Assessing Characteristic Scales Using Wavelets

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overheads for talk available at

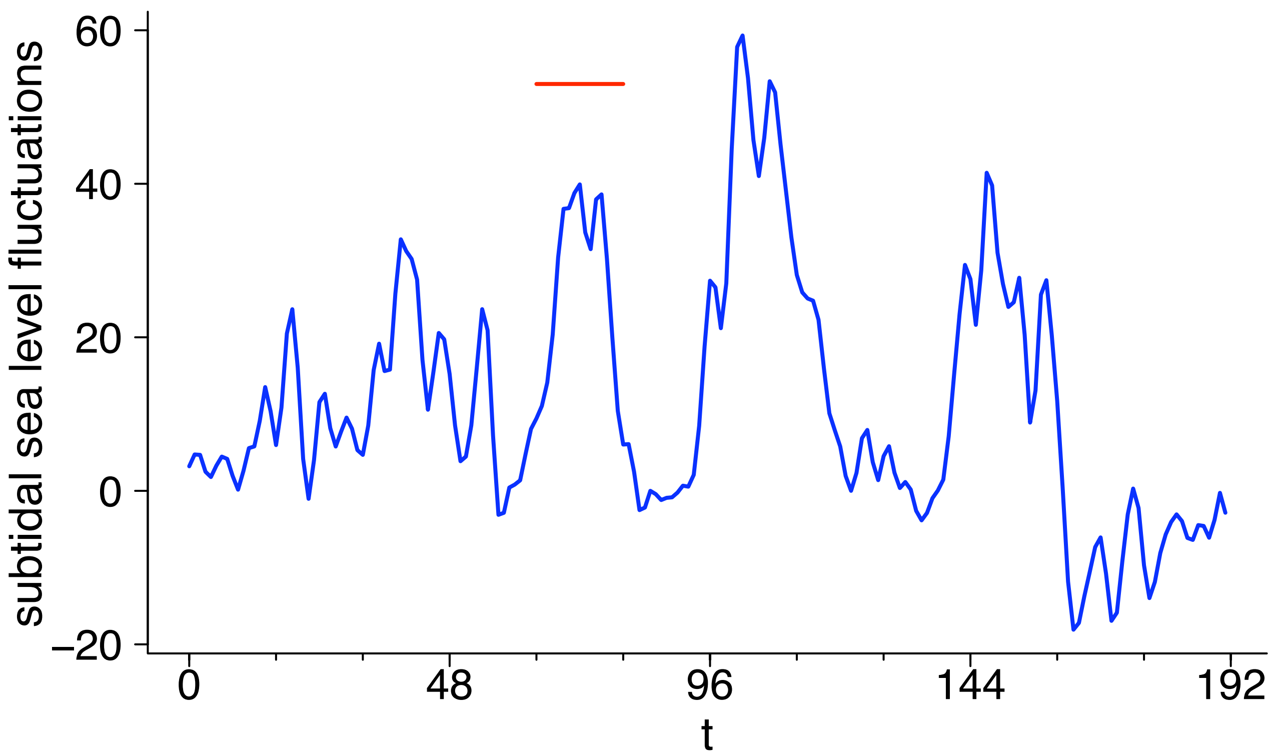
`http://faculty.washington.edu/dbp/talks.html`

Overview

- will discuss notion of characteristic scale, noting its lack of a standard definition
- will consider new definition in terms of discrete wavelet transform (DWT) as formulated by Daubechies, Mallat and others
- will present background on DWT, the wavelet variance (WV) and its sampling theory for intrinsically stationary Gaussian processes
- will then formulate definition of characteristic scale in terms of WV and present large-sample theory for its estimation
- will give four examples of time series of geophysical interest
- will conclude with comments on extensions of potential interest in telemedicine

Characteristic Scale: I

- geophysical time series often seem describable as a series of 'states' or 'events' whose durations tend to cluster around a value known as a characteristic scale (CS)



Characteristic Scale: II

- CS not defined outside of summary statistics used to extract it from a time series X_t, $t \in \mathbb{Z}$ (set of all integers)
- von Storch and Zwiers (1999) discuss several definitions when X_t is taken to be a stochastic process
- one definition quantifies 'memory' of process
- assuming $\mathbf{P}[X_{t+\tau} > 0 \,|\, X_t > 0] > 0.5$ for small lags $\tau > 0$, but $\mathbf{P}[X_{t+\tau} > 0 \,|\, X_t > 0] = 0.5$ at large lags, CS is smallest τ such that latter holds; i.e., CS is length of time needed for process to 'forget' its current positive state
- definition intuitively appealing, but of limited use: e.g., if X_t is a first order autoregressive process (AR(1)), $\tau = \infty$

Characteristic Scale: III

- second attempt at a definition (assumes X_t is stationary process)
- suppose $\text{var}\,\{X_t\} = \sigma^2$
- form sample mean of $X_1, X_2, \ldots, X_N$, i.e.,

$$\overline{X} = \frac{1}{N}\sum_{t=1}^{N} X_t, \text{ and consider its variance var}\,\{\overline{X}\}$$

- suppose $Y_1, Y_2, \ldots, Y_N$ are independent & identically distributed random variables, also with variance σ^2
- implies $\text{var}\,\{\overline{Y}\} = \sigma^2/N$
- equivalent sample size N' defined by setting $\text{var}\,\{\overline{X}\} = \sigma^2/N'$
- limit of N/N' as $N \to \infty$ defines a decorrelation time τ_D, a reasonable definition of CS for some – but not all – time series

Characteristic Scale: IV

- can show that
$$\tau_D = \sum_{k \in \mathbb{Z}} \rho_k,$$
where $\rho_k = \mathrm{cov}\{X_{t+k}, X_t\}/\sigma^2$ is autocorrelation sequence (ACS) for X_t
- for an AR(1) process, above reduces to $\tau_D = (1+\rho_1)/(1-\rho_1)$
- for other stationary processes, can use estimated ACS to estimate τ_D (doing so can be tricky!)
- von Storch and Zwiers (1999) note that variance of other sample statistics can be used in a similar manner to define a CS
- other definitions in literature involve structure functions (semi-variograms), fractal dimension, detrended fluctuation analysis, ...

Wavelet-based Definition of Characteristic Scale

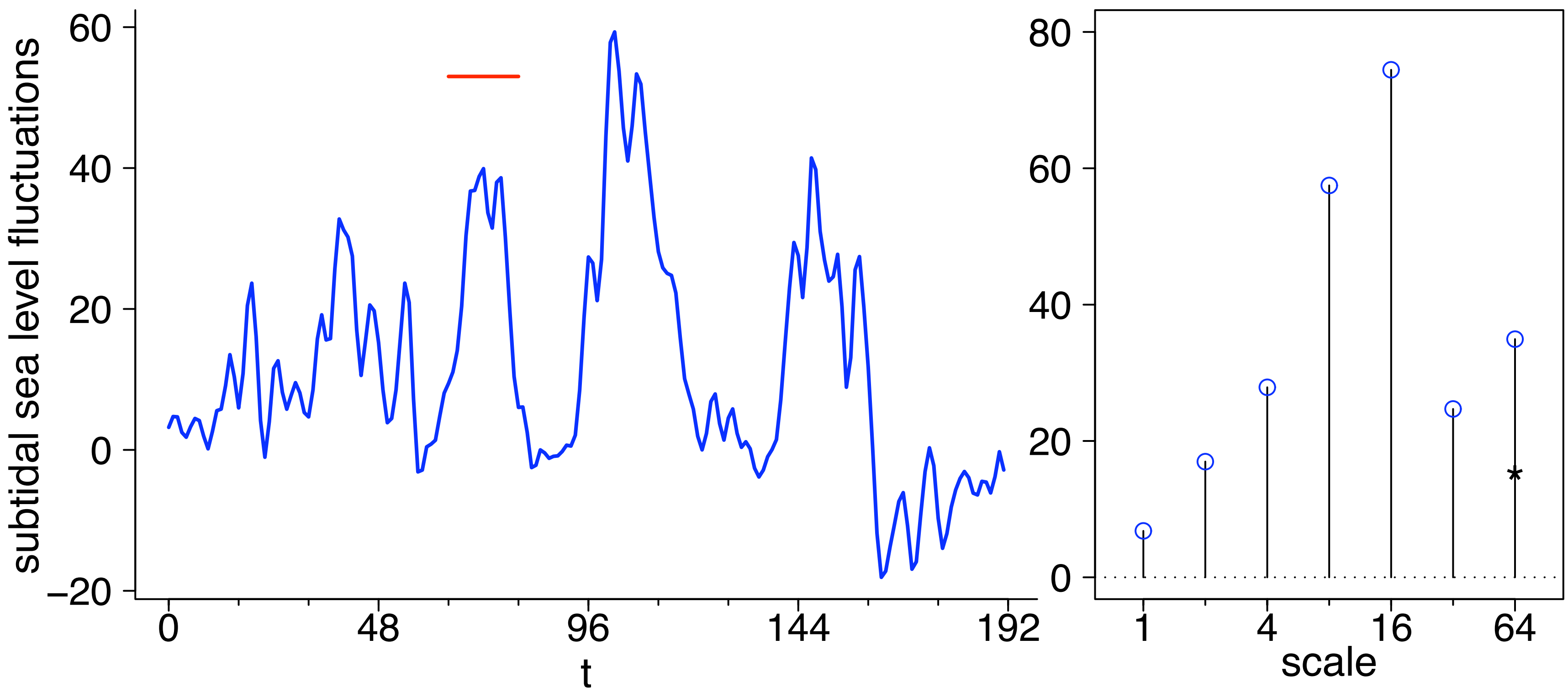
- theme of talk: a wavelet-based definition of CS
- discrete wavelet transform (DWT) describable as scale-based
- to fix ideas, consider Haar maximal overlap DWT, which yields wavelet coefficients $W_{\tau,t}^{\text{Haar}}$ for integer scale τ as follows:

