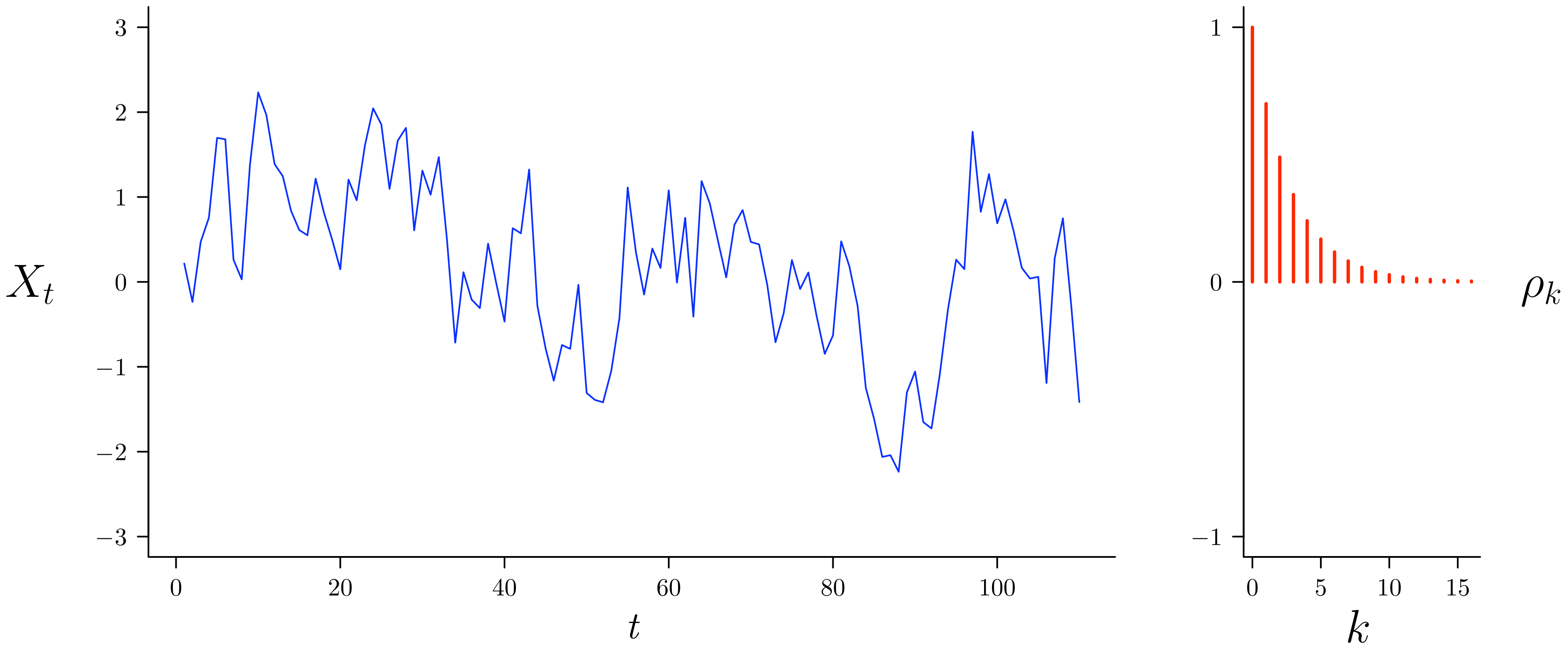
- with $\sigma_X^2 = 1$, here is a realization when $\phi = 0.7$



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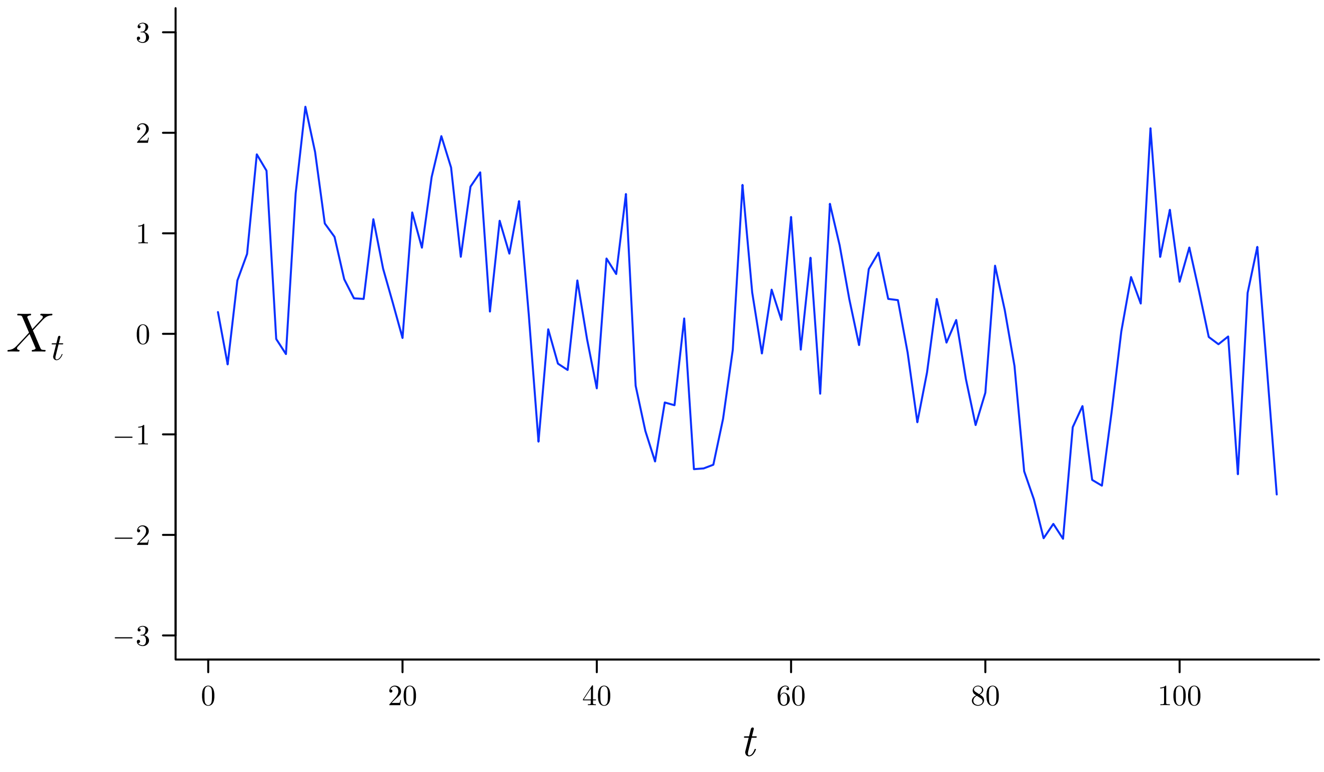
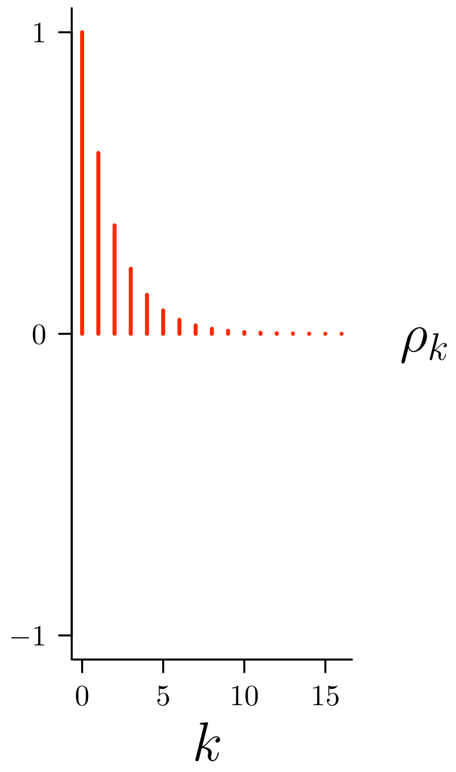
- with $\sigma_X^2 = 1$, here is a realization when $\phi = 0.6$





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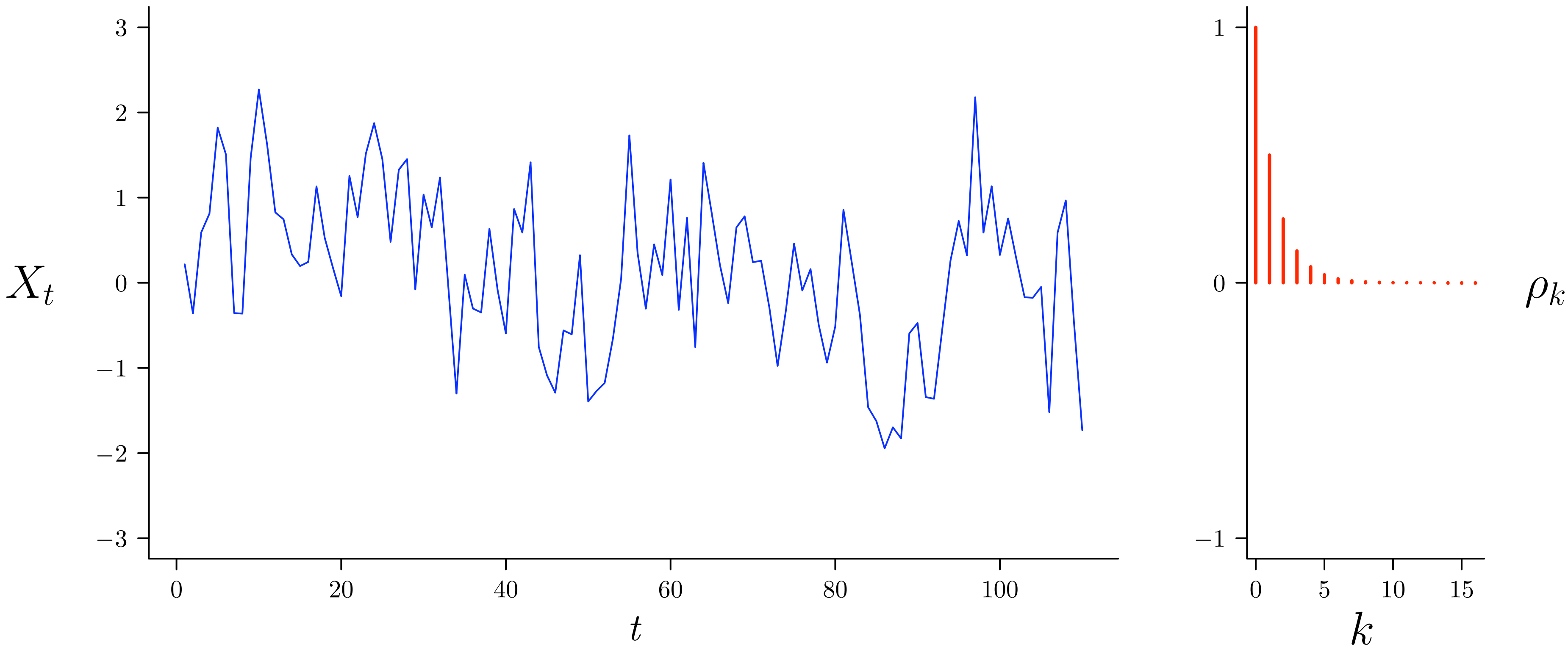
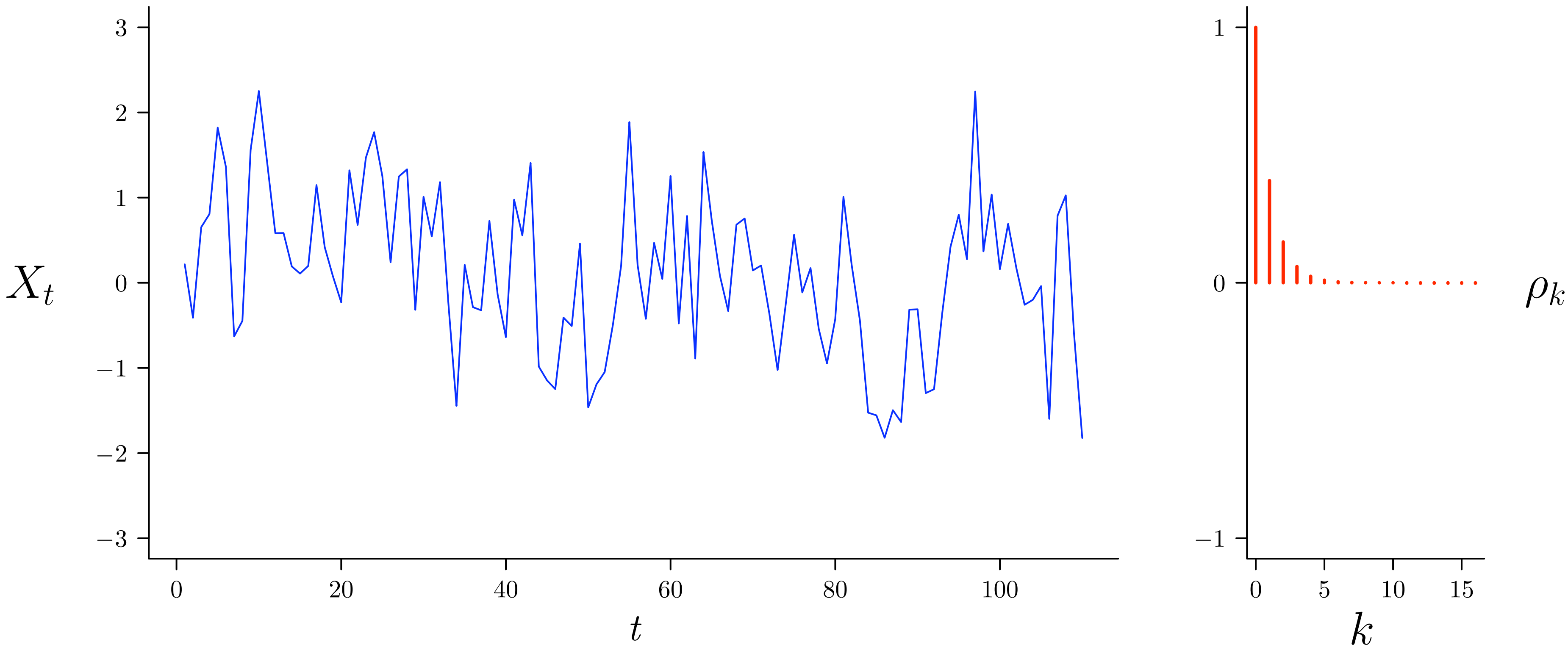
- easy to generate realizations of Gaussian AR(1) processes (see Kay, 1981, for details)
- with $\sigma_X^2 = 1$, here is a realization when $\phi = 0.5$



Examples of Realizations of AR(1) Processes

- easy to generate realizations of Gaussian AR(1) processes (see Kay, 1981, for details)
- with $\sigma_X^2 = 1$, here is a realization when $\phi = 0.4$



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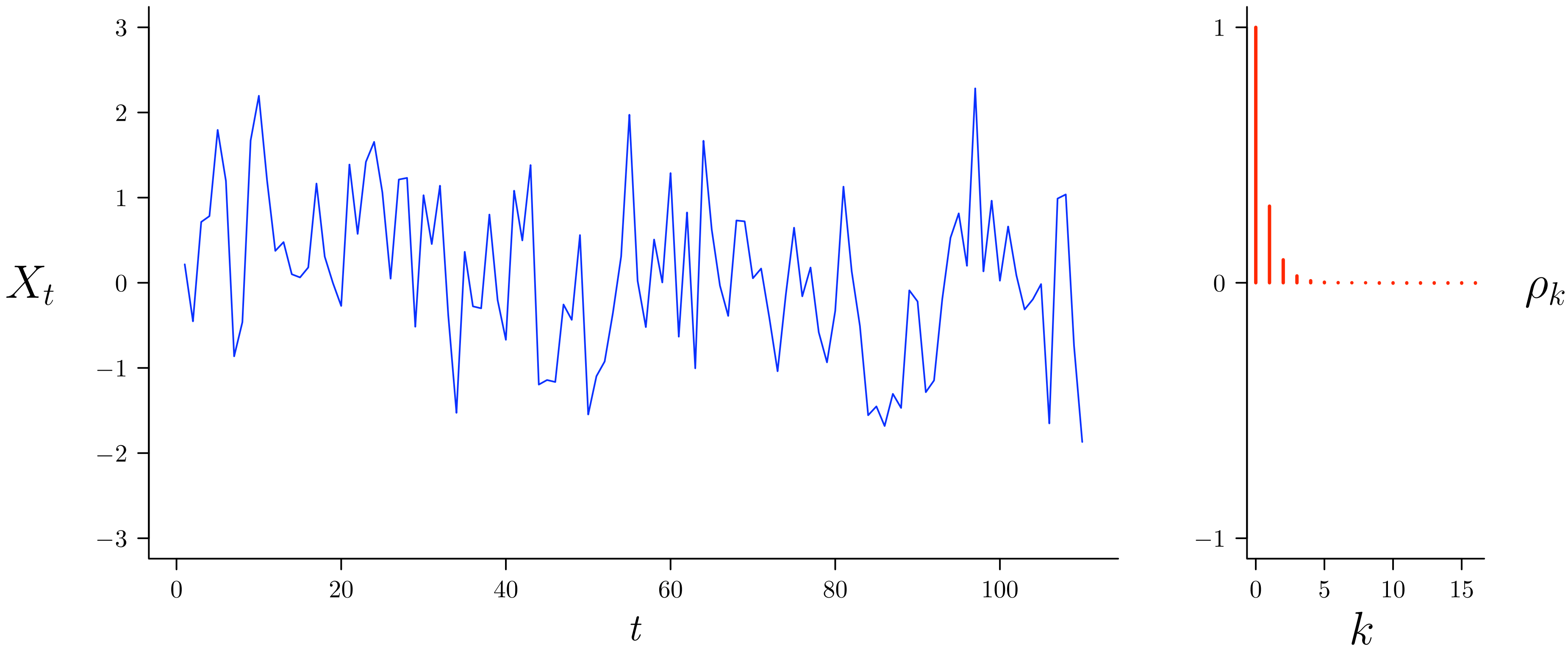
- with $\sigma_X^2 = 1$, here is a realization when $\phi = 0.3$



