Appendix

Proofs of theorems

Proof of Theorem 1. We can decompose h_l into a two-stage filter, the first of which has a transfer function whose squared modulus is $\mathcal{D}^{\frac{L}{2}}(f) = \left[4\sin^2(\pi f)\right]^{\frac{L}{2}}$. Letting $\widetilde{W}_t$ represent the output from this first stage, we have

$$\widetilde{W}_t = (1-B)^{\frac{L}{2}} Y_t = (1-B)^{\frac{L}{2}-d}\left[(1-B)^d Y_t\right] = (1-B)^{\frac{L}{2}-d} Z_t.$$

Hence $\widetilde{W}_t$ is a stationary process with zero mean. The second stage filter has a transfer function whose squared modulus is $\mathcal{C}$, which by construction can be factored into a filter of finite length (Daubechies, 1992, Ch. 6). The theorem follows by noting that filtering a zero-mean stationary process, i.e., $\widetilde{W}_t$, with a filter of finite length yields a zero-mean stationary process, i.e., W_t.

The proof of Theorem 2 requires a central limit theorem due to Ibragimov and five lemmata.

Ibragimov's Theorem. *Let ξ_t be a completely regular strictly stationary process such that $E(\xi_t) = 0$ and* $\operatorname{var}(\xi_t) < \infty$. *Let*

$$\sigma_n^2 \equiv \operatorname{var}\left(\sum_{t=1}^{n} \xi_t\right). \tag{7}$$

If $E(|\xi_t|^{2+\delta}) < \infty$ for some $\delta > 0$ and if $\sigma_n^2 \to \infty$, then $\xi_1 + \cdots + \xi_n$ is asymptotically normally distributed with zero mean and variance σ_n^2.

Proof of Ibragimov's Theorem. See Theorem 2.1, Ibragimov (1975).

Lemma 1. *If S is continuous and strictly positive, then any stationary process having S as its spectrum is completely regular.*

Proof of Lemma 1. See Theorem 1, p. 146, Ibragimov & Rozanov (1978).

Lemma 2. *Let $\zeta_1^2, \ldots, \zeta_n^2$ be a sample of size n of a strictly stationary process such that $E(|\zeta_t|^{4+\delta}) < \infty$ for some $\delta > 0$. Suppose that ζ_t^2 has spectrum S_{ζ^2} that is continuous on $[-\frac{1}{2}, \frac{1}{2}]$ and strictly positive. Then $\hat{\nu}_\zeta^2 \equiv \frac{1}{n}\sum \zeta_t^2$ is asymptotically normally distributed with mean $\nu_\zeta^2 \equiv E(\zeta_t^2)$ and variance $S_{\zeta^2}(0)/n$.*

Proof of Lemma 2. Let $\xi_t \equiv \zeta_t^2 - \nu_\zeta^2$, and define σ_n^2 as in Equation (7). Now $S_\xi = S_{\zeta^2}$, and the complete regularity of ξ_t follows from Lemma 1. Since $0 < S_\xi(0) < \infty$, we have

$$\sigma_n^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin^2(n\pi f)}{\sin^2(\pi f)} S_\xi(f)\, df = nS_\xi(0)(1 + o(1))$$

by a standard theorem, e.g., p. 322 of Ibragimov & Linnik (1971). Since $\sigma_n^2 \to \infty$ as $n \to \infty$, it follows from Ibragimov's Theorem that $\hat{\nu}_\zeta^2$ is asymptotically normally distributed with mean ν_ζ^2 and variance σ_n^2/n. Finally, since $\lim nS_{\zeta^2}(0)/\sigma_n^2 = 1$, Slutsky's theorem as given in Section 2c.4, part (x), p. 122 of Rao (1973) yields the lemma.

Lemma 3. *If G_t is a Gaussian stationary process with spectrum S_G, then G_t^2 is a stationary process with spectrum*

$$S_{G^2}(f) = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} S_G(f')S_G(f - f')\, df'. \tag{8}$$

Proof of Lemma 3. See p. 83 of Hannan (1970).

Lemma 4. *If S is a square integrable spectrum defined outside of $[-\frac{1}{2}, \frac{1}{2}]$ by periodic extension, then*

$$\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |S(f') - S(f' - \rho)|^2\, df'\right)^{\frac{1}{2}}$$

tends to zero with ρ.

Proof of Lemma 4. See Lemma 1.11, p. 37, Zygmund (1978).

Lemma 5. *Suppose that the spectra S_G and S_{G^2} are related by Equation (8). If S_G is finitely square integrable and strictly positive almost everywhere with respect to Lebesgue measure, then S_{G^2} is continuous on $[-\frac{1}{2}, \frac{1}{2}]$ and strictly positive.*

Proof of Lemma 5. For any $f \in (-\frac{1}{2}, \frac{1}{2})$ and ρ such that $f + \rho \in [-\frac{1}{2}, \frac{1}{2}]$, the Schwarz inequality can be used to show that

$$|S_{G^2}(f) - S_{G^2}(f+\rho)| \leq 2 \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} S_G^2(f')\, df' \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(S_G(f') - S_G(f'-\rho)\right)^2 \, df' \right)^{\frac{1}{2}}.$$

By Lemma 4 the second integral above tends to zero as $\rho \to 0$, which establishes that S_{G^2} is continuous on $(-\frac{1}{2}, \frac{1}{2})$. A similar argument holds for the points $\pm\frac{1}{2}$. Next, we need to show that $\inf S_{G^2}(f) > 0$. Suppose not. Since S_{G^2} is continuous on a closed bounded interval the infimum is attained at, say, f_1; thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} S_G(f') S_G(f_1 - f')\, df' = 0.$$

Since S_G is strictly positive almost everywhere, the above integrand is also such. A standard result in measure theory says that, if h is nonnegative on $[-\frac{1}{2}, \frac{1}{2}]$, then $h(f) = 0$ almost everywhere if and only if $\int h(f)\, df = 0$; see, e.g., Corollary 4.10, p. 34, Bartle (1966). This fact establishes a contradiction, from which we conclude that S_{G^2} is strictly positive.

Proof of Theorem 2. Since W_t is a Gaussian stationary process with zero mean and spectrum S_W by Theorem 1, the process W_t^2 is strictly stationary with spectrum S_{W^2} related to S_W as in Equation (8); moreover, $E(|W_t|^{4+\delta}) < \infty$ for any $\delta > 0$. By Lemma 5, S_{W^2} is continuous on $[-\frac{1}{2}, \frac{1}{2}]$ and strictly positive. The conditions of Lemma 2 are now satisfied for $\zeta_t^2 = W_t^2$ with $n = N_W$, thus yielding the theorem for W_t. The result for V_t follows from an identical argument.

Additional References

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