$$2W_{\tau,t}^{\text{Haar}} = \frac{1}{\tau}\sum_{l=0}^{\tau-1} X_{t-l} - \frac{1}{\tau}\sum_{l=0}^{\tau-1} X_{t-\tau-l},$$

  i.e., a difference in adjacent averages spanning τ values
- if 'event' near X_t is of duration τ, $|W_{\tau,t}^{\text{Haar}}|$ will tend to be large
- summary of 'largeness' of $|W_{\tau,t}^{\text{Haar}}|$ is var $\{W_{\tau,t}^{\text{Haar}}\}$, which is known as the wavelet variance (WV)
- large var $\{W_{\tau,t}^{\text{Haar}}\}$ should provide basis for useful CS definition

Wavelet Variance for Subtidal Sea-level Time Series

- right-hand plot shows decomposition of sample variance of X_t via empirical wavelet variances at scales 1, 2, 4, 8, 16, 32 and 64, plus a term (the asterisk) summarizing all higher scales



Background on Wavelet Variance: I

- wavelet variance well-defined for an intrinsically stationary process X_t of integer order $d \geq 0$, so let's review this concept
- if $d = 0$, X_t is a stationary process; i.e., both $E\{X_t\}$ and $\text{cov}\,\{X_{t+\tau}, X_t\} = s_\tau$ – its autocovariance sequence (ACVS) – exist and are finite and independent of t
- if $d > 0$, subjecting X_t to dth order backward difference filter yields a stationary process with ACVS $s_\tau^{(d)}$, namely,

$$X_t^{(d)} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k} = \begin{cases} X_t - X_{t-1} \text{ for } d = 1; \\ X_t - 2X_{t-1} + X_{t-2} \text{ for } d = 2; \\ \quad \vdots, \end{cases}$$

whereas $X_t^{(d-1)}, \ldots, X_t^{(1)}$ & $X_t^{(0)} = X_t$ are all nonstationary (an example being a random walk process)

Background on Wavelet Variance: II

- let $h_{1,l}, l = 0, 1, \ldots, L_1 - 1$, be unit-level Daubechies wavelet filter of even width L_1 with normalization $\sum_l h_{1,l}^2 = 1/2$
- $L_1 = 2$ case is the Haar filter: $h_{1,0} = \frac{1}{2}$, $h_{1,1} = -\frac{1}{2}$
- for $L_1 \geq 4$, use of $h_{1,l}$ equivalent to subjecting output from order $\frac{L_1}{2}$ backward difference filter to width $\frac{L_1}{2}$ low-pass filter
- can use unit level filter to construct jth level filter $h_{j,l}$ of width $L_j = (2^j - 1)(L_1 - 1) + 1$
- jth level wavelet coefficient process for X_t given by

$$W_{j,t} = \sum_{l=0}^{L_j - 1} h_{j,l} X_{t-l}$$

Background on Wavelet Variance: III

- coefficient $W_{j,t}$ is proportional to changes in adjacent weighted averages with effective scale (span) of $\tau_j = 2^{j-1}$
- assuming X_t is a dth order intrinsically stationary process and $L_1 \geq 2d$, wavelet coefficient process $W_{j,t}$ is stationary
- wavelet variance ν_j^2 for X_t at scale τ_j is variance of $W_{j,t}$:

$$\nu_j^2 = \text{var}\{W_{j,t}\}$$

- if X_t is stationary,

$$\text{var}\{X_t\} = \sum_{j=1}^{\infty} \nu_j^2,$$

in which case ν_j^2 is contribution to overall variance due to changes in adjacent weighted averages over scale τ_j

Background on Wavelet Variance: IV

- on the hand, $\sum_{j=1}^{J} \nu_j^2 \to \infty$ as $J \to \infty$ if X_t is intrinsically stationary of order $d \geq 1$, but ν_j^2 can still be interpreted as measuring variability of changes in adjacent weighted averages
- can express ν_j^2 in terms of ACVS $s_\tau^{(d)}$ for underlying stationary component $X_t^{(d)}$ for X_t:

$$\nu_j^2 = s_0^{(d)} \sum_{l=0}^{L_j-d-1} \left(h_{j,l}^{(d)}\right)^2 + 2 \sum_{\tau=1}^{L_j-d-1} s_\tau^{(d)} \sum_{l=0}^{L_j-d-1-\tau} h_{j,l}^{(d)} h_{j,l+\tau}^{(d)},$$

where $h_{j,l}^{(d)}$ is dth-order cumulative summation of $h_{j,l}$ (i.e., $h_{j,l}^{(1)} = \sum_{n=0}^{l} h_{j,n}$, while $h_{j,l}^{(2)} = \sum_{n=0}^{l} h_{j,n}^{(1)}$ and so forth)

Estimation Theory for Wavelet Variance: I

- given a time series $X_0, X_1, \ldots, X_{N-1}$, can compute level j wavelet coefficients for indices $L_j - 1 \leq t \leq N-1$ assuming $M_j = N - L_j + 1 > 0$
- sufficient (but not necessary) condition for $W_{j,t}$ to be a zero mean stationary process is $L_1 > 2d$ ($W_{j,t}$ is necessarily stationary if $L_1 = 2d$, but might not have zero mean)
- assuming L_1 chosen such that $W_{j,t}$ is a zero mean stationary process, then $\nu_j^2 = E\{W_{j,t}^2\}$ and hence

$$\hat{\nu}_j^2 = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} W_{j,t}^2$$

  is unbiased estimator of wavelet variance ν_j^2

Estimation Theory for Wavelet Variance: II

- can deduce second moment properties of $\hat{\nu}_j^2$ under assumption that $W_{j,t}$'s are multivariate Gaussian: for $j \leq k$, have

$$\operatorname{cov}\{\hat{\nu}_j^2, \hat{\nu}_k^2\} = \frac{2}{M_j} \sum_{\tau=-(M_k-1)}^{M_k-1} \left(1 - \frac{|\tau|}{M_k}\right) s_{j,k,\tau}^2 + \frac{2}{M_j M_k} \sum_{t=L_j-1}^{L_k-2} \sum_{u=L_k-1}^{N-1} s_{j,k,t-u}^2,$$

where $s_{j,k,\tau}$ is cross-covariance sequence for bivariate stationary processes $W_{j,t}$ & $W_{k,t}$:

$$s_{j,k,\tau} = \operatorname{cov}\left\{W_{j,t+\tau}, W_{k,t}\right\} = \sum_{l=0}^{L_j-d-1} h_{j,l}^{(d)} \sum_{m=0}^{L_k-d-1} h_{k,m}^{(d)} s_{\tau-l+m}^{(d)}$$

Estimation Theory for Wavelet Variance: III

- if M_j is not too small (≥ 100 or so), useful approximation is

$$\operatorname{cov}\{\hat{\nu}_j^2, \hat{\nu}_k^2\} \approx \frac{2A_{j,k}}{M_j}, \text{ where } A_{j,k} = \sum_{\tau=-\infty}^{\infty} s_{j,k,\tau}^2$$