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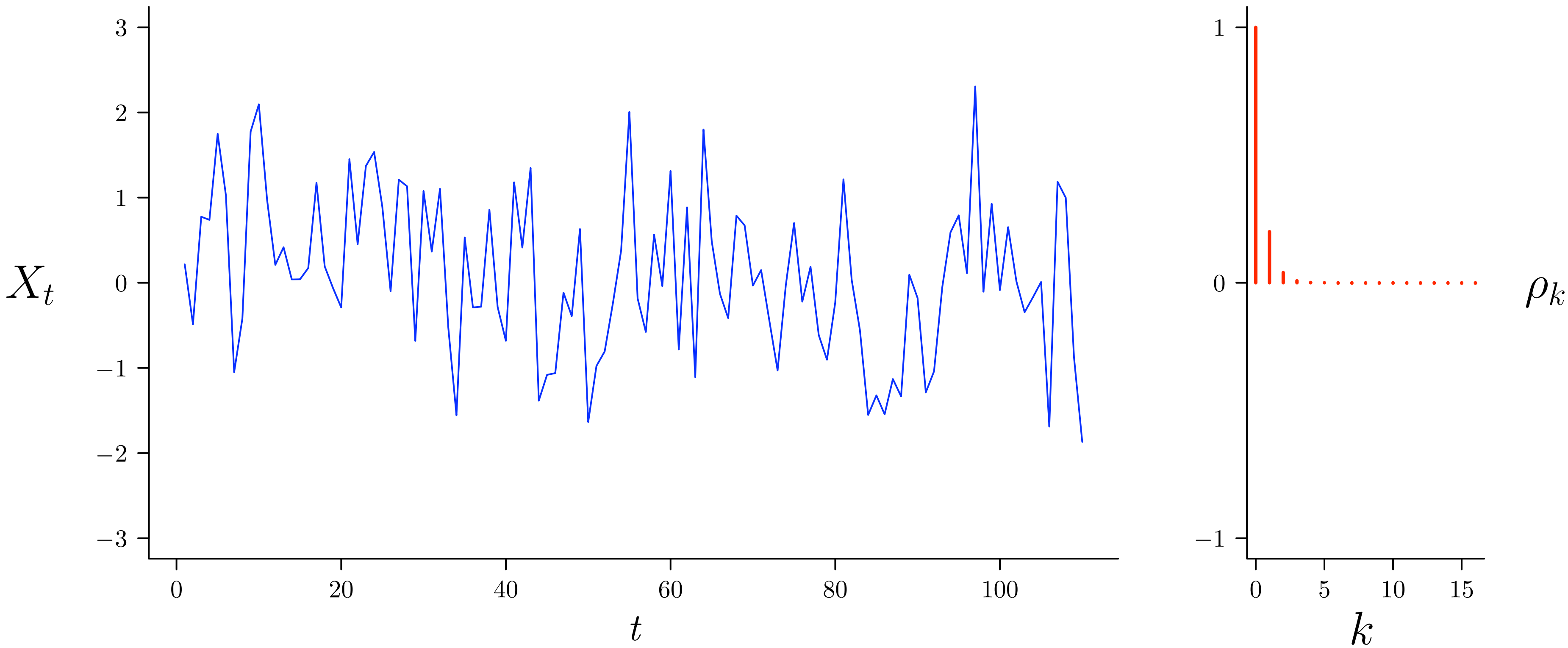
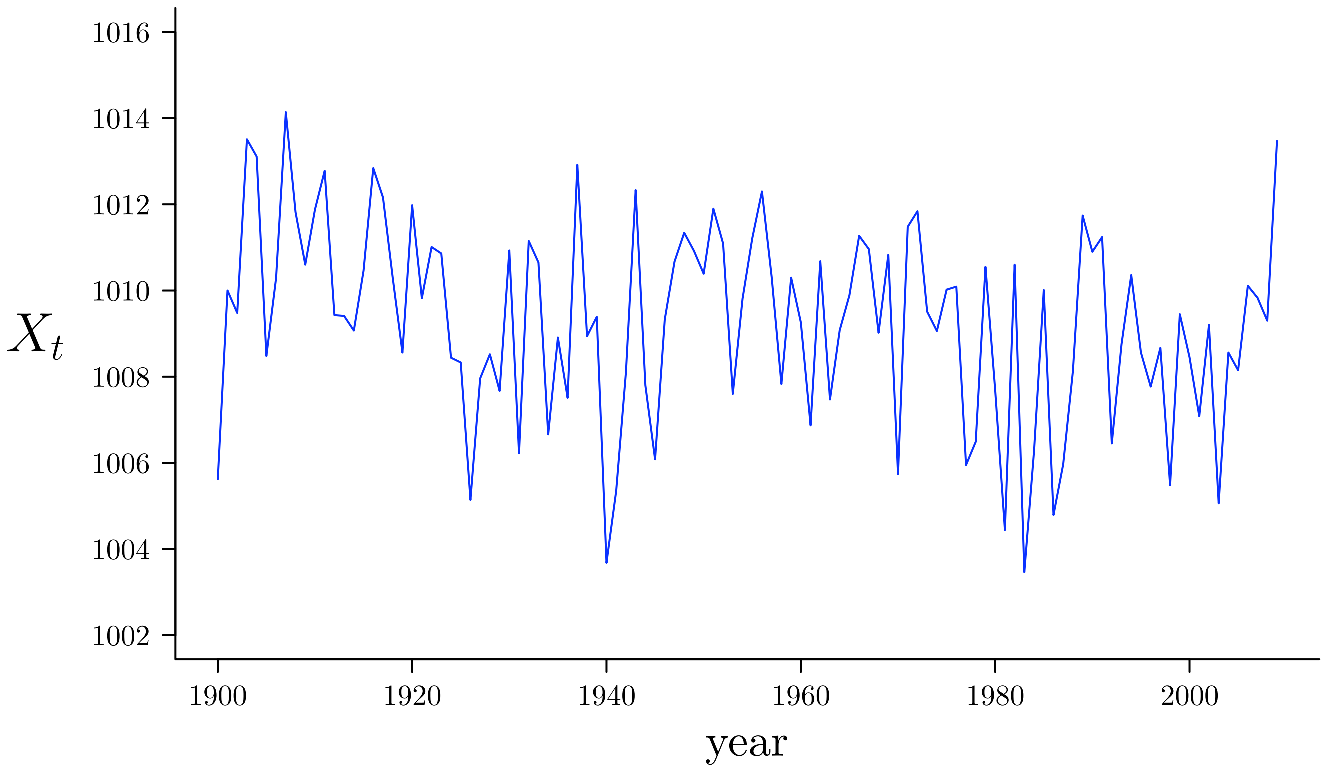
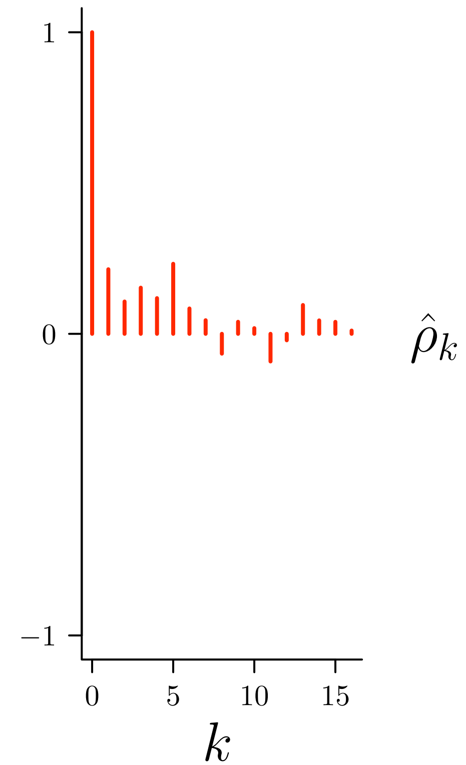
- with $\sigma_X^2 = 1$, here is a realization when $\phi = 0.2$



Examples of Realizations of AR(1) Processes

- easy to generate realizations of Gaussian AR(1) processes (see Kay, 1981, for details)
- for comparison, here is NPI with ϕ estimated by $\hat{\rho}_1 \doteq 0.21$





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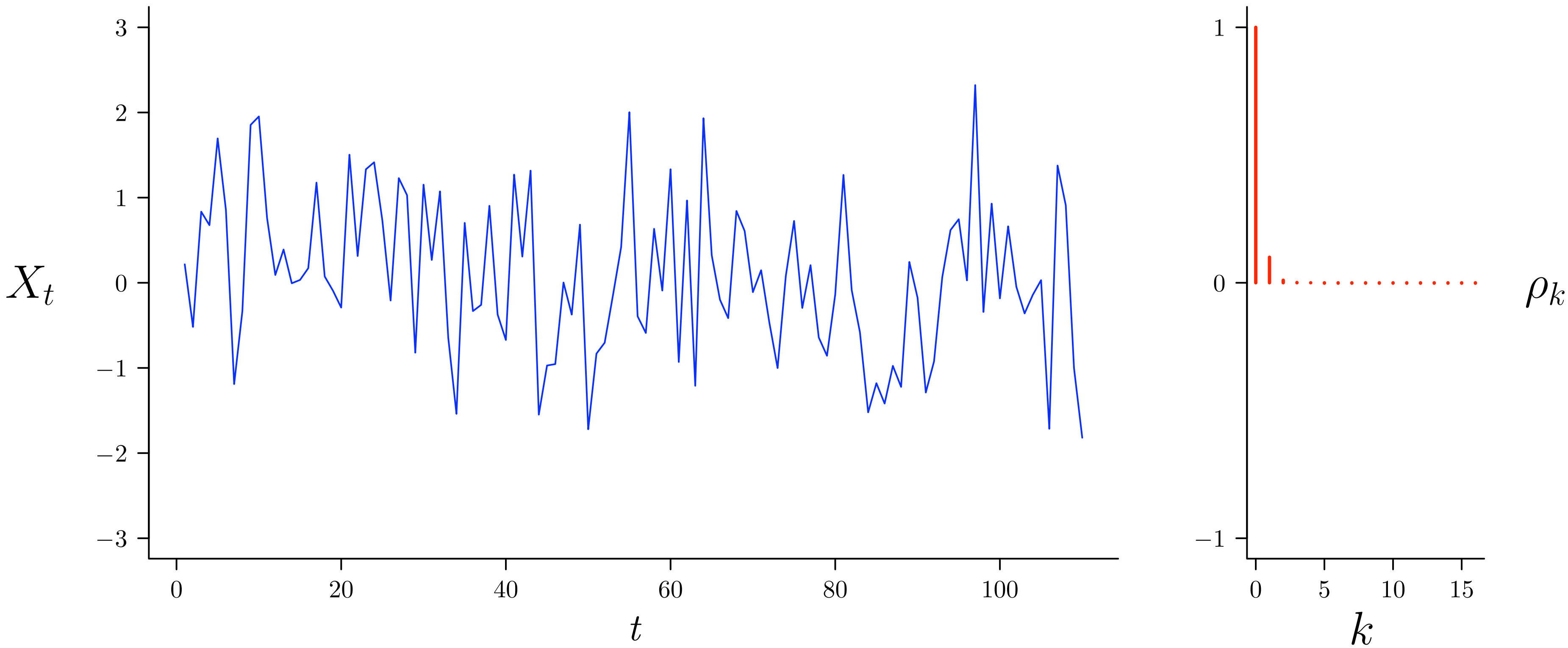

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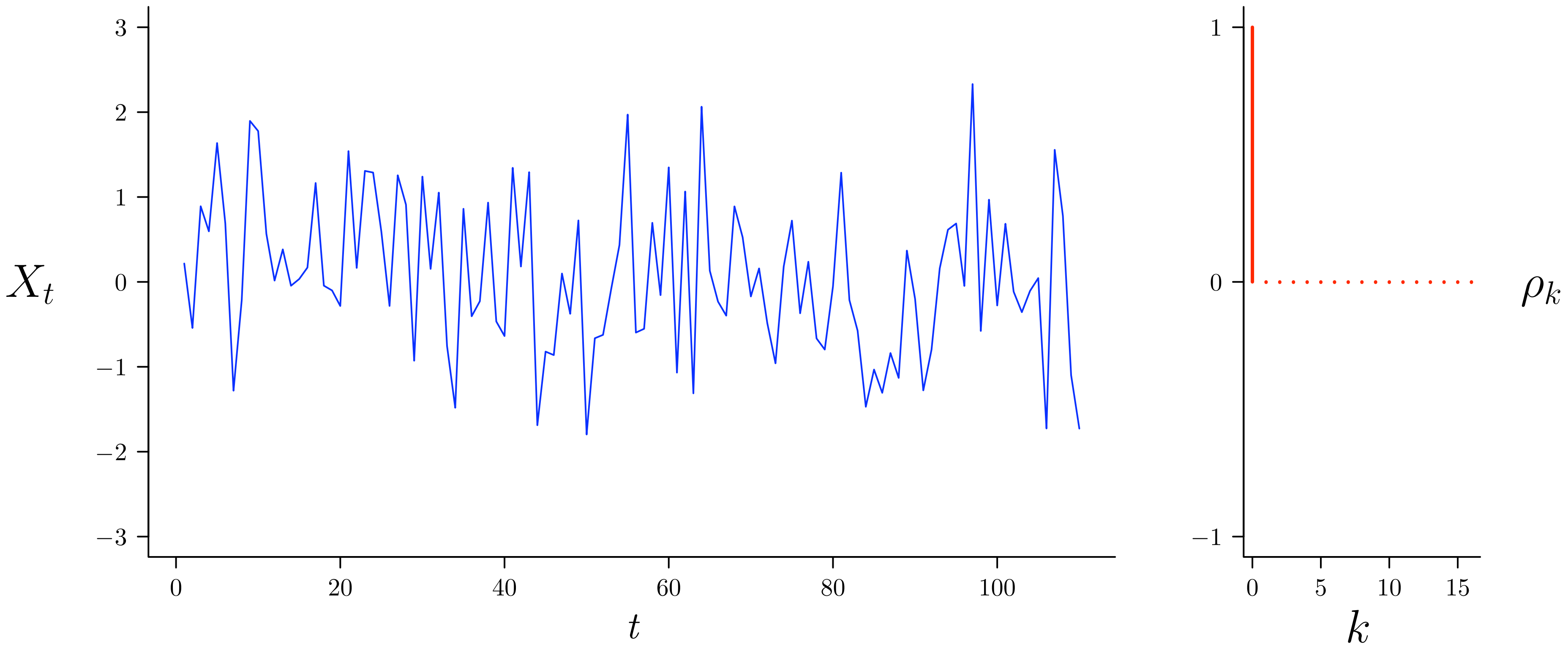

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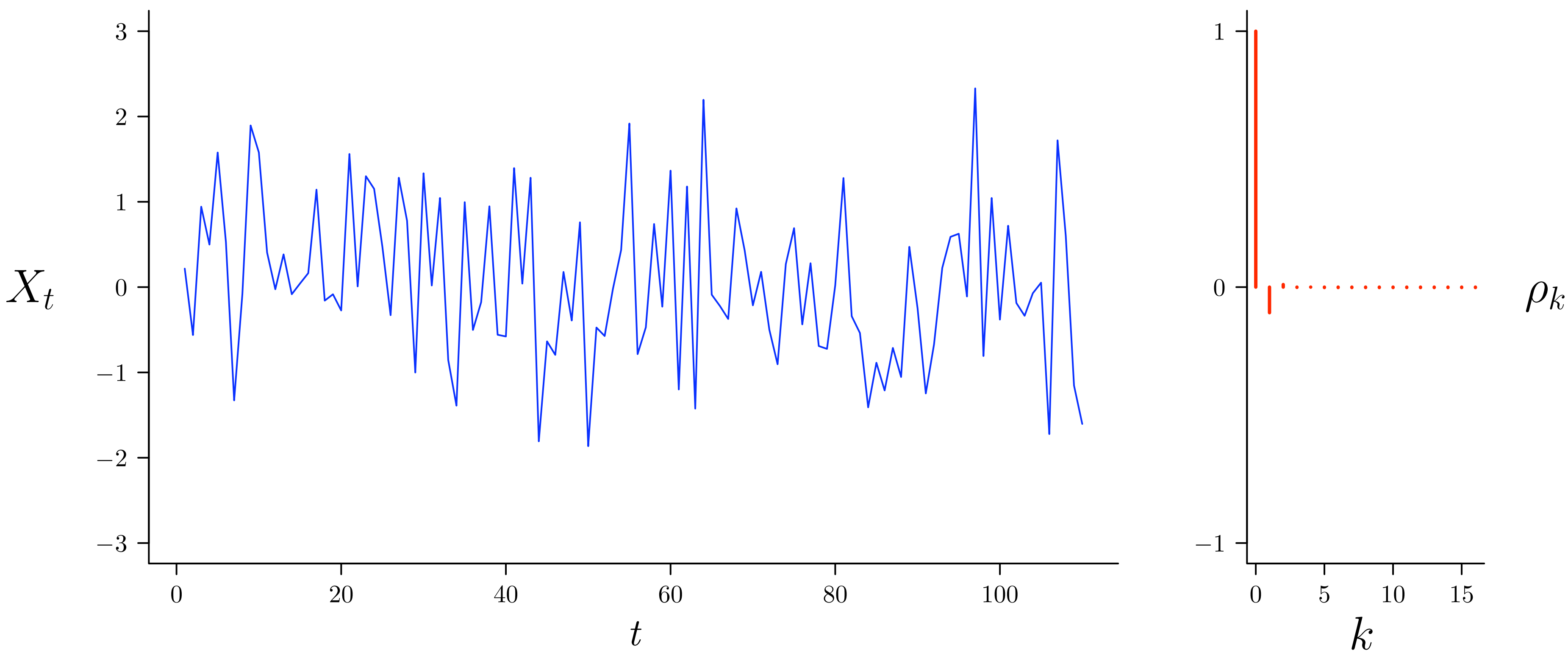