- can estimate $A_{j,k}$ (and hence $\operatorname{cov}\{\hat{\nu}_j^2, \hat{\nu}_k^2\}$) via

$$\hat{A}_{j,k} = \frac{1}{2}\left(\hat{\nu}_j^2\hat{\nu}_k^2 + 2\sum_{\tau=1}^{M_k-1} \hat{s}_{j,\tau}\hat{s}_{k,\tau}\right),$$

where $\hat{s}_{j,\tau}$ is biased estimator of ACVS:

$$\hat{s}_{j,\tau} = \frac{1}{M_j}\sum_{t=L_j-1}^{N-1-\tau} W_{j,t+\tau}W_{j,t}, \qquad 0 \leq \tau \leq M_j - 1$$

Estimation Theory for Wavelet Variance: IV

- under mild conditions, $\hat{\nu}_j^2$ is asymptotically normal with mean ν_j^2 and large-sample variance $2A_{j,j}/M_j$
- to summarize, $\hat{\nu}_j^2$ is an unbiased estimator of ν_j with tractable statistical properties
- in particular, can determine var $\{\hat{\nu}_j^2\}$ and cov $\{\hat{\nu}_j^2, \hat{\nu}_k^2\}$ by substituting readily computable estimates $\hat{A}_{j,k}$ for $A_{j,k}$ in expressions above

Wavelet-Based Definition of Characteristic Scale: I

- interpretation of WV as scaled-based analysis of variance motivates the following definition for CS (basic idea is to consider τ_j's where ν_j^2 is large compared to neighboring values)
- suppose X_t is intrinsically stationary with WVs such that $\nu_j^2 \geq \nu_{j\pm1}^2$ for some $j \geq 2$, with strict inequality holding in at least one case
- fit a quadratic $y_k = a + bx_k + cx_k^2$ passing through $(x_k, y_k) = (\log_2(\tau_k), \log_2(\nu_k^2))$, $k = j-1, j, j+1$
- define a *local CS* as location at which quadratic is maximized:

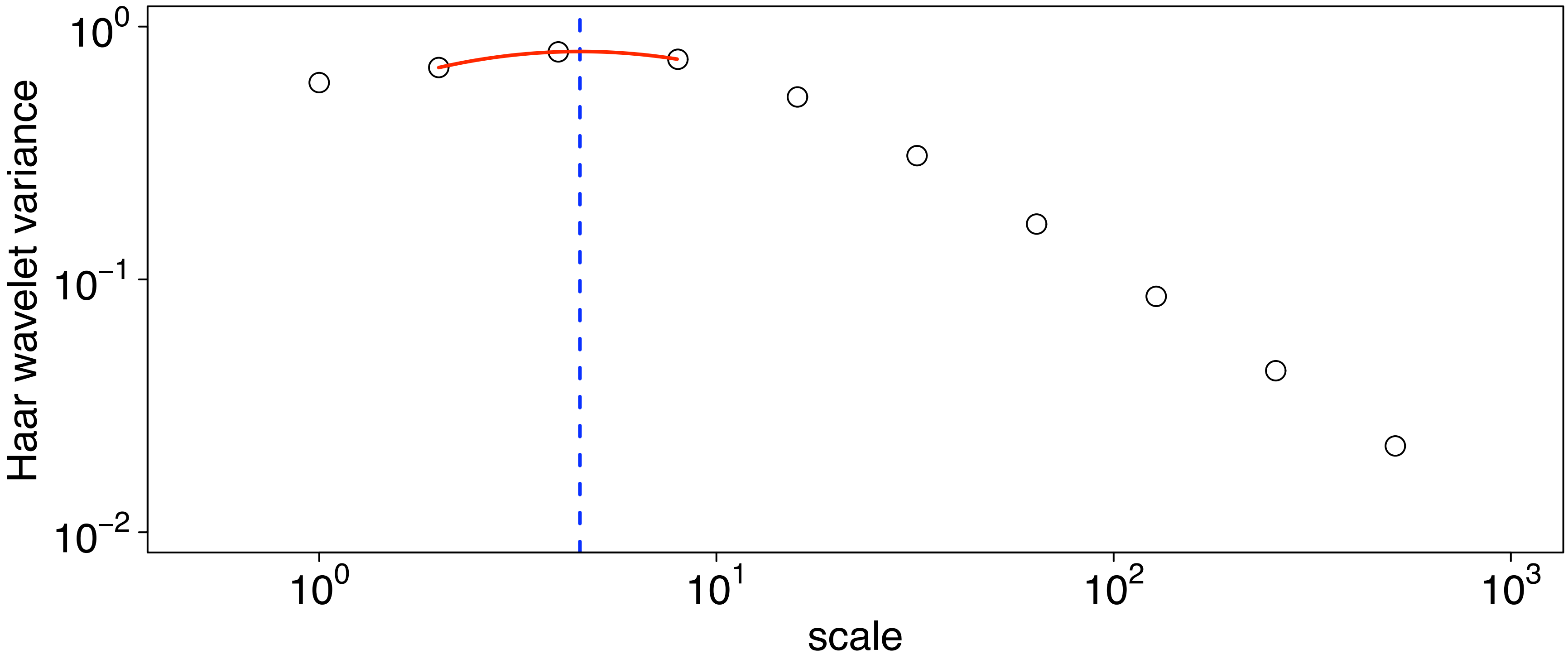
$$\tau_{c,j} = 2^{-\beta_1/\beta_2}\tau_j, \text{ where } \beta_1 = \frac{y_{j+1} - y_{j-1}}{2} \ \& \ \beta_2 = y_{j+1} - 2y_j + y_{j-1}$$

Wavelet-Based Definition of Characteristic Scale: II

- note: $\tau_j/\sqrt{2} \leq \tau_{c,j} \leq \tau_j\sqrt{2}$
- note: $\nu^2_{j-1} < \nu^2_j = \nu^2_{j+1} > \nu^2_{j+2}$ yields $\tau_{c,j} = \tau_{c,j+1} = \tau_j\sqrt{2}$
- in addition, if X_t has local CS $\tau_{c,j}$ such that $\nu^2_j > \nu^2_k$ for all $k \in \mathbb{Z}^+$ (the positive integers) excluding $k = j-1, j, j+1$, then X_t is said to have *global CS* $\tau_c = \tau_{c,j}$
- let's look at some examples of theoretical WV curves possessing local and global CSs

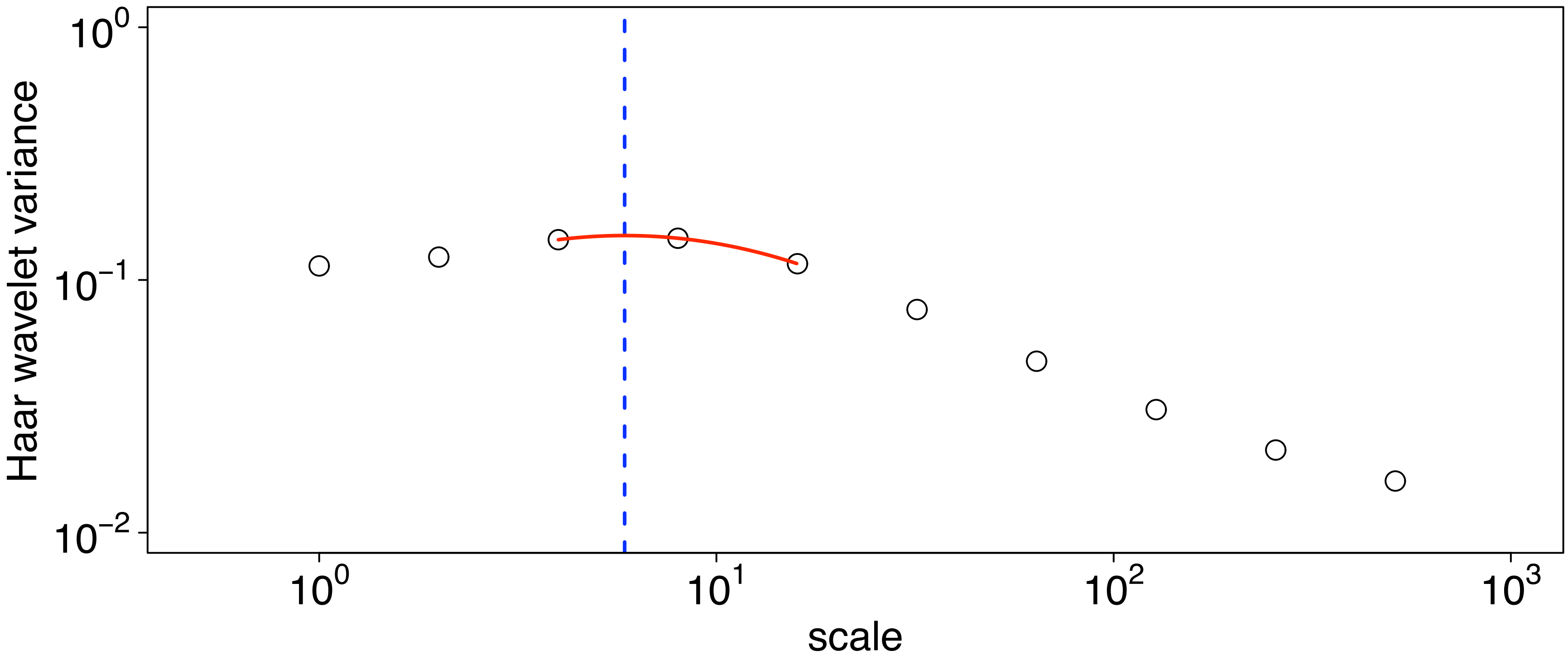
First-Order Autoregressive Process

- AR(1) process defined by $X_t = \phi X_{t-1} + \epsilon_t$, where here $\phi = 0.7$ and ϵ_t is white noise (τ_c and decorrelation time $\tau_D = \frac{1+\phi}{1-\phi}$ similar, with agreement getting better as $\phi \to 1$)



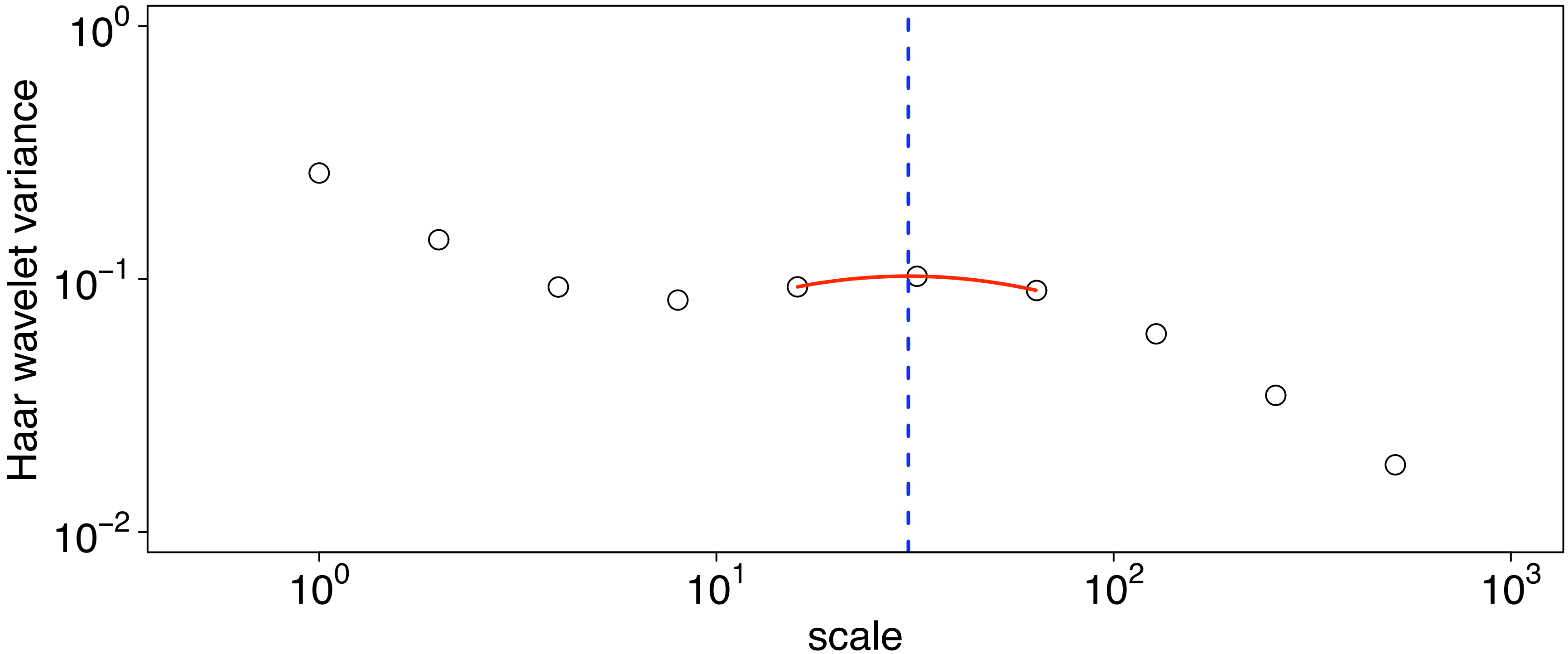
Sum of AR(1) & Fractionally Differenced Processes

- FD process has 'long-memory' – CS measure $\tau_D = \sum_{k \in \mathbb{Z}} \rho_k = \infty$ (reflecting asymptotic decay rate of ρ_k); by contrast, wavelet-based τ_c is finite (concentrates on localized properties of X_t)



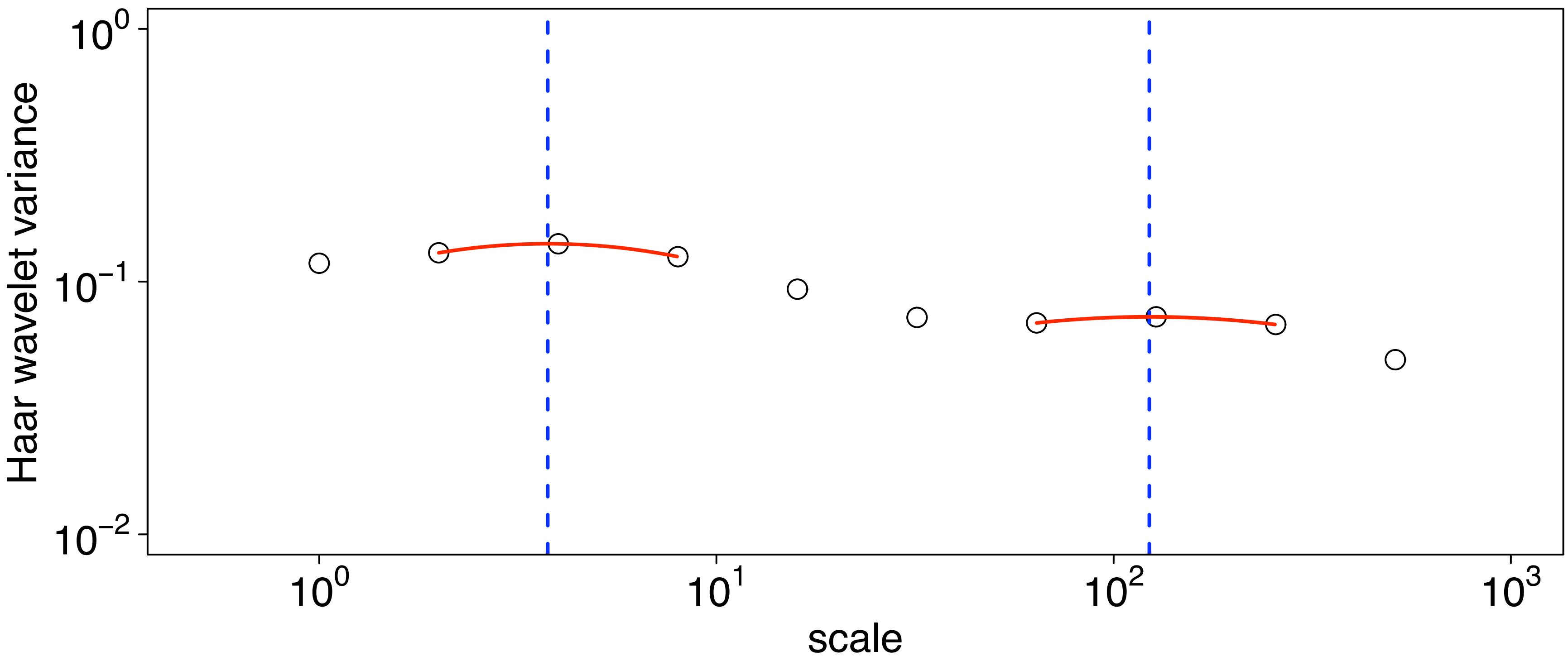
Sum of AR(1) & White Processes

- example of process with a local CS that is not also a global CS



Sum of Two AR(1) Processes

- example of process with two local CSs, one of which is also a global CS; AR(1) parameters are $\phi = 0.65$ and $\phi = 0.99$