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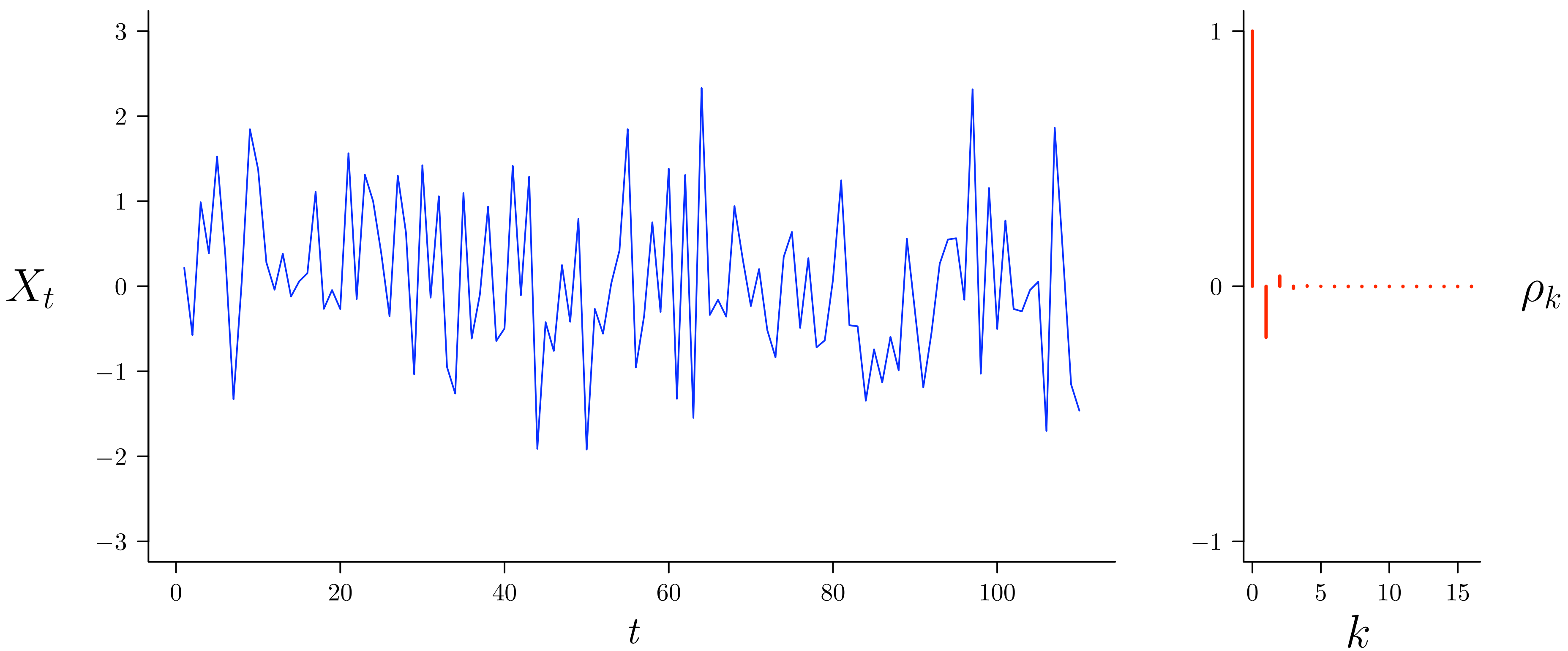

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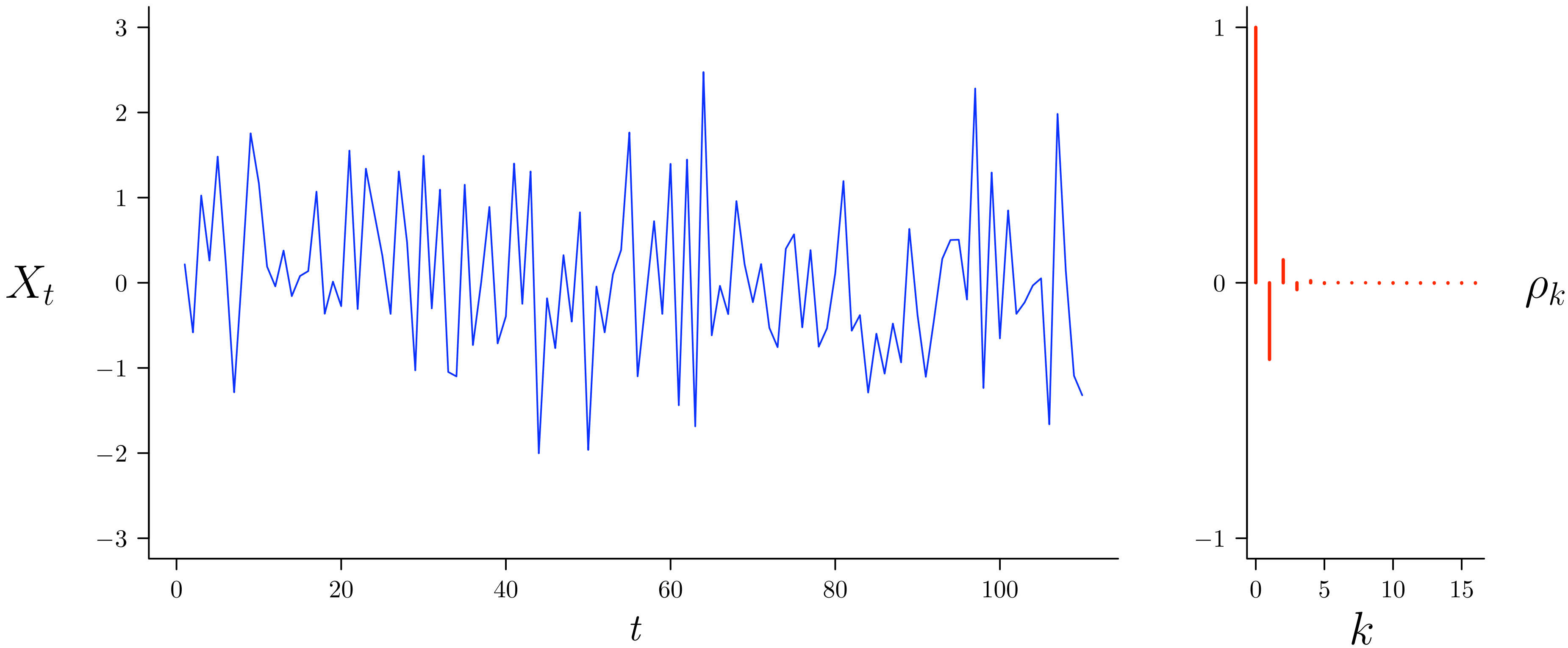

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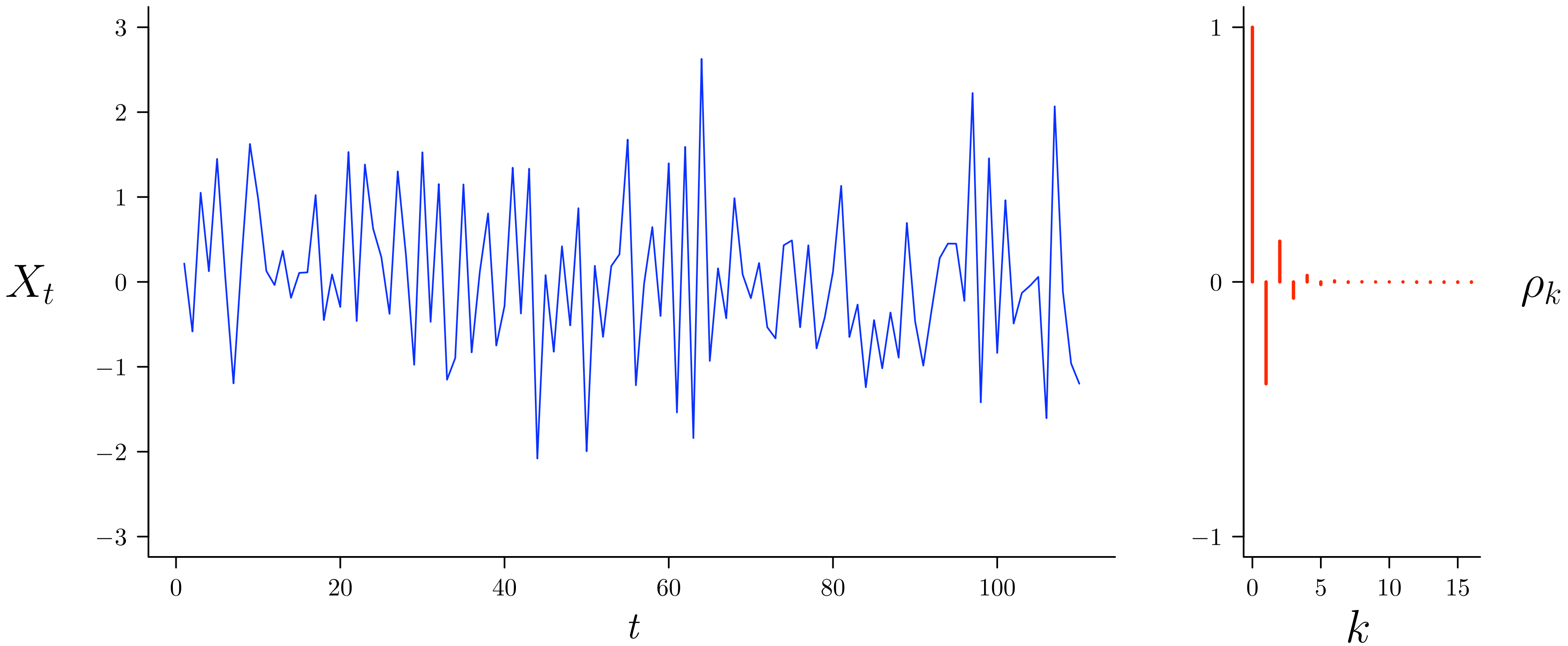

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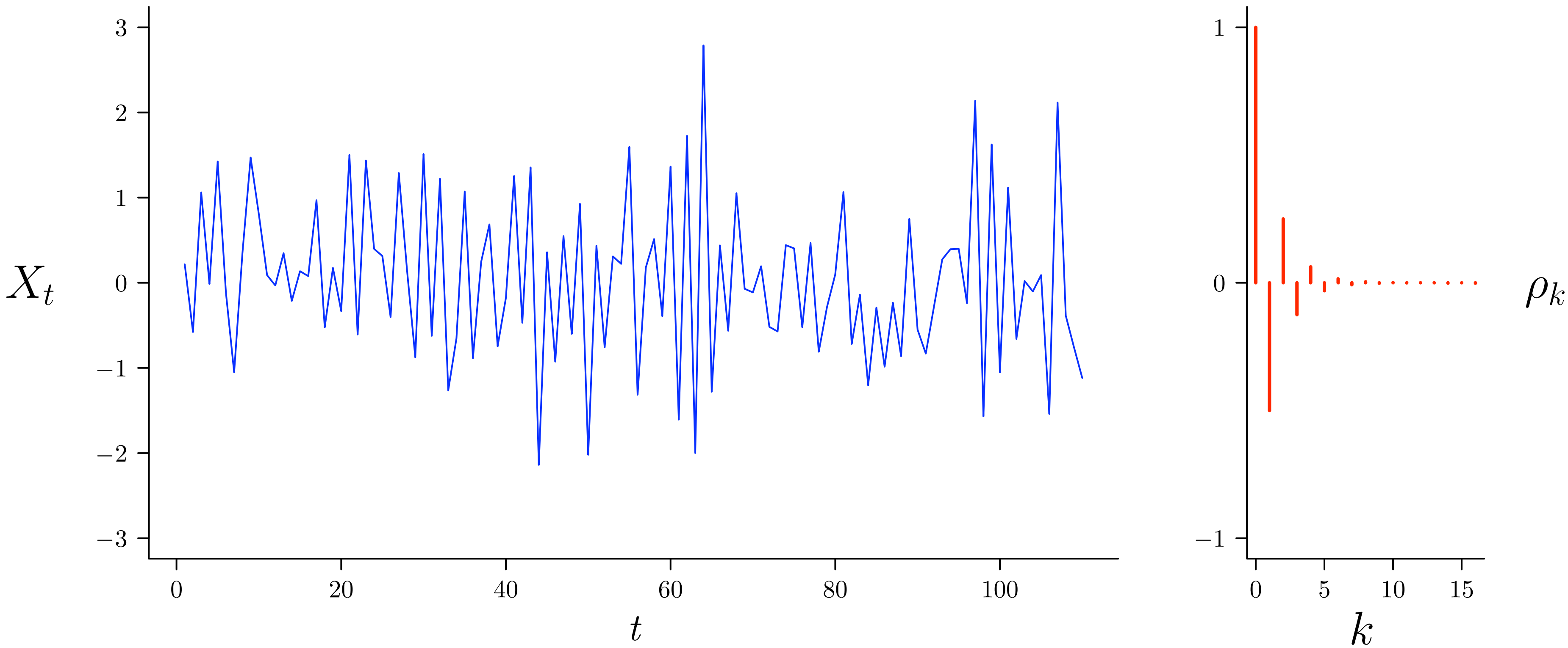

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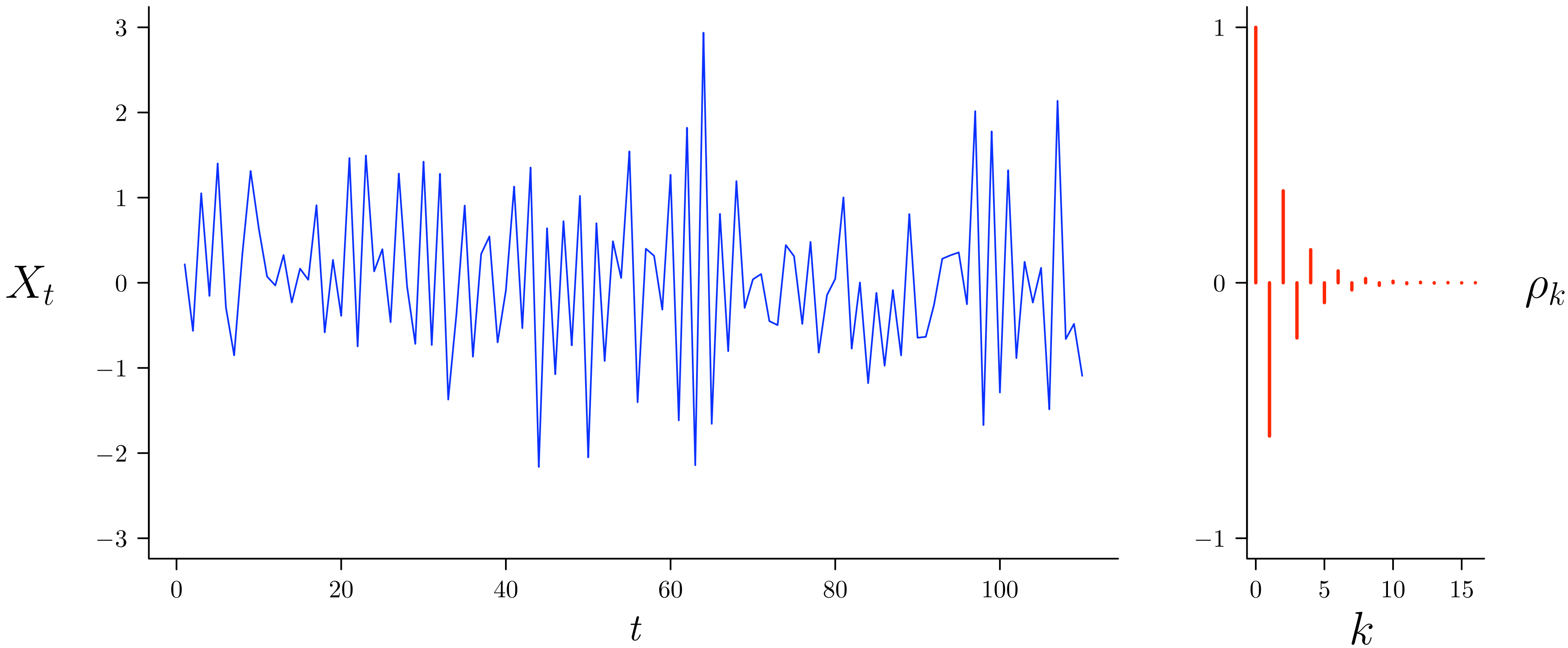