Comments on Definition of Wavelet-Based CS

- natural definition of CS would involve WV defined over a continuum of scales from a continuous wavelet transform (CWT)
- for AR(1) processes, CWT-based CS and DWT-based τ_c using Haar wavelet are virtually the same
- DWT-based CS is preferable due to its greater degree of statistical tractability
- τ_c defined in terms of quadratic fit in log/log space, which is preferable to linear/linear, linear/log or log/linear spaces, also because of considerations involving agreement with CWT and statistical tractability

Estimation of Wavelet-Based Characteristic Scale

- suppose we have a time series that is a realization of a portion $X_0, X_1, \ldots, X_{N-1}$ of an intrinsically stationary process of order d
- given an appropriately chosen wavelet filter and an unbiased estimator $\hat{\nu}_j^2$ for levels $j = 1, \ldots, J_0$, suppose there is some $1 < j < J_0$ such that $\hat{\nu}_j^2 \geq \hat{\nu}_{\pm j}^2$ (with strict inequality holding in at least one case)
- conditional on observed pattern and estimating $y_k = \log_2(\nu_k^2)$ using $\hat{y}_k = \log_2(\hat{\nu}_k^2)$, obvious estimator of $\tau_{c,j}$ is

$$\tau_{c,j} = 2^{-\hat{\beta}_1/\hat{\beta}_2}\tau_j, \text{ where } \hat{\beta}_1 = \frac{\hat{y}_{j+1} - \hat{y}_{j-1}}{2} \ \& \ \hat{\beta}_2 = \hat{y}_{j+1} - 2\hat{y}_j + \hat{y}_{j-1}$$

Statistical Properties of CS Estimator: I

- assuming $[\hat{\nu}^2_{j-1}, \hat{\nu}^2_j, \hat{\nu}^2_{j+1}]^T$ is multivariate Gaussian with mean & covariance given by large-sample theory, delta method says

$$\left[\log_2\left(\hat{\nu}^2_{j-1}\right), \log_2\left(\hat{\nu}^2_j\right), \log_2\left(\hat{\nu}^2_{j+1}\right)\right]^T = \left[\hat{y}_{j-1}, \hat{y}_j, \hat{y}_{j+1}\right]^T$$

is approximately Gaussian with mean

$$\left[\log_2\left(\nu^2_{j-1}\right), \log_2\left(\nu^2_j\right), \log_2\left(\nu^2_{j+1}\right)\right]^T = \left[y_{j-1}, y_j, y_{j+1}\right]^T$$

and large-sample covariance matrix Σ whose (m, n)th element is

$$\frac{\operatorname{cov}\{\hat{\nu}^2_{j'+m}, \hat{\nu}^2_{j'+n}\}}{\nu^2_{j'+m}\nu^2_{j'+n}\log^2(2)} + 2\frac{\operatorname{var}\{\hat{\nu}^2_{j'+m}\}\operatorname{var}\{\hat{\nu}^2_{j'+n}\} + (\operatorname{cov}\{\hat{\nu}^2_{j'+m}, \hat{\nu}^2_{j'+n}\})^2}{\nu^4_{j'+m}\nu^4_{j'+n}\log^2(2)};$$

here $j' = j - 1$ and $0 \le (m, n) \le 2$

Statistical Properties of CS Estimator: II

- note that

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_{j-1} \\ \hat{y}_j \\ \hat{y}_{j+1} \end{bmatrix} \equiv H \begin{bmatrix} \hat{y}_{j-1} \\ \hat{y}_j \\ \hat{y}_{j+1} \end{bmatrix}$$

- hence $[\hat{\beta}_1, \hat{\beta}_2]^T$ is asymptotically Gaussian with mean $[\beta_1, \beta_2]^T$ and covariance $H\Sigma H^T$
- letting $\hat{\kappa} = -\hat{\beta}_1/\hat{\beta}_2$, delta method says $\hat{\kappa}$ is asymptotically Gaussian with mean $-\beta_1/\beta_2$ and large-sample variance

$$\sigma^2_{\hat{\kappa}} = \frac{\operatorname{var}\{\hat{\beta}_1\}}{\beta_2^2} + \frac{\beta_1^2 \operatorname{var}\{\hat{\beta}_2\}}{\beta_2^4} + \frac{\operatorname{var}\{\hat{\beta}_1\}\operatorname{var}\{\hat{\beta}_2\} + 2(\operatorname{cov}\{\hat{\beta}_1, \hat{\beta}_2\})^2}{\beta_2^4} + \frac{3\beta_1^2(\operatorname{var}\{\hat{\beta}_2\})^2}{\beta_2^6} - \frac{2\beta_1 \operatorname{cov}\{\hat{\beta}_1, \hat{\beta}_2\}}{\beta_2^3}$$

Statistical Properties of CS Estimator: III

- final use of delta method says $\hat{\tau}_{c,j}$ is asymptotically normal with mean $\tau_{c,j}$ and large-sample variance $\tau^2_{c,j}\,\sigma^2_{\hat{\kappa}}\,\log^2_e(2)$
- approximate 95% confidence interval (CI) for $\tau_{c,j}$ given by

$$\left[2^{-1.96\sigma_{\hat{\kappa}}}\hat{\tau}_{c,j},\, 2^{1.96\sigma_{\hat{\kappa}}}\hat{\tau}_{c,j}\right]$$

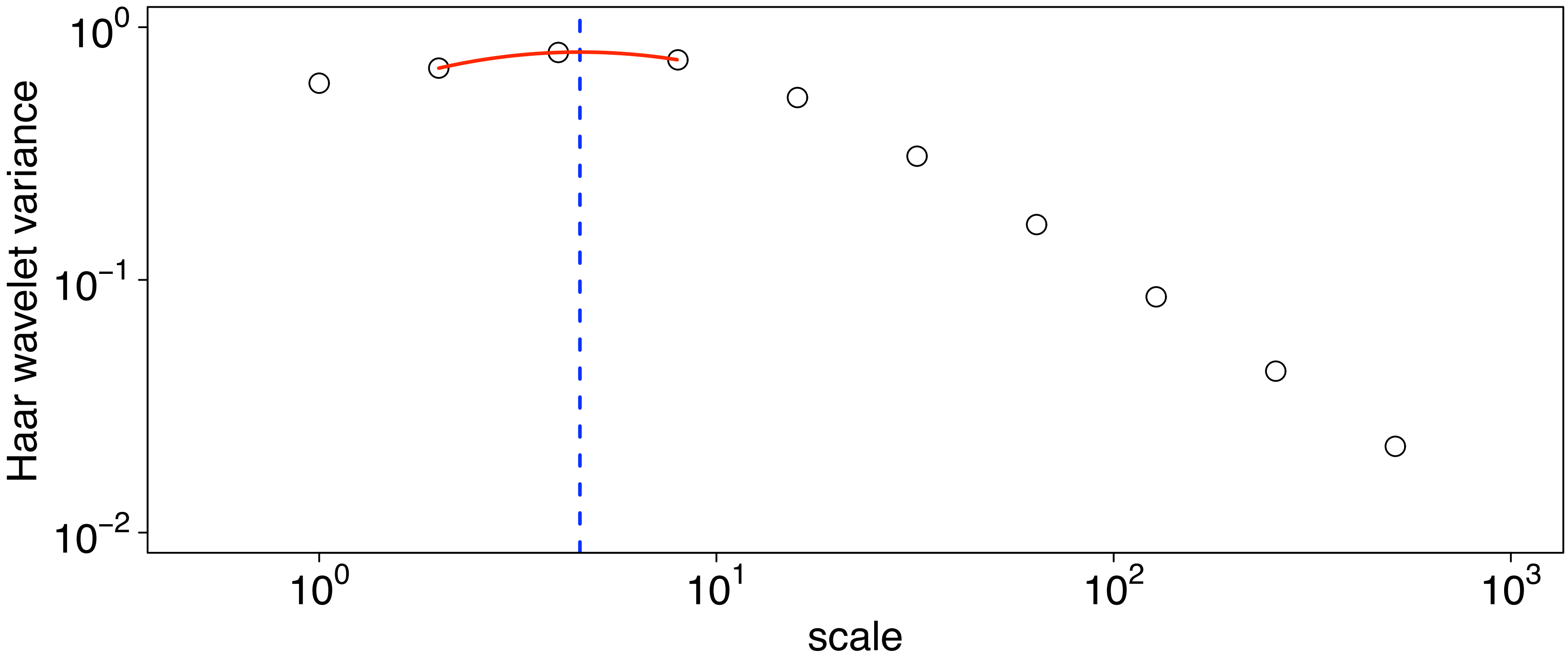
- in practical applications, can use 'plug-in' estimate for $\sigma^2_{\hat{\kappa}}$
- caveat: approach conditioned upon $\hat{\nu}^2_j \geq \hat{\nu}^2_{j\pm1}$ correctly indicating the presence of a local CS (asymptotically OK)
- sanity check: generate realizations of trivariate Gaussian with mean $[\hat{\nu}^2_{j-1}, \hat{\nu}^2_j, \hat{\nu}^2_{j+1}]^T$ & covariance matrix given by large sample theory, and look at proportion of realizations with maximum at j (if large, have some faith observed pattern actually exists)

Monte Carlo Experiments: I

- for each of four processes and for sample sizes N ranging from 256 to 8192, generated 1000 realizations using exact simulation method for AR and FD processes
- recorded number of realizations M for which peak there was a peak in Haar WV curve at either proper level j or $j \pm 1$
- for each of these M realizations, estimated CS and computed 95% CI using plug-in procedure
- following tables show average of estimated CSs and percentage of times CIs trapped true CS

Monte Carlo Experiments: II

- AR(1) process ($\phi = \rho_1 = 0.7$)



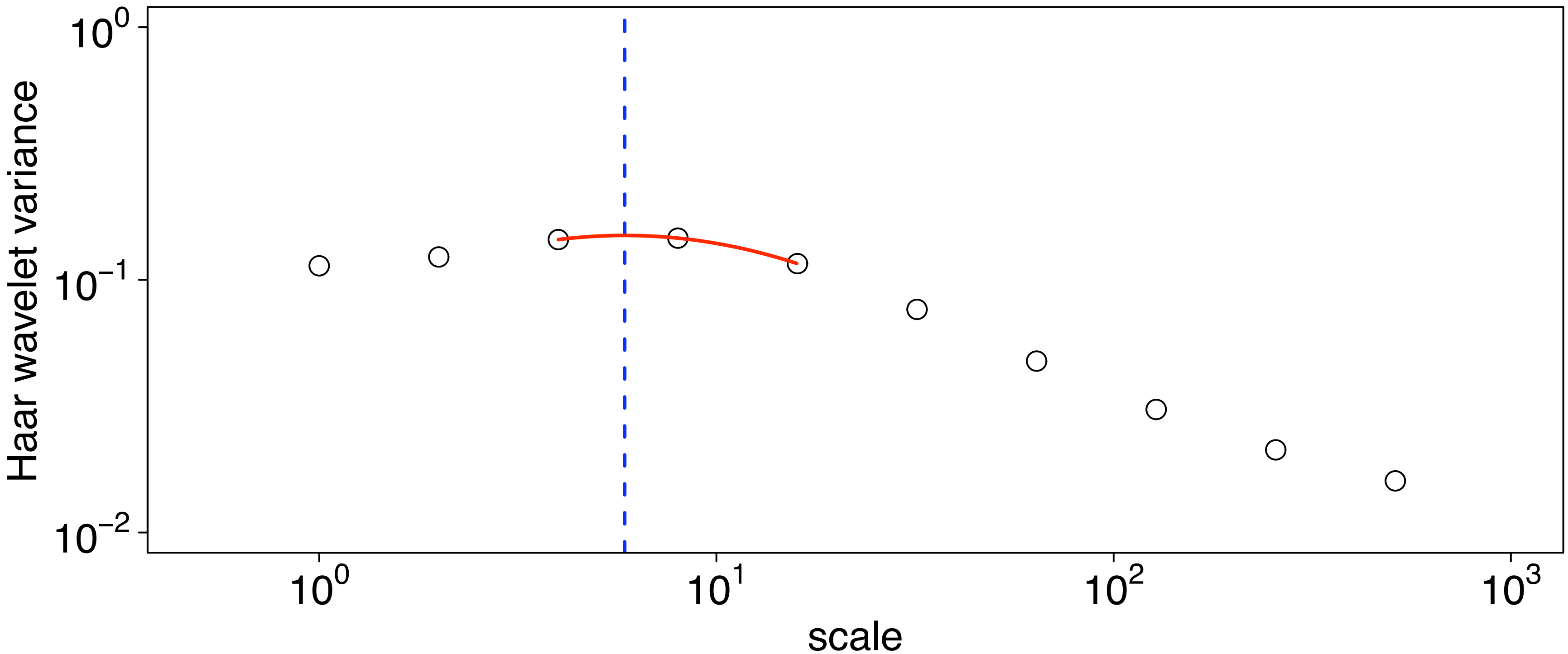
Monte Carlo Experiments: III

- results for AR(1) process ($\phi = \rho_1 = 0.7$)

N	τ_c	$\hat{\tau}_c$	M	% coverage
256	4.53	4.75	953	88.6
512		4.69	992	88.2
1024		4.68	999	87.9
2048		4.66	1000	87.1
4096		4.60	1000	90.0
8192		4.57	1000	94.4

Monte Carlo Experiments: IV

- sum of AR(1) & FD processes ($\phi = 0.75$ & $\delta = 0.45$)



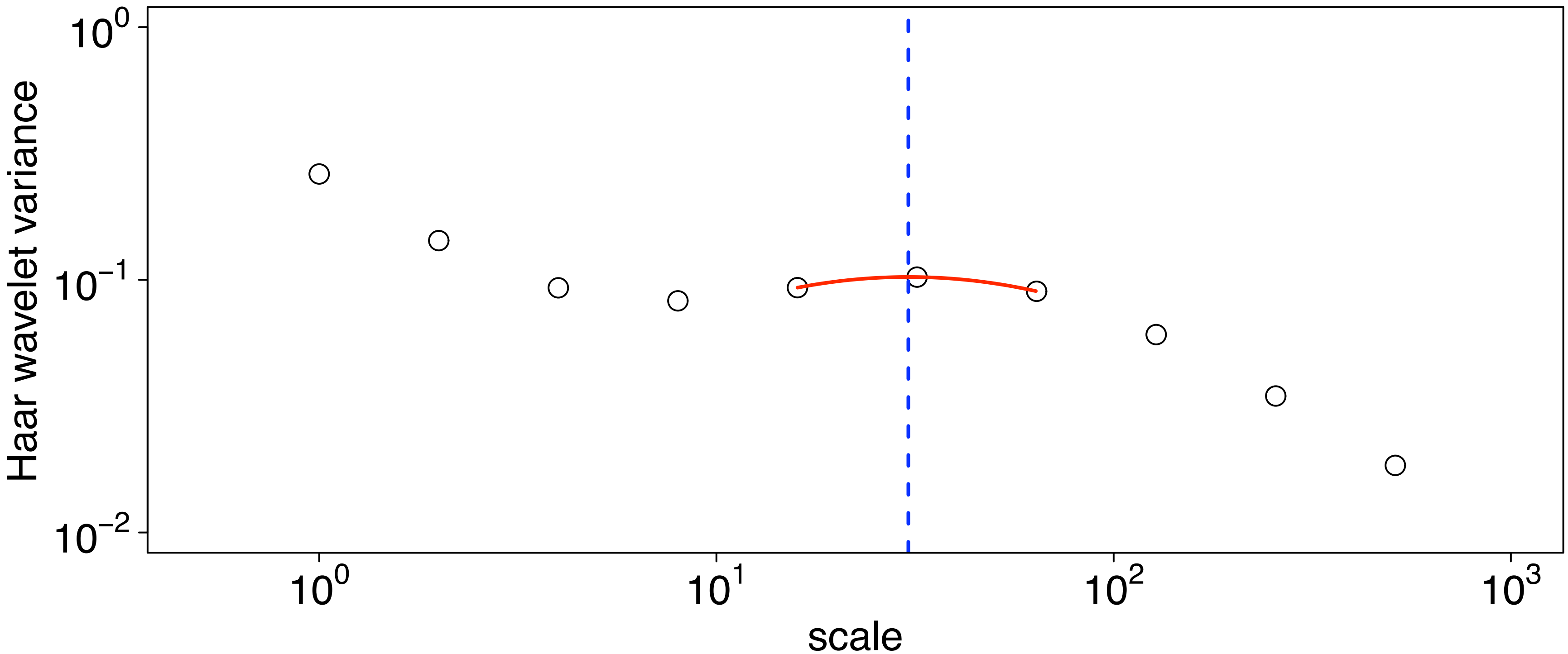
Monte Carlo Experiments: V

- results for sum of AR(1) & FD processes ($\phi = 0.75$ & $\delta = 0.45$)