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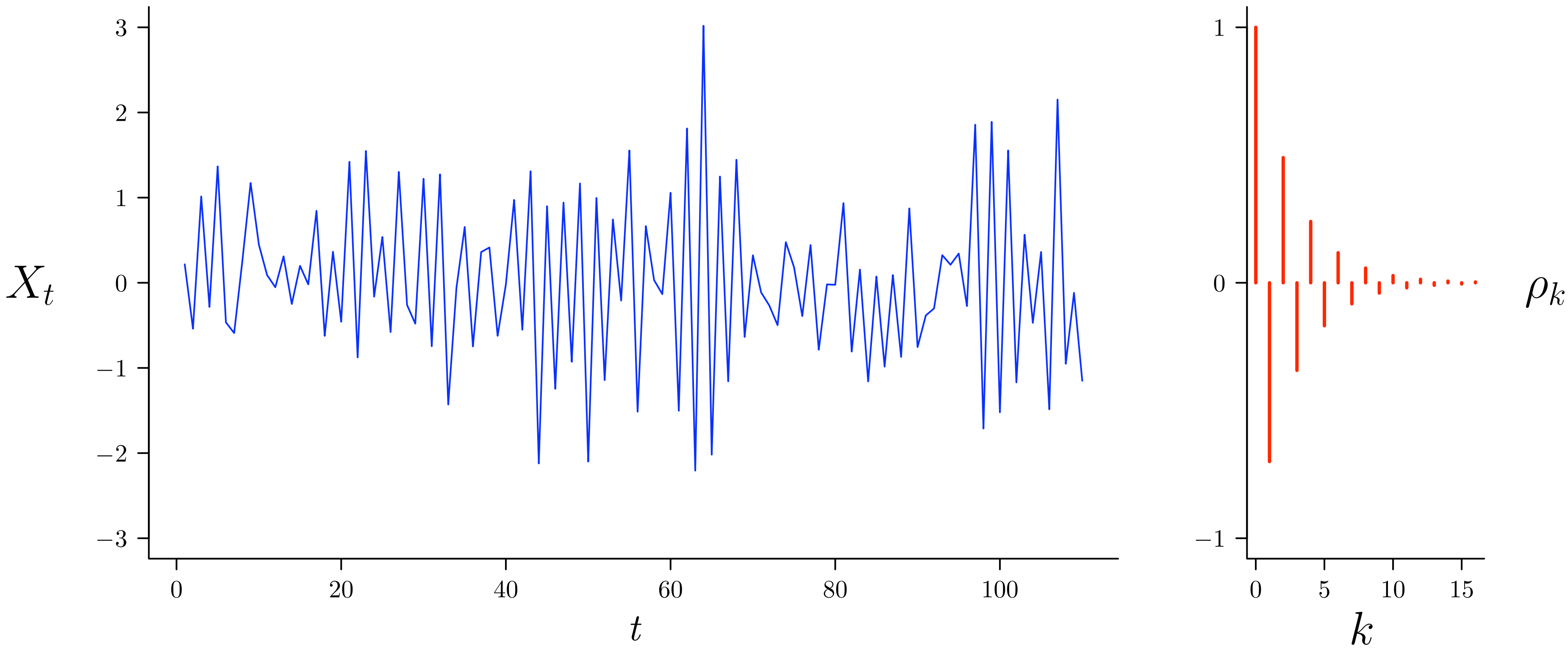

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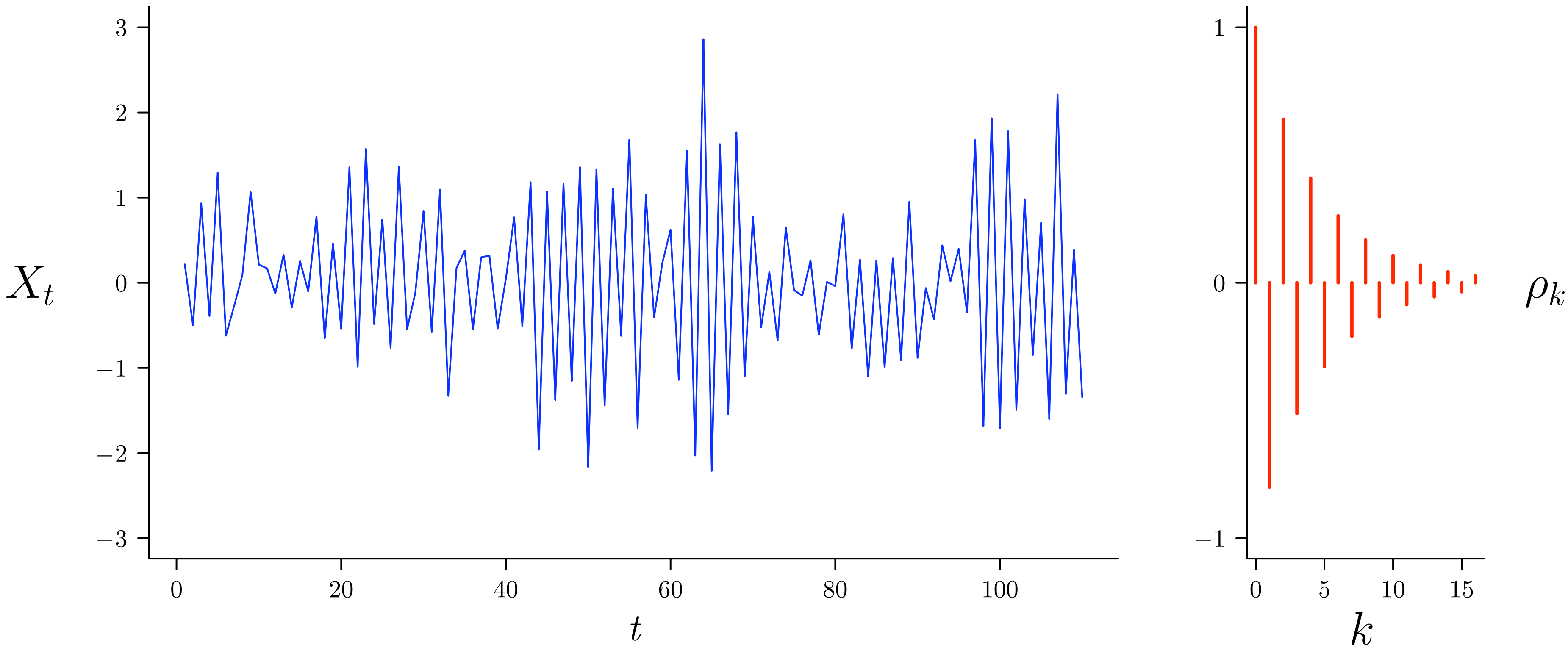

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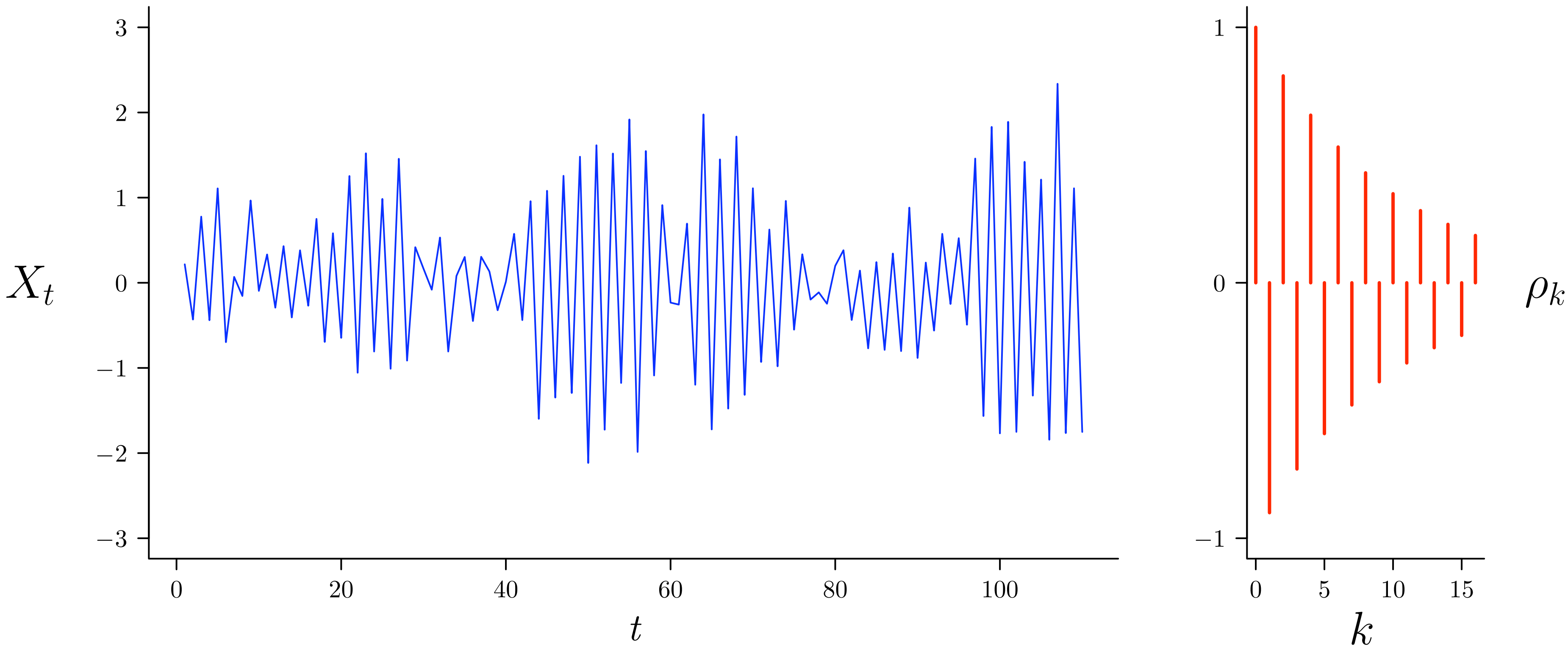

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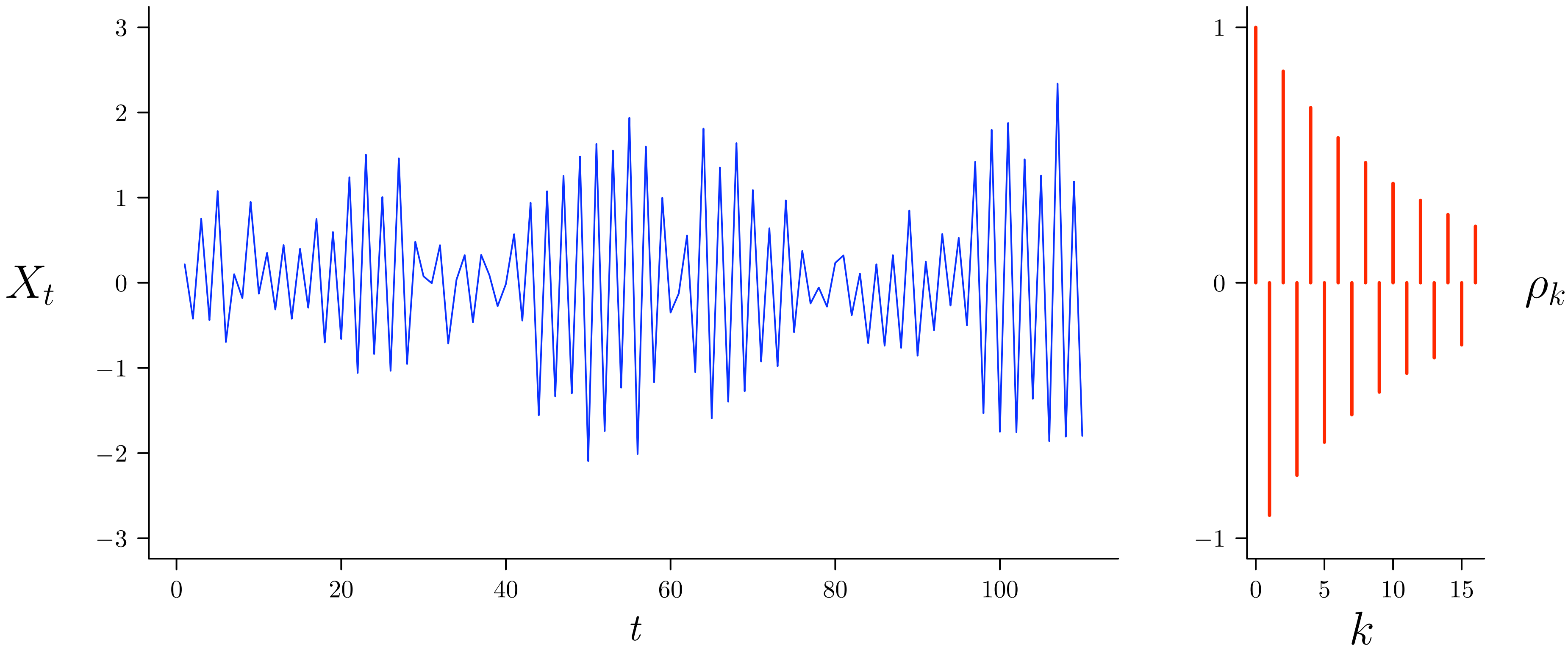

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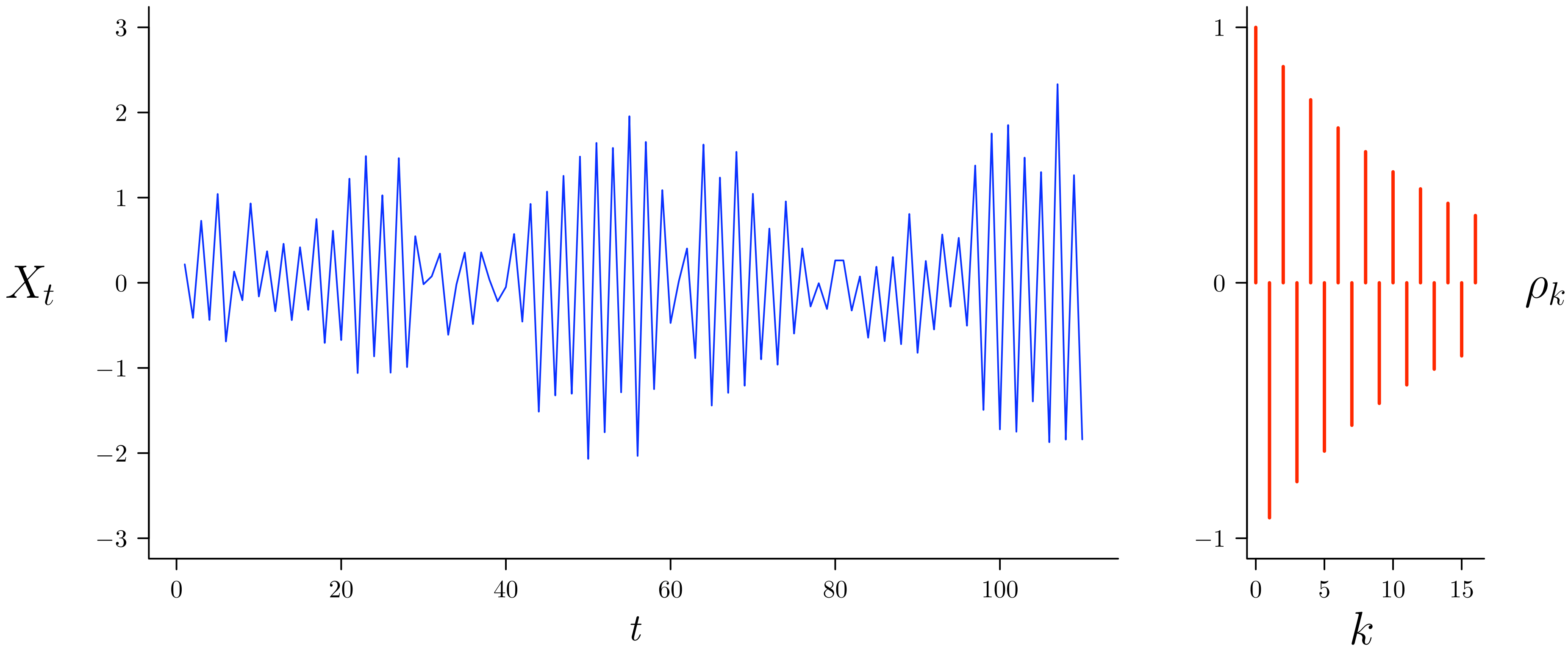