N	τ_c	$\hat{\tau}_c$	M	% coverage
256	5.87	6.58	942	89.2
512		6.16	981	89.2
1024		5.88	1000	90.8
2048		5.84	1000	90.5
4096		5.81	1000	92.6
8192		5.83	1000	93.8

Monte Carlo Experiments: VI

- sum of AR(1) & white noise processes ($\phi = 0.95$)



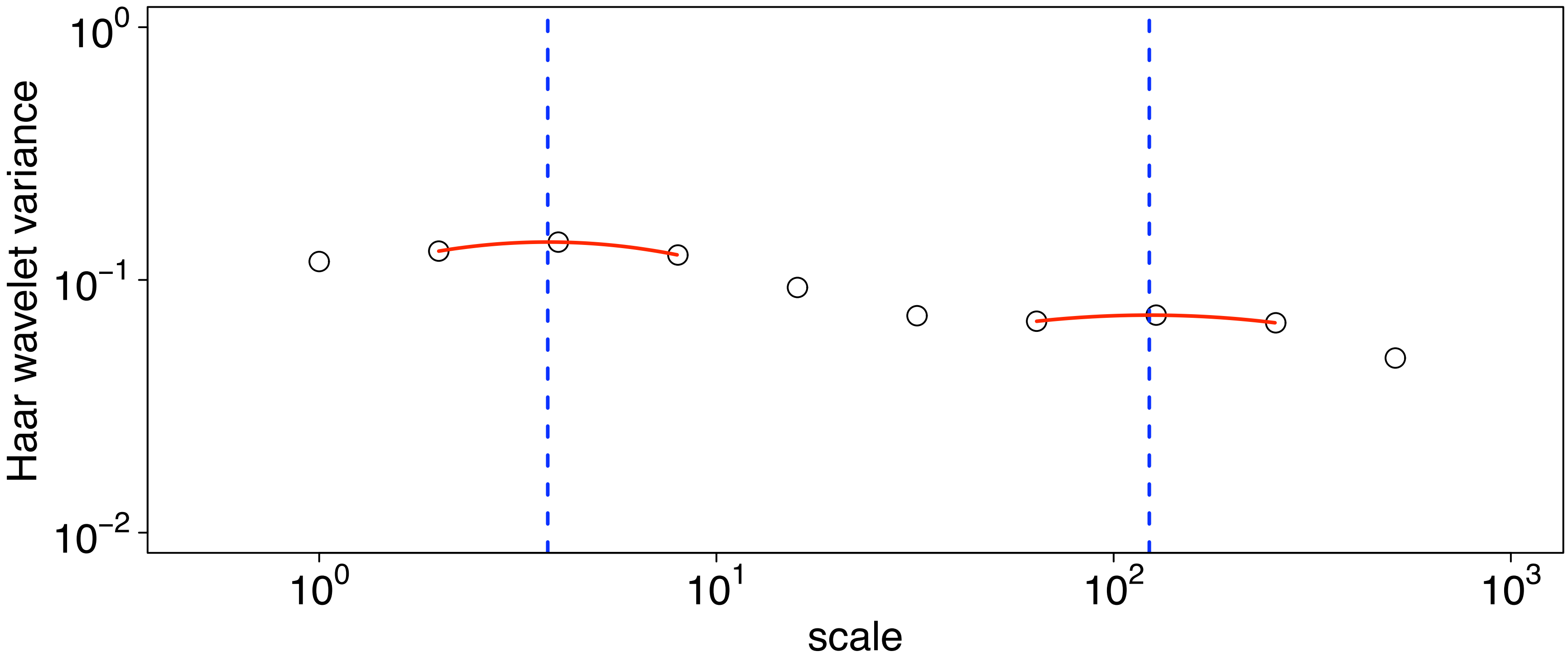
Monte Carlo Experiments: VII

- results for sum of AR(1) & white noise processes ($\phi = 0.95$)

N	τ_c	$\hat{\tau}_c$	M	% coverage
256	30.42	33.51	798	87.0
512		33.51	838	86.8
1024		33.51	900	89.1
2048		32.49	964	88.5
4096		31.73	999	91.4
8192		31.41	1000	93.1

Monte Carlo Experiments: VIII

- sum of two AR(1) processes ($\phi = 0.65$ & 0.99)



Monte Carlo Experiments: IX

- first CS of sum of two AR(1) processes ($\phi = 0.65$ & 0.99)

N	τ_c	$\hat{\tau}_c$	M	% coverage
256	3.76	4.08	935	90.4
512		4.00	971	91.0
1024		3.87	997	92.6
2048		3.84	999	96.7
4096		3.80	1000	97.2
8192		3.76	1000	96.1

Monte Carlo Experiments: X

- second CS of sum of two AR(1) processes ($\phi = 0.65$ & 0.99)

N	τ_c	$\hat{\tau}_c$	M	% coverage
256	122.96	—	—	—
512		89.31	454	86.8
1024		139.31	690	88.0
2048		149.60	703	88.9
4096		152.97	757	89.8
8192		153.43	775	86.7

- note: $8192/\tau_c \doteq 67$; i.e., small number of replications of CS

Monte Carlo Experiments: XI

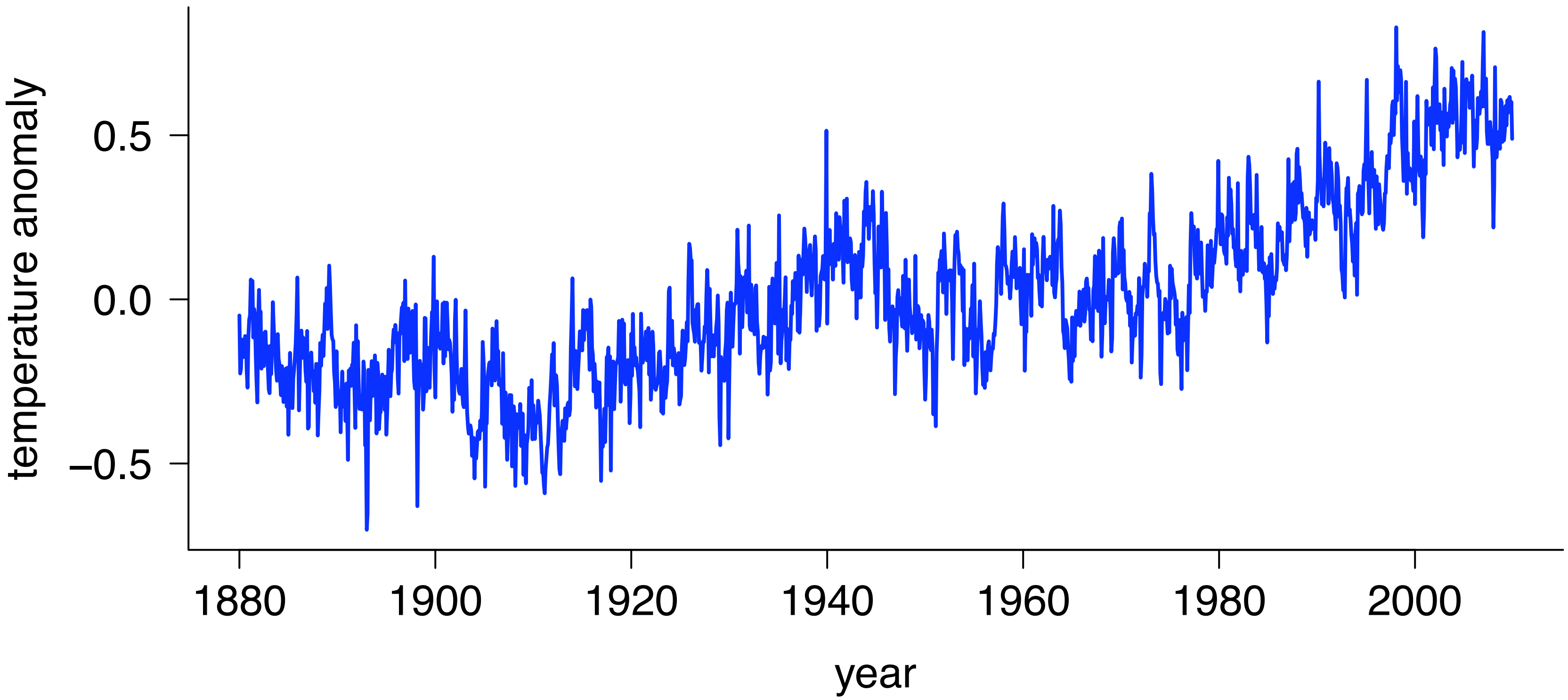
- tendency for $\hat{\tau}_c$ to be positively biased
- closeness of coverage percentage to nominal 95% tends to depend on true τ_c: the smaller τ_c is, the better the coverage rate
- for small sample sizes, cover rate tends to be below 95%
- coverage rates tend to improve with increasing sample size, as asymptotic theory would suggest
- bottom line: experiments indicate large-sample theory gives useful – but admittedly not perfect – approximations to variability in $\hat{\tau}_c$ for moderate sample sizes
- note: similar results hold in experiments using Daubechies wavelet filters of lengths $L_1 = 4$ and 8