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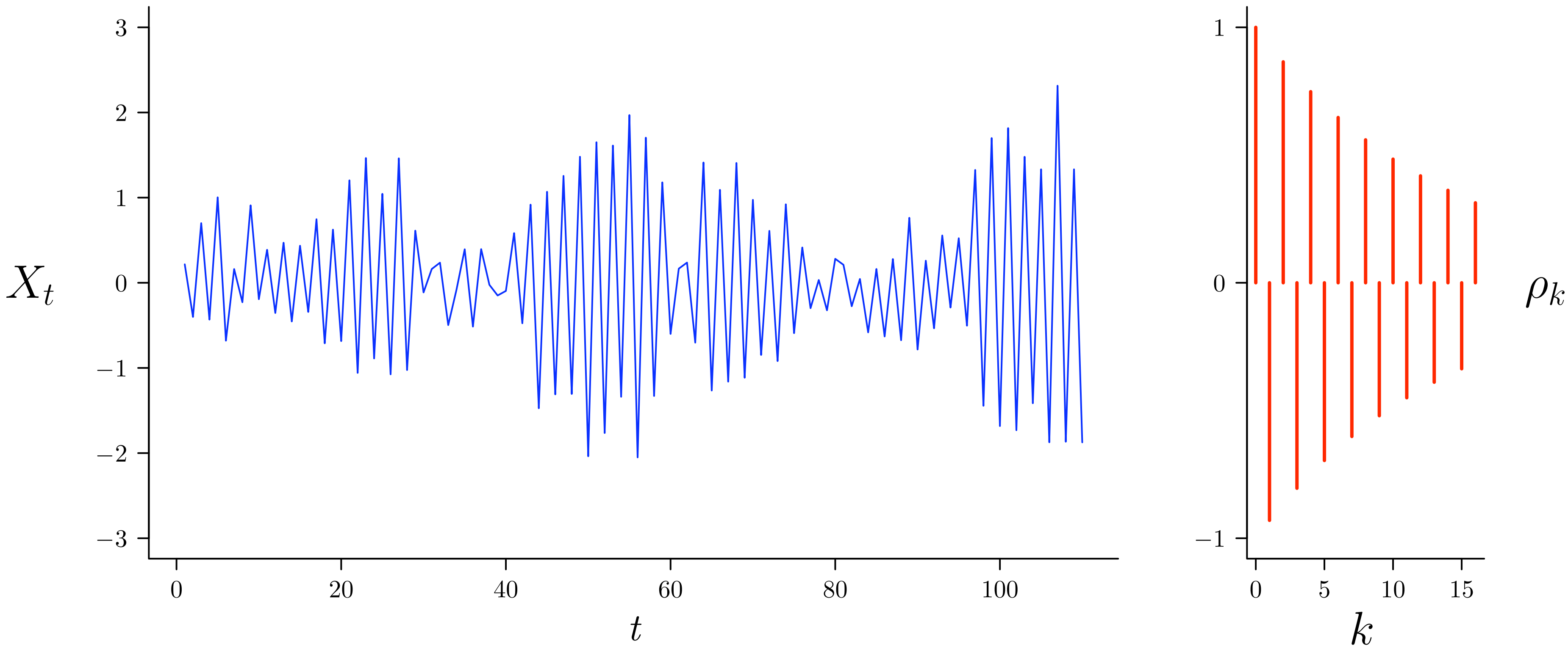

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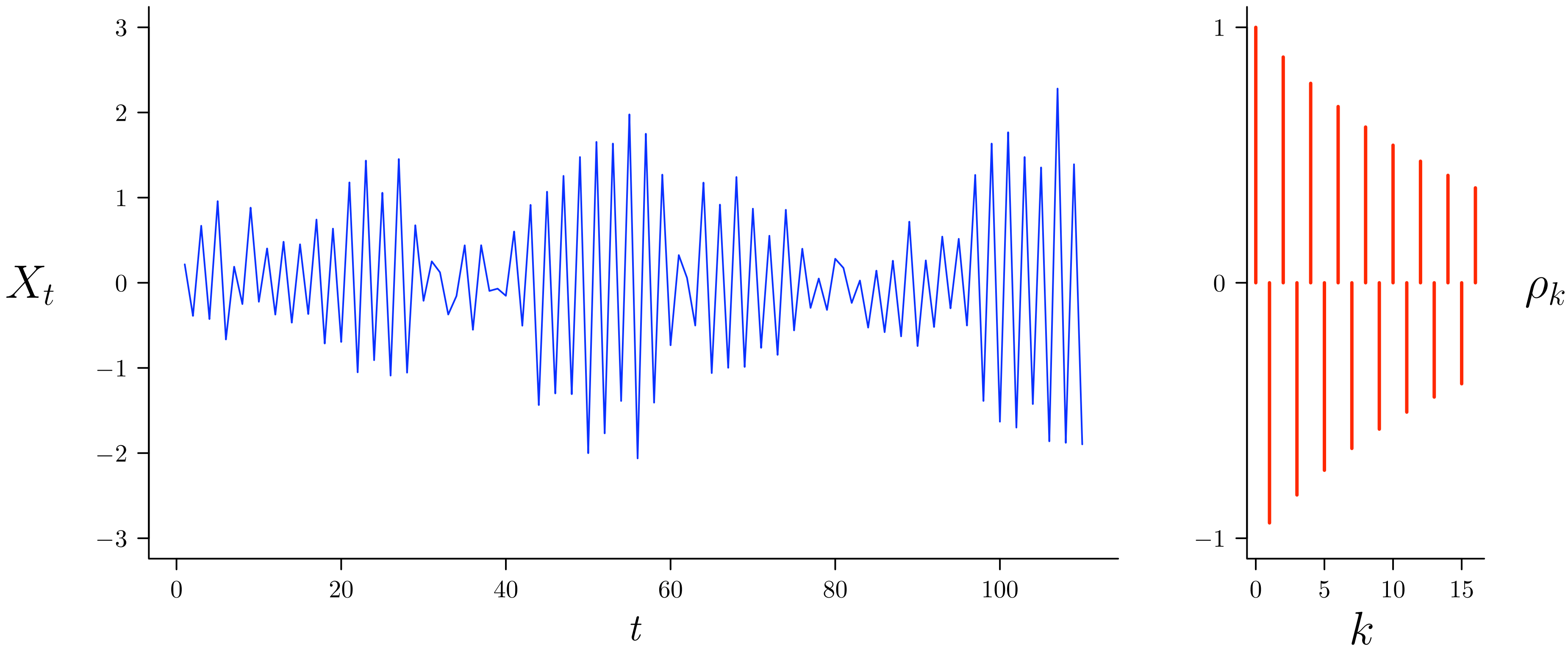

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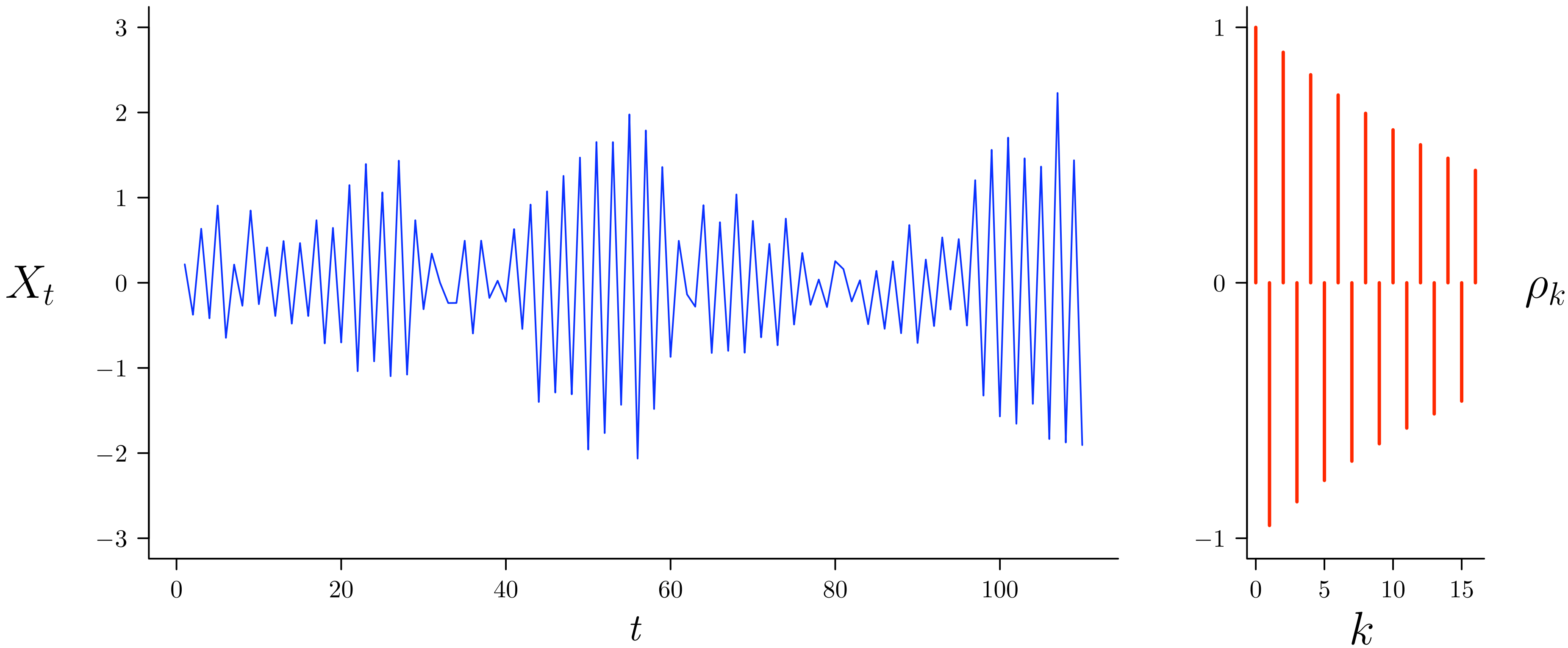

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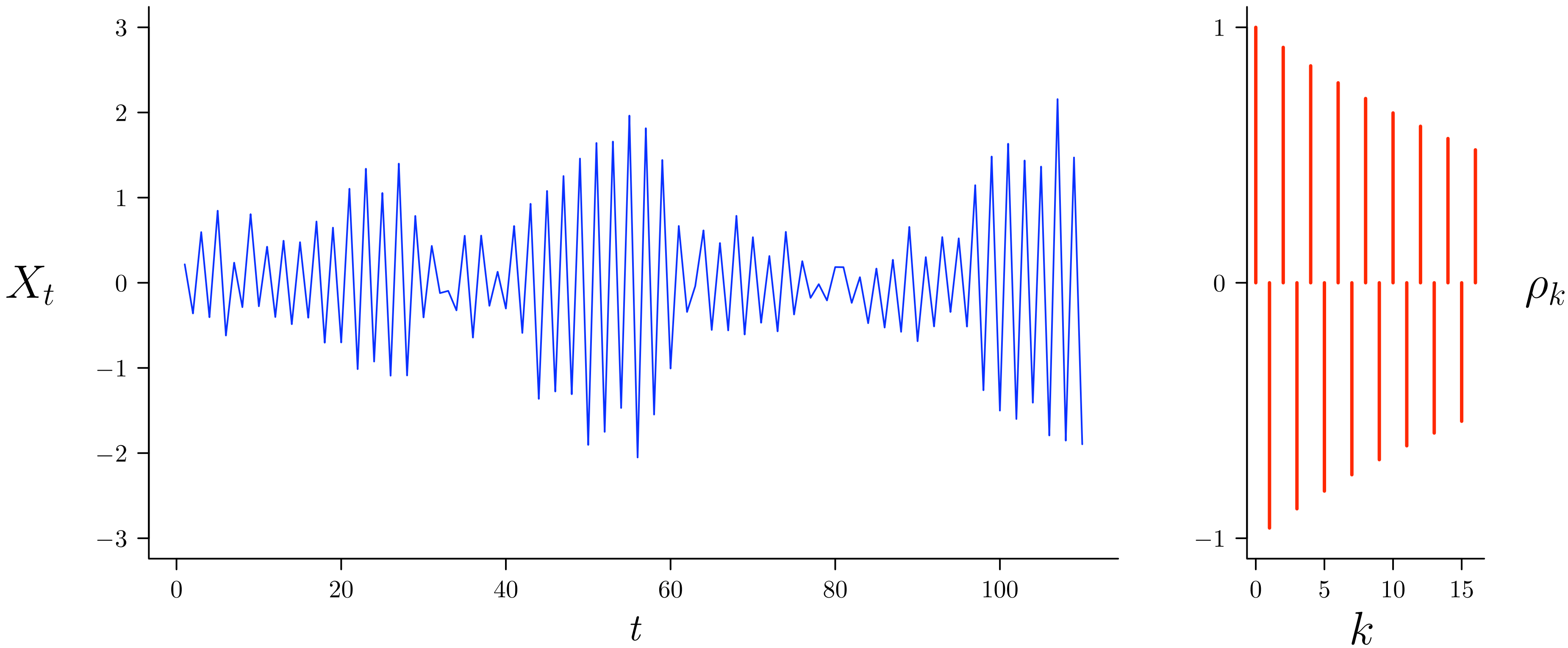

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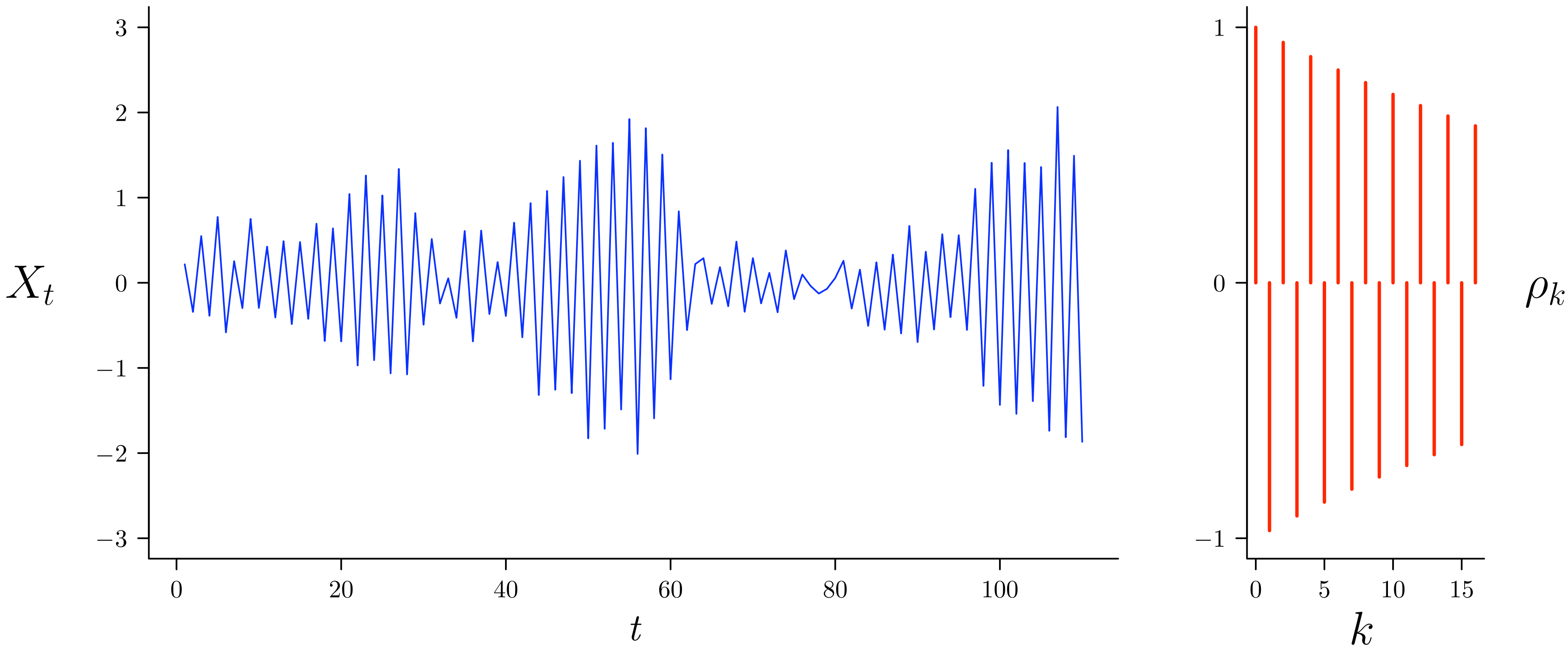

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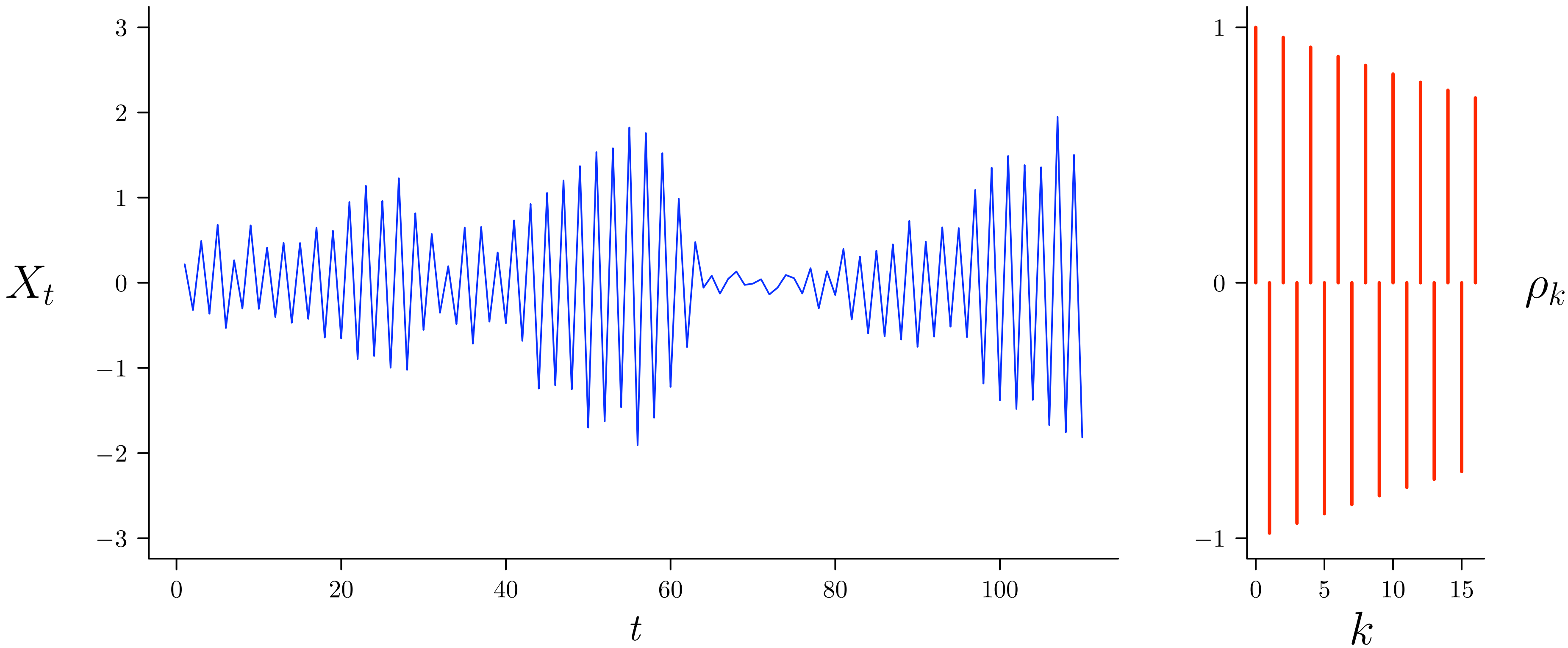

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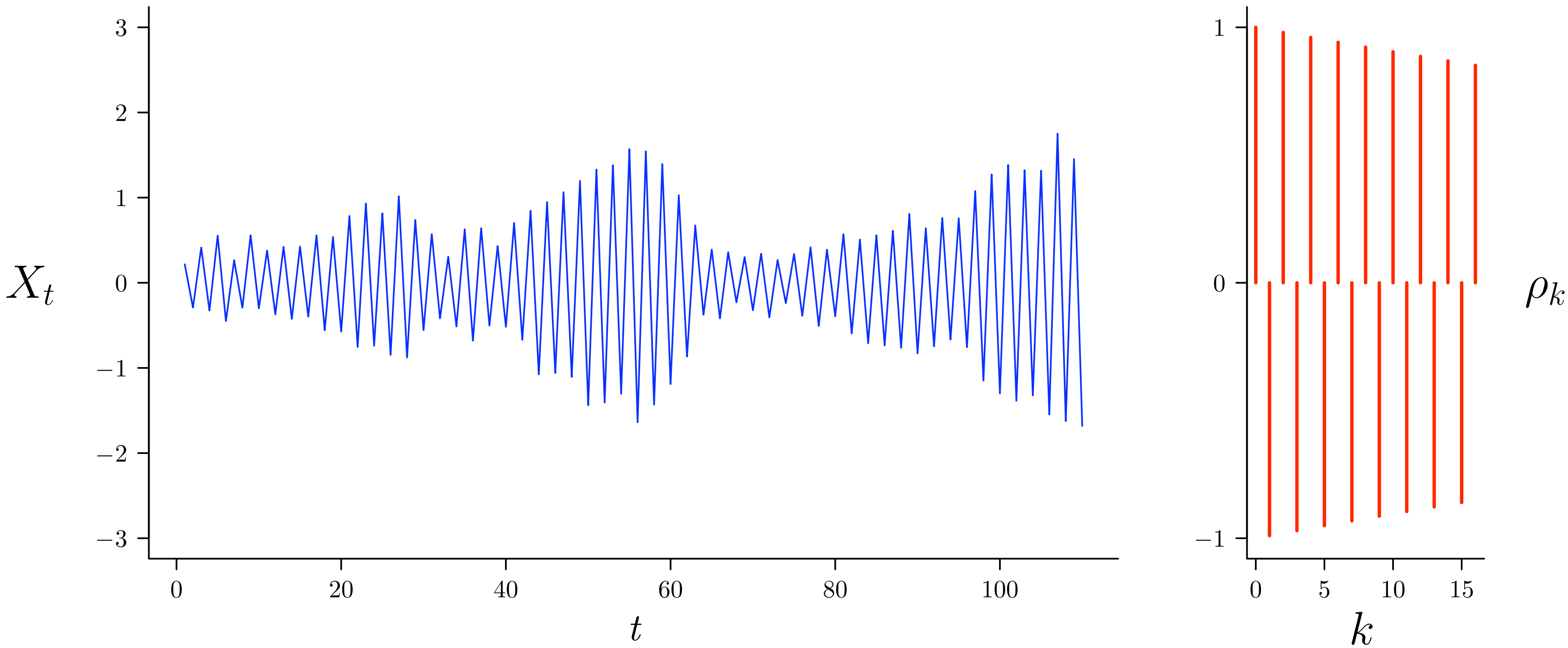


Quote from Review of Recent Paper (2010)

"... the paper ... can serve as a cautious reminder to those (probably more than 90% of climate researchers) who blindly use the AR(1) model ... for climate data"

- Q: has red noise been 'oversold' within climate community?
- points out need for statistical tests that flag time series for which red noise might be too simplistic of a model
- will now review a concept that, for Gaussian processes, provides a clear distinction between AR(1) processes and all other types of correlated stationary processes – to do so, we first need to consider 'predicting' X_t using other RVs in the process

Forward/Backward Prediction of X_t: I

- given a realization of a portion $X_{t-1}, X_{t-2}, \ldots, X_{t-k}$ of a zero mean Gaussian stationary process, suppose we want to predict X_t based upon some function, say $g(X_{t-1}, X_{t-2}, \ldots, X_{t-k})$, of these k RVs
- the 'best' predictor of X_t takes the form

$$g(X_{t-1}, X_{t-2}, \ldots, X_{t-k}) = \sum_{j=1}^{k} \phi_{k,j} X_{t-j} \equiv \overrightarrow{X}_t(k),$$

where here 'best' means that

$$E\{[X_t - g(X_{t-1}, X_{t-2}, \ldots, X_{t-k})]^2\}$$

is minimized over all possible functions of $X_{t-1}, X_{t-2}, \ldots, X_{t-k}$ (note: the $\phi_{k,j}$ depend just on the ACS for the process)

Forward/Backward Prediction of X_t: II

- given a realization of a portion $X_{t+1}, X_{t+2}, \ldots, X_{t+k}$ of RVs coming *after* X_t, the best 'predictor' of X_t is given by

$$\overleftarrow{X}_t(k) \equiv \sum_{j=1}^{k} \phi_{k,j} X_{t+j},$$

  i.e., coefficients $\phi_{k,j}$ in $\overleftarrow{X}_t(k)$ are the same as those in $\overrightarrow{X}_t(k)$

- given $\overrightarrow{X}_t(k)$ and $\overleftarrow{X}_t(k)$, form the corresponding forward and backward prediction errors:

$$\overrightarrow{\epsilon}_t(k) \equiv X_t - \overrightarrow{X}_t(k) \text{ and } \overleftarrow{\epsilon}_t(k) \equiv X_t - \overleftarrow{X}_t(k);$$

  note: define $\overrightarrow{X}_t(0) = \overleftarrow{X}_t(0) = 0$ so that $\overrightarrow{\epsilon}_t(0) = \overleftarrow{\epsilon}_t(0) = X_t$

Partial Autocorrelation Sequence (PACS): I

- focusing now on $X_{t-k}, X_{t-k+1}, \ldots, X_{t-1}, X_t$, we can interpret $\phi_{k,k}$ in the following interesting manner:

$$\phi_{k,k} = \frac{\operatorname{cov}\left\{\overrightarrow{\epsilon}_t(k-1), \overleftarrow{\epsilon}_{t-k}(k-1)\right\}}{\left(\operatorname{var}\left\{\overrightarrow{\epsilon}_t(k-1)\right\}\operatorname{var}\left\{\overleftarrow{\epsilon}_{t-k}(k-1)\right\}\right)^{1/2}};$$

because

$$\begin{aligned}\overrightarrow{\epsilon}_t(k-1) &= X_t - \overrightarrow{X}_t(k-1)\\ \overleftarrow{\epsilon}_{t-k}(k-1) &= X_{t-k} - \overleftarrow{X}_{t-k}(k-1),\end{aligned}$$

and because both $\overrightarrow{X}_t(k-1)$ and $\overleftarrow{X}_t(k-1)$ depend on just

$$X_{t-k+1}, \ldots, X_{t-1},$$

can regard $\phi_{k,k}$ as correlation between X_{t-k} and X_t after 'adjustment' by the intervening $k-1$ RVs $X_{t-k+1}, \ldots, X_{t-1}$

Partial Autocorrelation Sequence (PACS): II

- $\phi_{k,k}$, $k = 1, 2, \ldots$, is known as the partial ACS (PACS)
- Ramsey (1974): under a Gaussian assumption, a stationary process is an AR(1) process if and only if its PACS is identically zero for all $k \geq 2$
- can thus use estimators of $\hat{\phi}_{2,2}$, $\hat{\phi}_{3,3}$, ... to test null hypothesis

  H_0: time series is a realization of an AR(1) process

  versus nonspecific alternative

  H_1: time series is a realization of another stationary process

Partial Autocorrelation Sequence (PACS): III

- given time series that is a realization of $X_0, X_1, \ldots, X_{N-1}$, can estimate $\phi_{k,k}$ by fitting kth order AR process, i.e.,

$$X_t = \sum_{j=1}^{k} \phi_{k,j} X_{t-j} + \epsilon_t,$$

  using a variety of methods (Yule–Walker, Burg, forward least squares (LS), forward/backward LS, maximum likelihood, ...)

- AR(k) coefficients $\{\phi_{k,j} : j = 1, \ldots, k\}$ and PACS $\{\phi_{j,j} : j = 1, \ldots, k\}$ are equivalent to one another
- large-sample theory says PACS estimators $\hat{\phi}_{2,2}$, $\hat{\phi}_{3,3}$, ..., $\hat{\phi}_{K,K}$ are approximately IID normal with mean zero and variance $1/N$ for an AR(1) process and fixed K (Kay & Makhoul, 1983)

Portmanteau Tests for White Noise: I

- can formulate tests for red noise analogous to portmanteau tests for white noise (Box and Pierce, 1970; Ljung and Box, 1978)
- large-sample statistical theory says that, for a time series coming from a white noise process and for fixed K, standard ACS estimators $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_K$ are approximately IID normal with mean zero and variance $1/N$
- Box–Pierce and Ljung–Box–Pierce portmanteau test statistics for white noise versus nonspecific alternative are given by

$$Q_K = N \sum_{k=1}^{K} \hat{\rho}_k^2 \text{ and } \widetilde{Q}_K = N(N+2) \sum_{k=1}^{K} \frac{\hat{\rho}_k^2}{N-k}$$

Portmanteau Tests for White Noise: II

- for either Q_K or $\widetilde{Q}_K$, reject null hypothesis of white noise at significance level α when statistic exceeeds $(1 - \alpha) \times 100\%$ percentage point for chi-square distribution with K degrees of freedom
- literature recommends setting $K = \max\{2, \min\{20, N/10\}\}$
- Baragona and Battaglia (2000) and Kwan (2003) consider analogous tests for white noise based on PACS estimators rather than ACS estimators (for a white noise process and for fixed K, PACS estimators $\hat{\phi}_{1,1}, \hat{\phi}_{2,2}, \ldots, \hat{\phi}_{K,K}$ are asymptotically IID normal with mean zero and variance $1/N$, i.e., the same result as for $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_K$)

Omnibus Portmanteau-based Tests for Red Noise

- since large sample properties of
  - $\hat{\rho}_k$ for $k \geq 1$ under white noise hypothesis and
  - $\hat{\phi}_{k,k}$ for $k \geq 2$ under red noise hypothesis

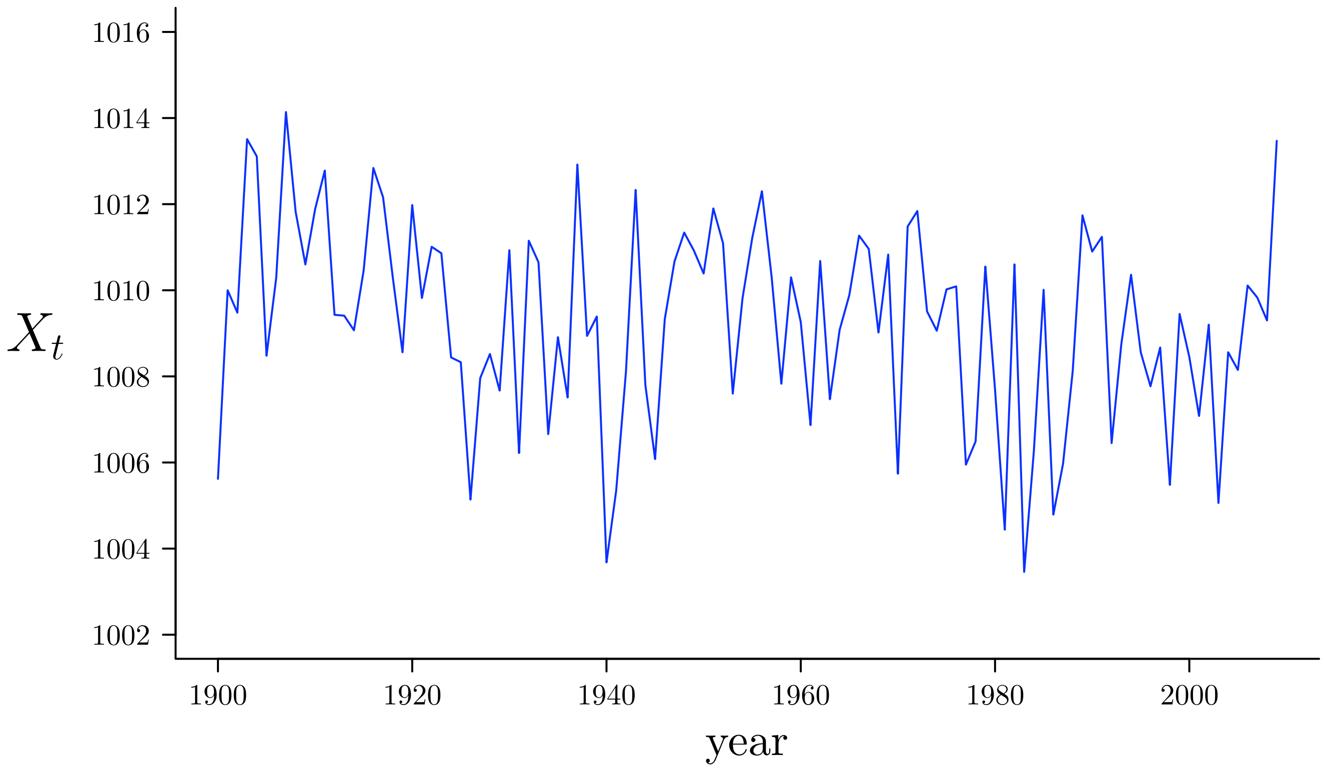
  are identical, can test hypothesis of red noise versus nonspecific alternative using the following analogs of Q_K or $\widetilde{Q}_K$:

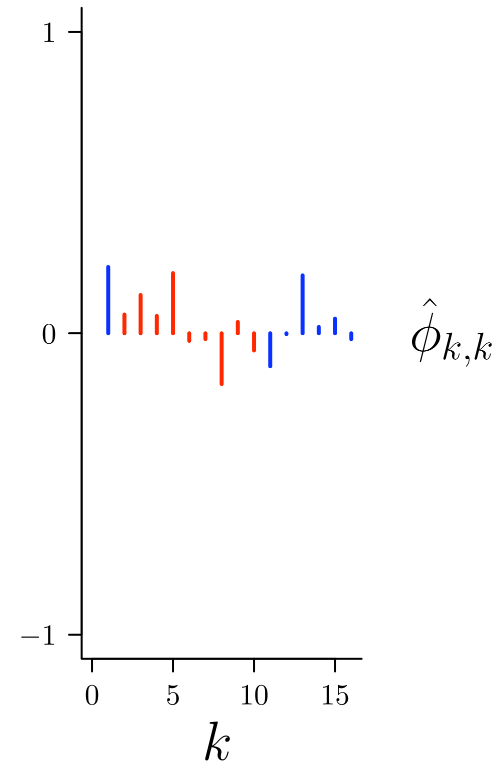
$$T_K = N \sum_{k=2}^{K} \hat{\phi}_{k,k}^2 \text{ and } \widetilde{T}_K = N(N+2) \sum_{k=2}^{K} \frac{\hat{\phi}_{k,k}^2}{N-k}$$

- for either test statistic, reject null hypothesis of red noise at significance level α when statistic exceeeds $(1-\alpha) \times 100\%$ percentage point for chi-square distribution with $K-1$ degrees of freedom

Use of Tests on NPI Time Series

- for NPI series with $K = 10$, get $T_{10} \doteq 10.69$ and $\widetilde{T}_{10} \doteq 11.47$, yielding $\hat{\alpha} = 0.30$ and 0.24, so cannot reject red noise hypothesis at any reasonable significance level

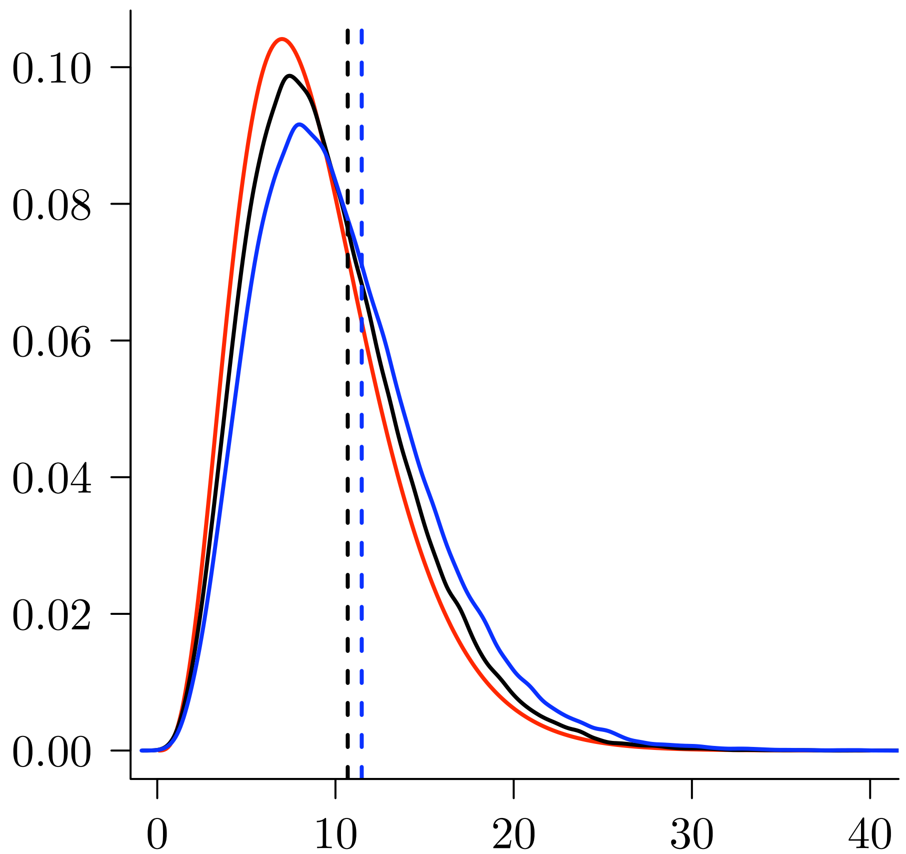




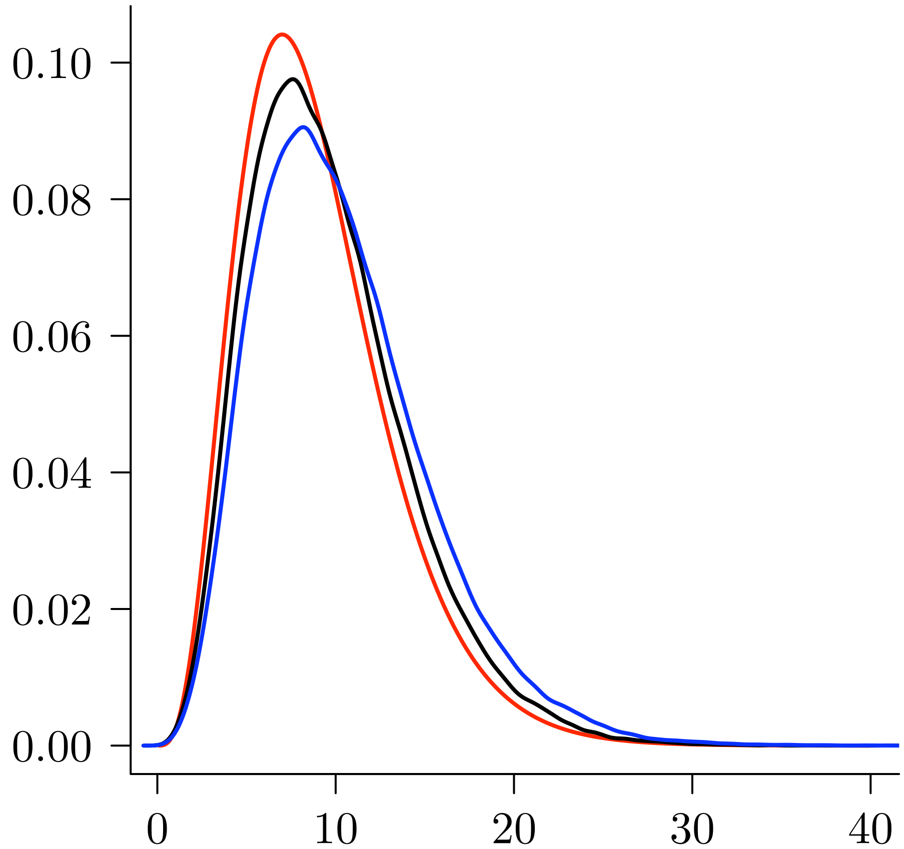
Assessing χ^2 Approximation

- generated $100,000$ Gaussian AR(1) time series of length $N = 110$ using ϕ estimated from NPI data and computed T_{10} and $\widetilde{T}_{10}$ for each series
- used these as input to `density` function in `R` to estimate probability density functions (PDFs) for comparison with χ^2_9 PDF
- repeated above, but with $\phi = 0.9$, to see if results depend on ϕ

PDFs for T_{10}, $\widetilde{T}_{10}$ and χ^2_9 with $\phi \doteq 0.21$

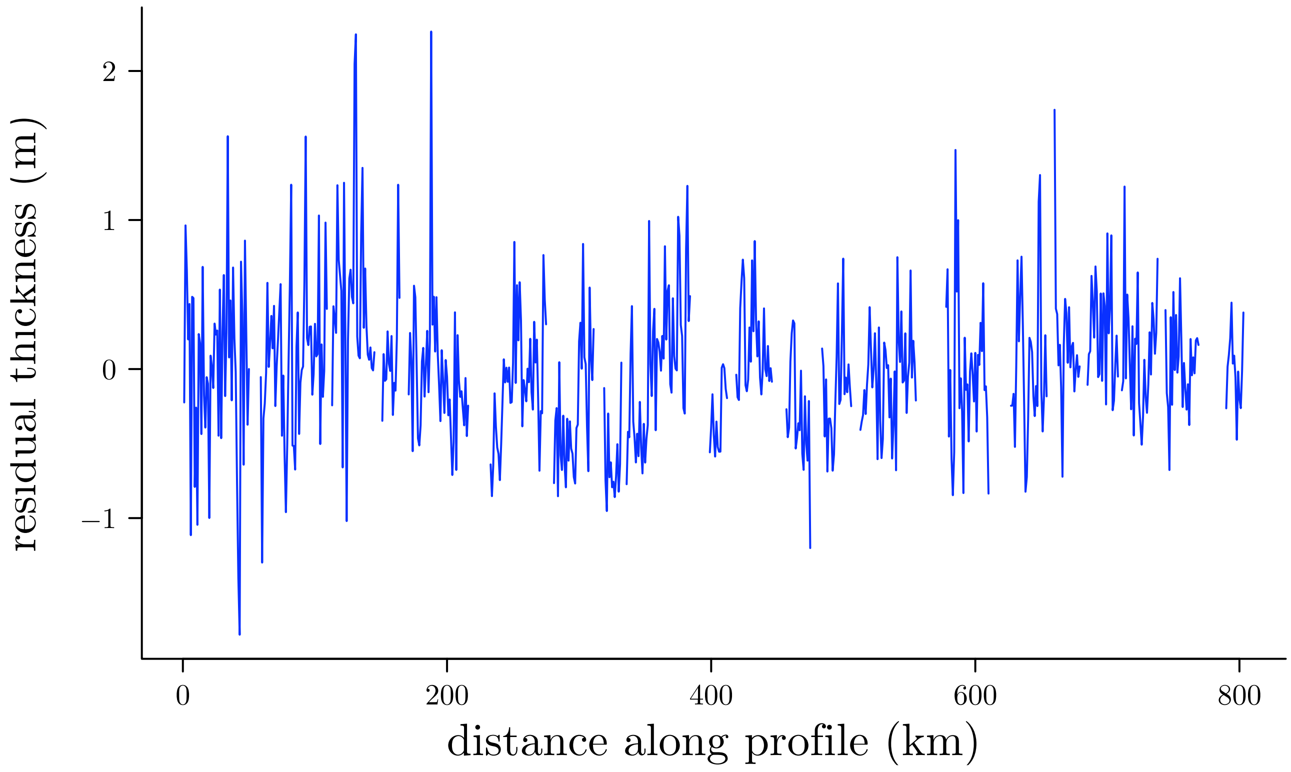


PDFs for T_{10}, $\widetilde{T}_{10}$ and χ^2_9 with $\phi = 0.9$



Extension to Handle 'Gappy' Time Series: I

- climatology time series often have missing values, as is true for a 'time' series of Arctic sea-ice thickness measurements (172 out of 803 observations missing – 21% of data)

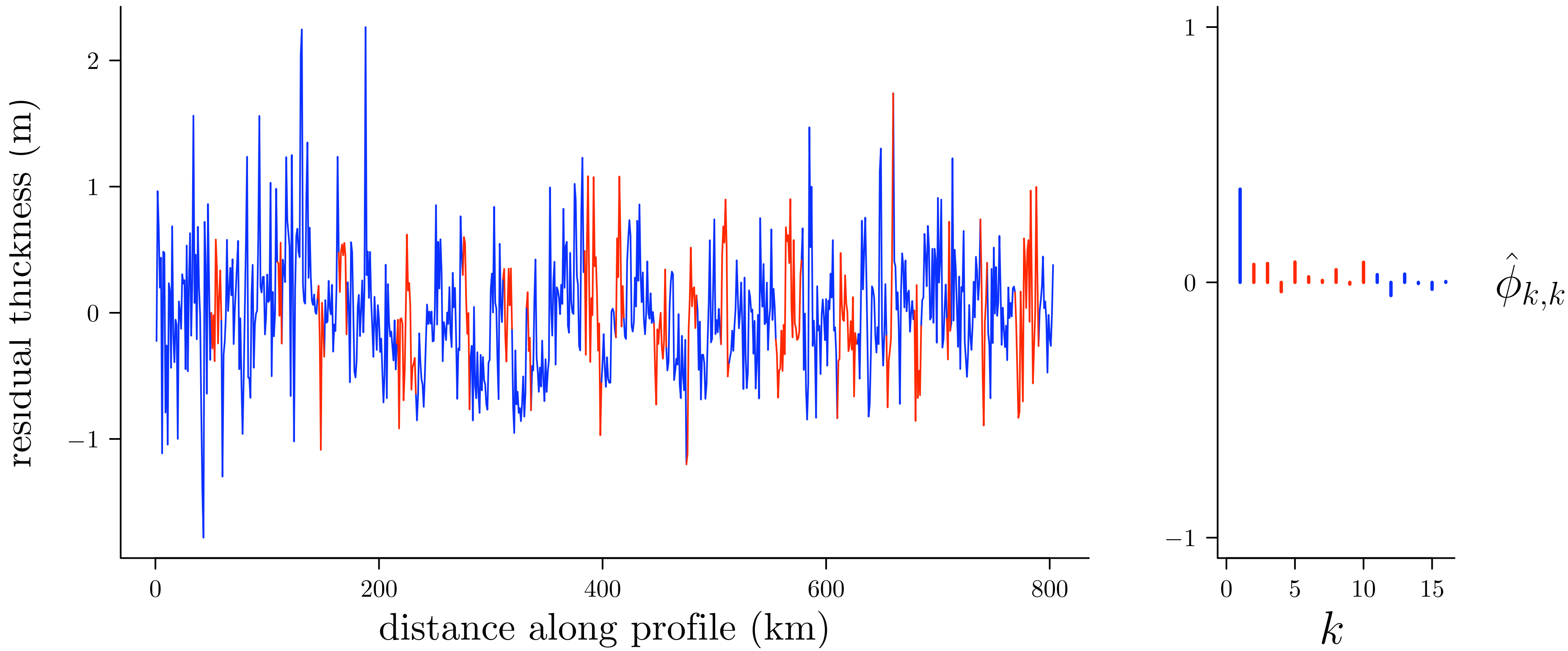


Extension to Handle 'Gappy' Time Series: II

- assuming null hypothesis of red noise to be true, can estimate model parameters for gappy time series using maximum likelihood (Jones, 1980), yielding $\hat{\phi} \doteq 0.36$ (± 0.04)
- let $\mathbf{X}_O$ and $\mathbf{X}_M$ be vectors of RVs containing observed and missing parts of time series
- using $E\{\mathbf{X}_M \,|\, \mathbf{X}_O\}$ and $\text{var}\,\{\mathbf{X}_M \,|\, \mathbf{X}_O\} = \Sigma_{M\,|\,O}$ formed under null hypothesis and conditioned on AR(1) parameter estimates, can generate realizations of missing data
  - formally requires Cholesky factorization of $\Sigma_{M\,|\,O}$
  - due to special properties of AR(1) model, can reduce to factorization of a set of much smaller matrices
- can compute T_K and $\widetilde{T}_K$ for many such realizations

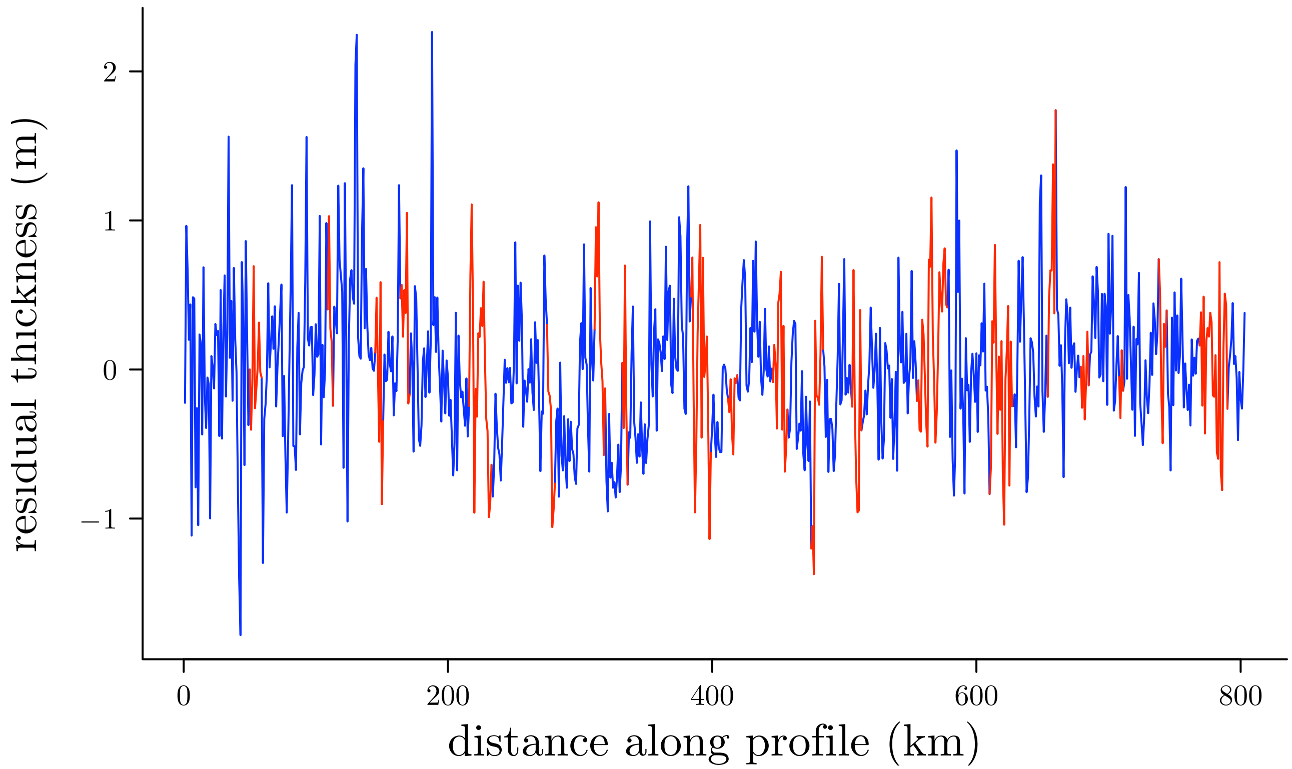
Extension to Handle 'Gappy' Time Series: III

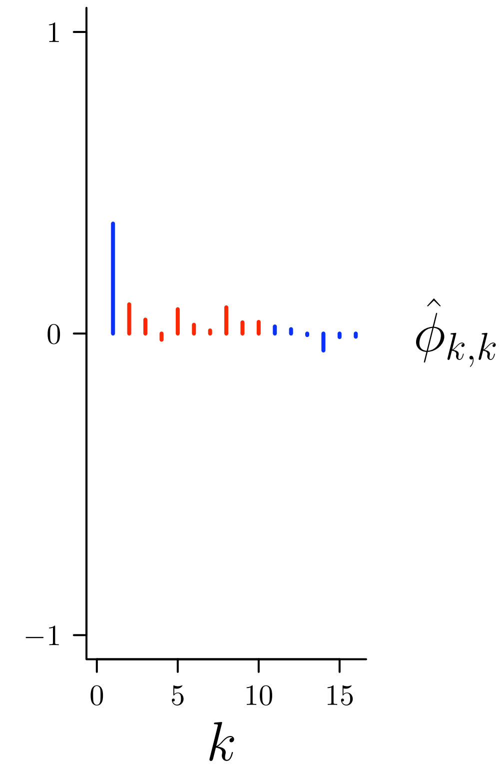
- get $T_{10} \doteq 21.97$ and $\widetilde{T}_{10} \doteq 22.18$ for this particular stochastic interpolation of Arctic sea-ice series, yielding $\hat{\alpha} \doteq 0.009$ and 0.008



Extension to Handle 'Gappy' Time Series: III

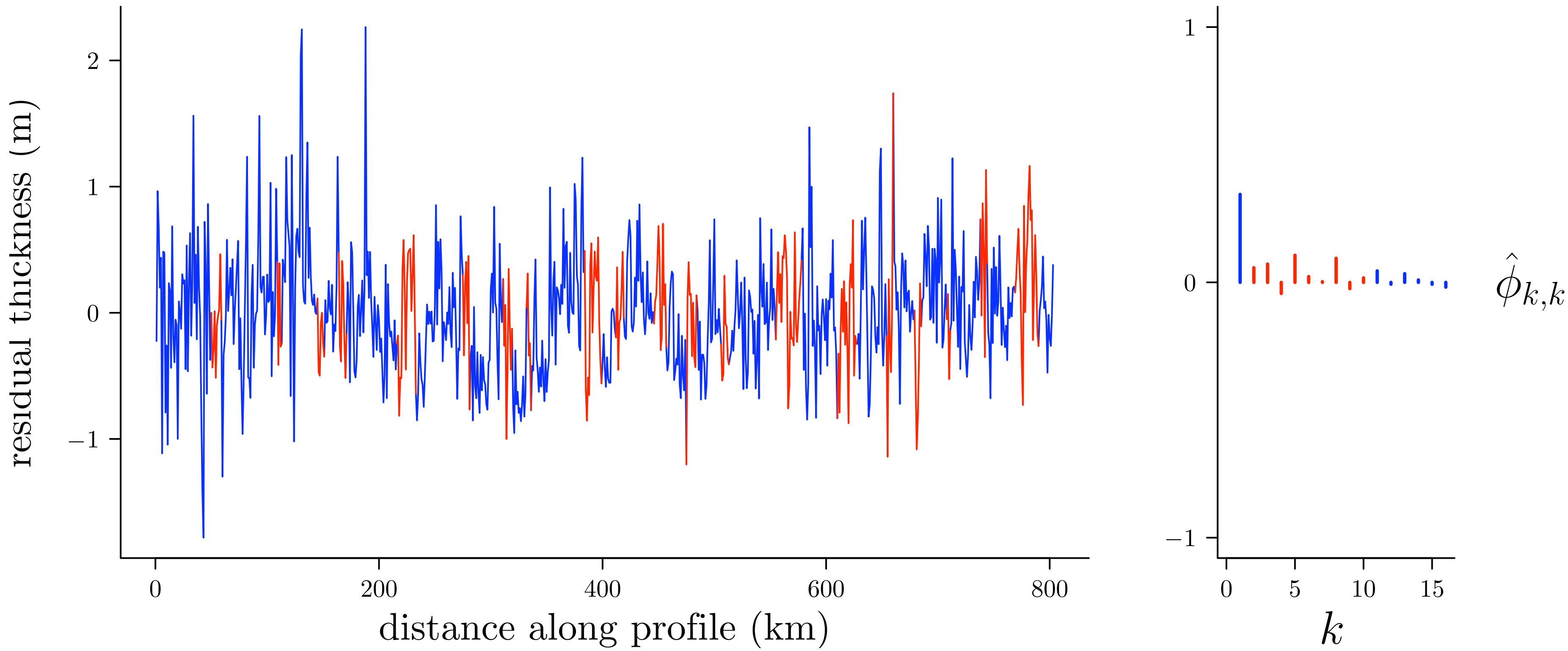
- get $T_{10} \doteq 23.99$ and $\widetilde{T}_{10} \doteq 24.20$ for this particular stochastic interpolation of Arctic sea-ice series, yielding $\hat{\alpha} \doteq 0.004$ and 0.004





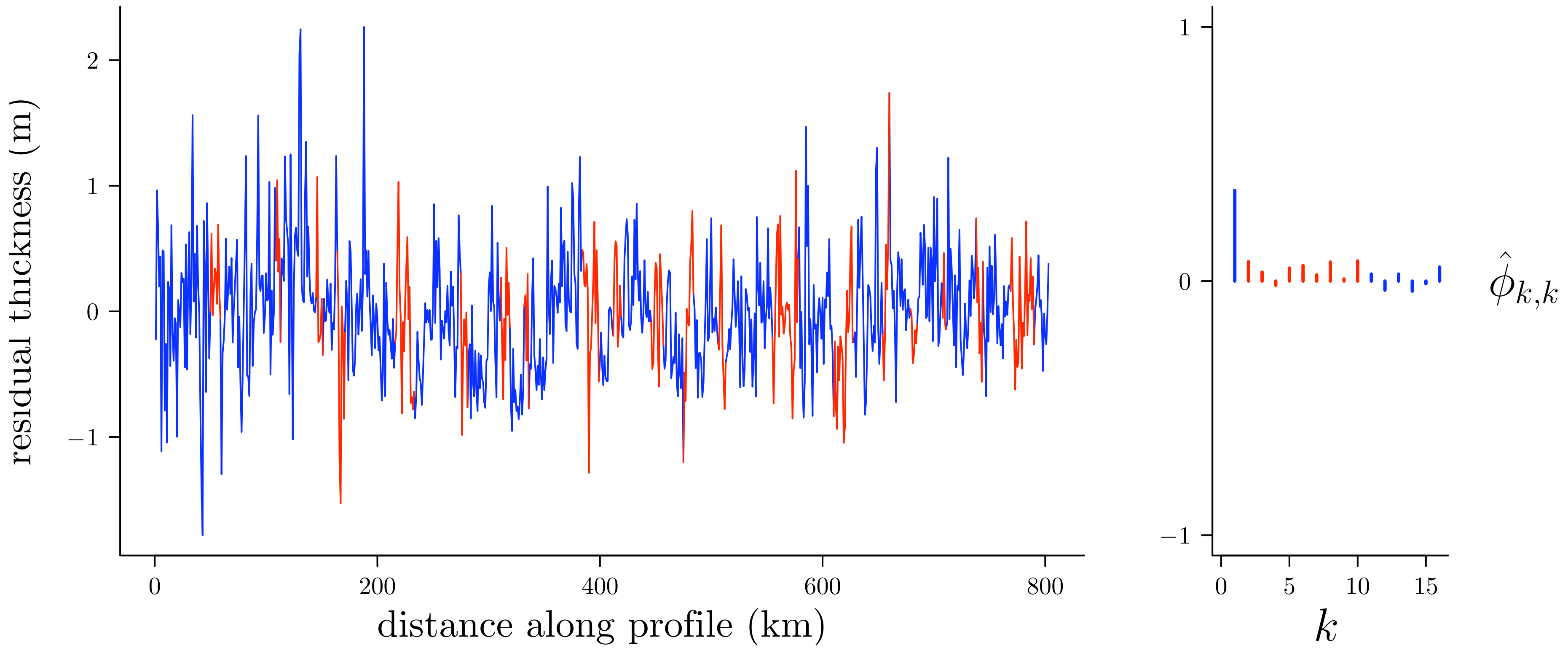
Extension to Handle 'Gappy' Time Series: III

- get $T_{10} \doteq 25.68$ and $\widetilde{T}_{10} \doteq 25.91$ for this particular stochastic interpolation of Arctic sea-ice series, yielding $\hat{\alpha} \doteq 0.002$ and 0.002



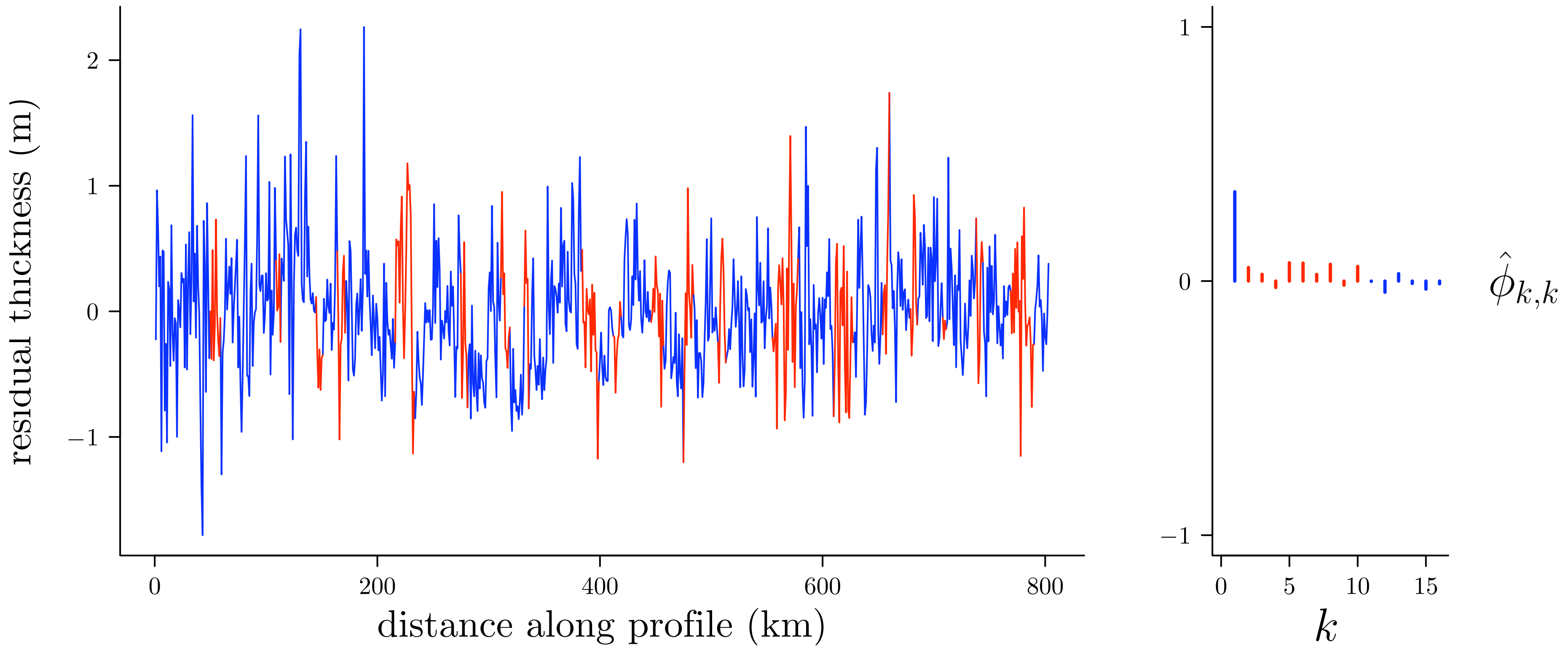
Extension to Handle 'Gappy' Time Series: III

- get $T_{10} \doteq 20.94$ and $\widetilde{T}_{10} \doteq 21.16$ for this particular stochastic interpolation of Arctic sea-ice series, yielding $\hat{\alpha} \doteq 0.013$ and 0.012



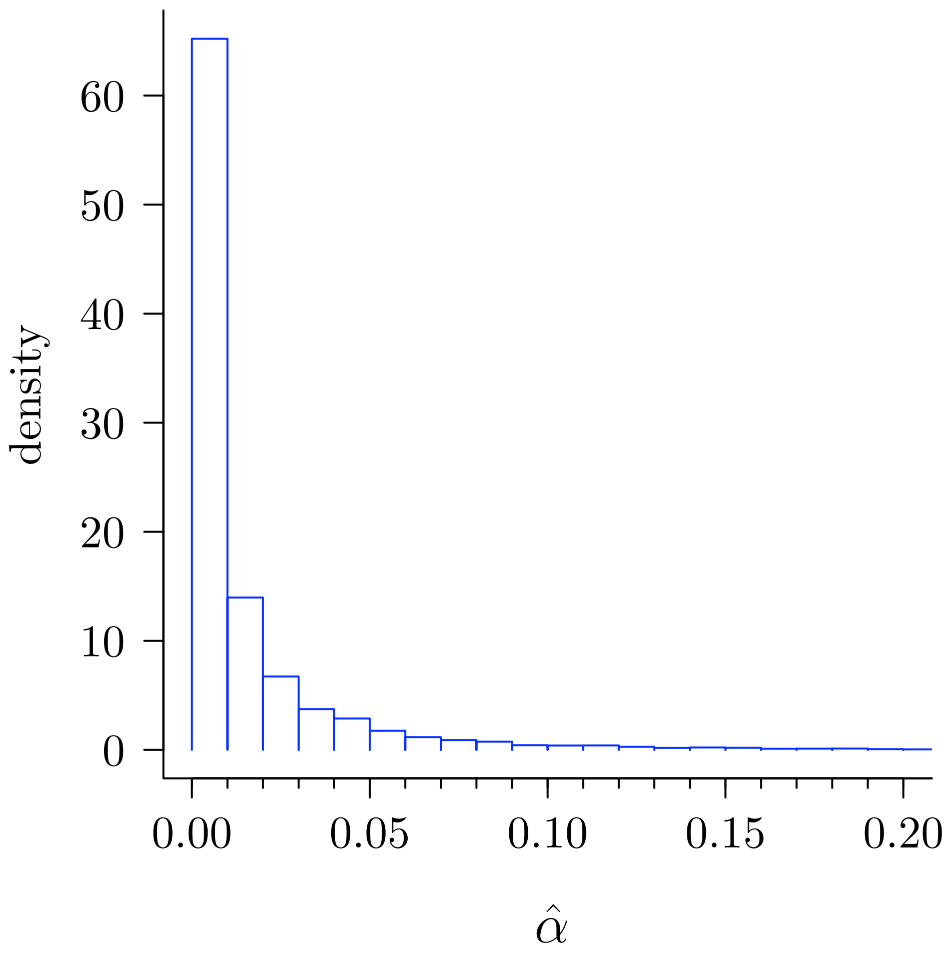
Extension to Handle 'Gappy' Time Series: III

- get $T_{10} \doteq 18.44$ and $\widetilde{T}_{10} \doteq 18.63$ for this particular stochastic interpolation of Arctic sea-ice series, yielding $\hat{\alpha} \doteq 0.030$ and 0.029



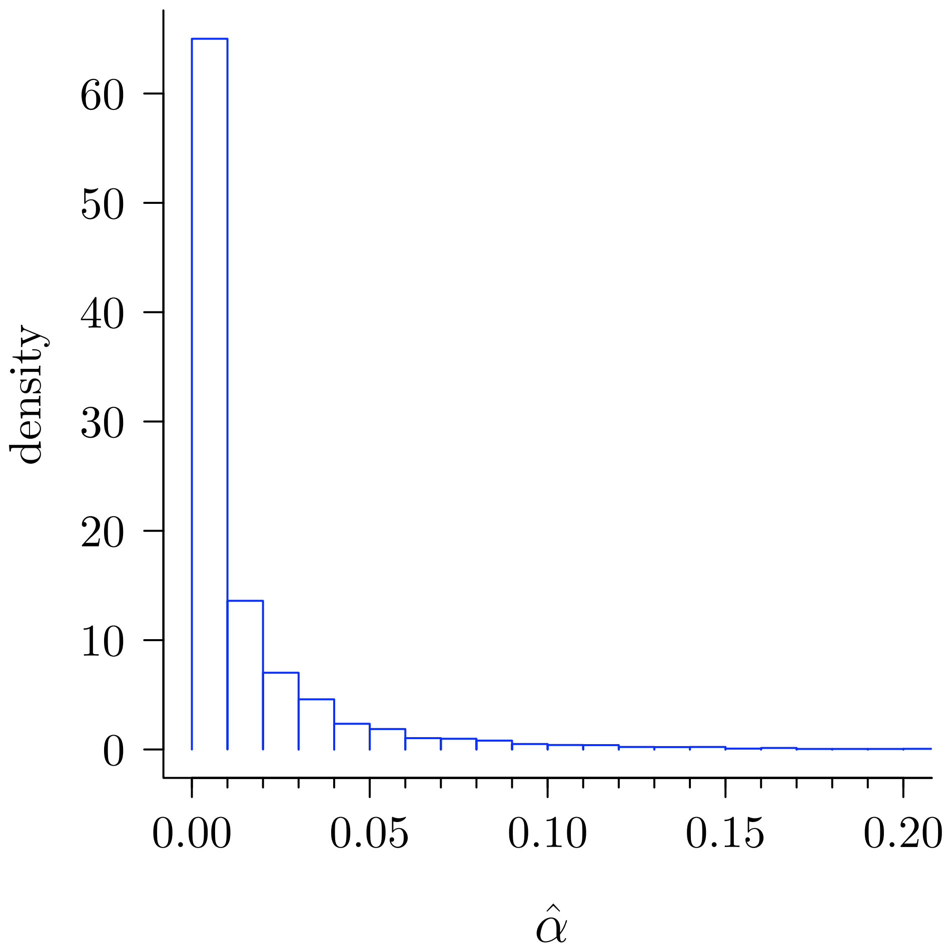
Extension to Handle 'Gappy' Time Series: IV

- histogram of 10,000 $\hat{\alpha}$'s for T_{10} (one for $\widetilde{T}_{10}$ virtually the same)



Extension to Handle 'Gappy' Time Series: V

- variation: account for uncertainty in ML parameter estimates



Future Directions

- critique: tests based on gap-filling biased toward *not* rejecting null hypothesis
- another approach for handling gappy time series (under study):
  - base tests on PACS estimates from fitting AR(k) models via maximum likelihood (Jones, 1980)
  - method slow (numerical optimization over k parameters)
  - approach based on constraining ML estimators via Levinson–Durbin recursions leads to sequence of one-dimensional optimization problems and hence potential speed-up
  - small sample properties of simplified approach under study
- lots of other avenues to look into!

Thanks to . . .

- conference organizers for opportunity to talk
- numerous folks at CSIRO who made my visit possible

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