Four Real-World Examples

- global temperature record
- coherent structures in river flow
- Madden–Julian atmospheric oscillation
- medium multiyear Arctic sea ice

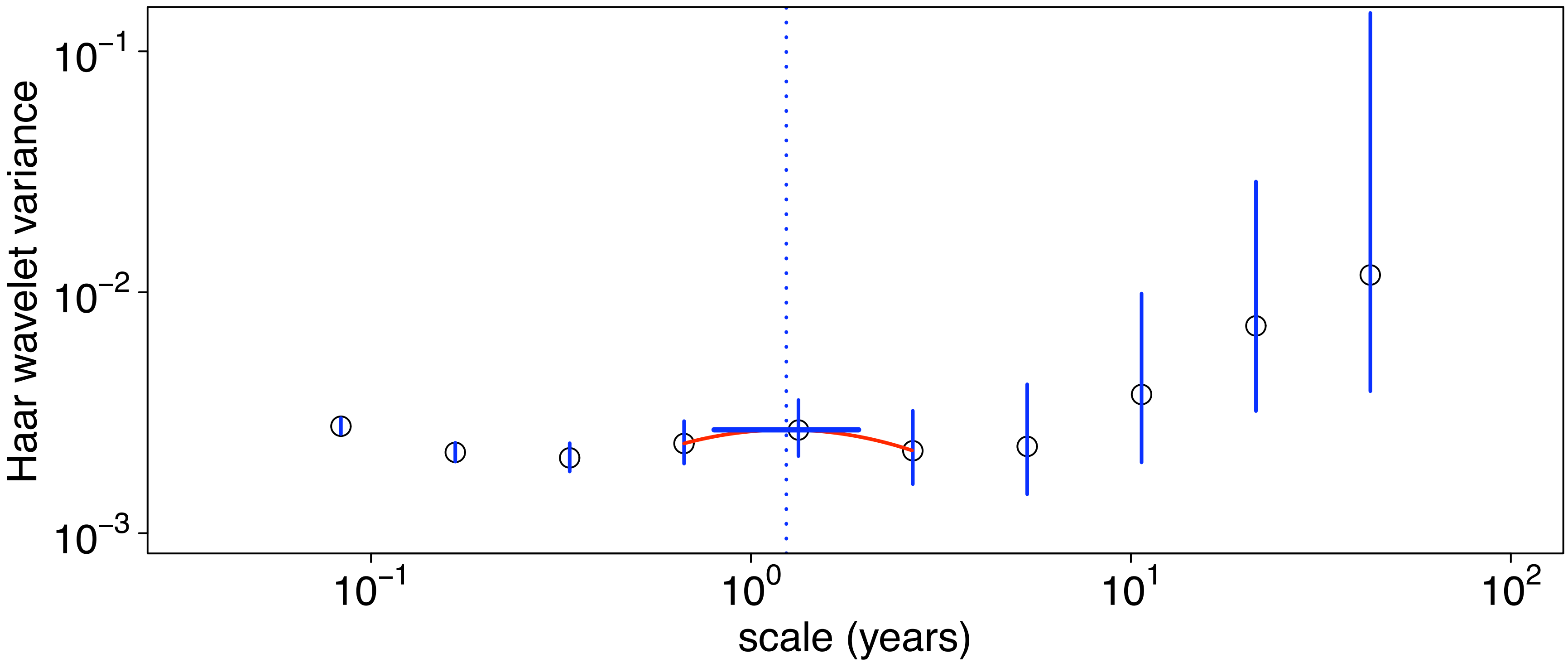
Global Temperature Record: I

- monthly global temperature anomalies (land and water combined) from Jan. 1880 to Dec. 2009 ($N = 1560$; data obtained from NOAA Web site); note prominent trend upwards



Global Temperature Record: II

- $\hat{\tau}_{c,5} \doteq 14.9$ months with associated 95% CI of $[9.6, 23.0]$

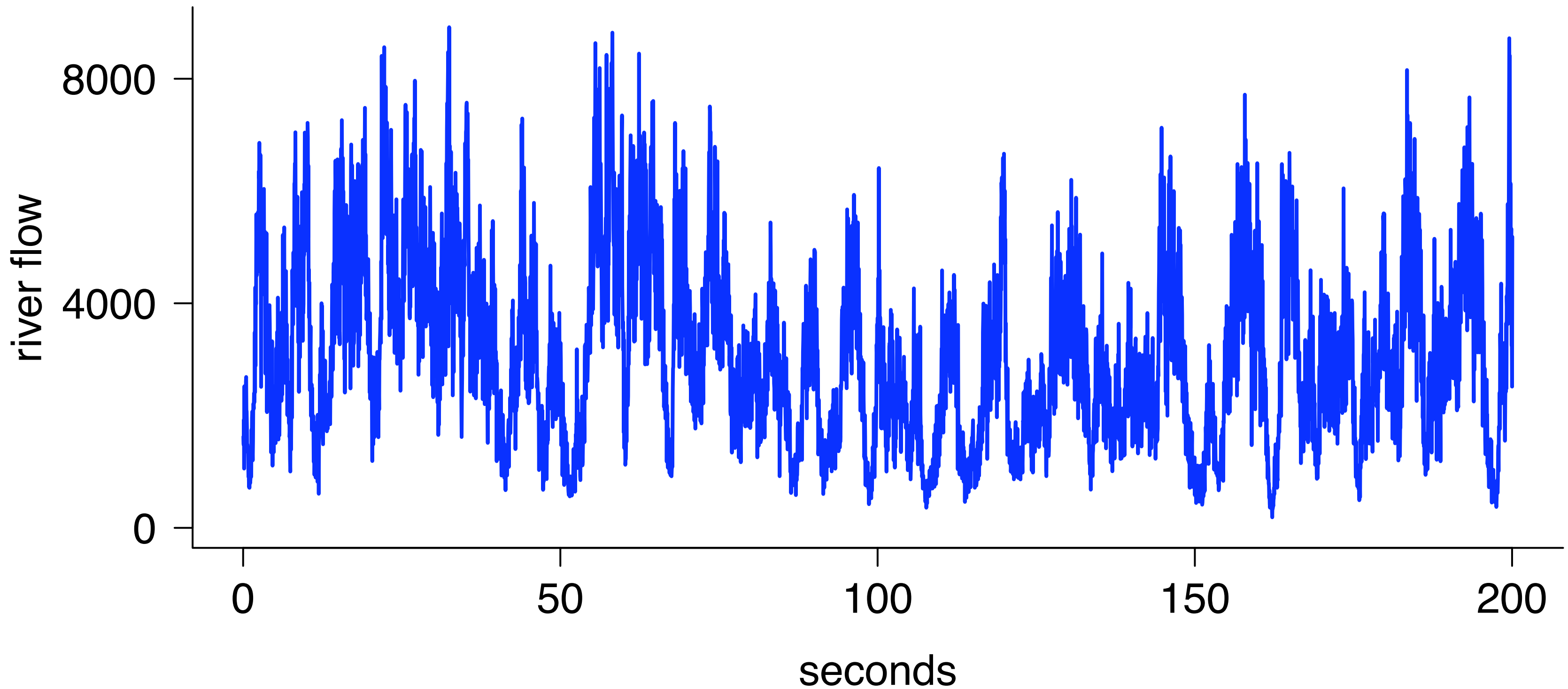


Global Temperature Record: III

- Tsonis et al. (1998) detrended series using singular spectrum analysis and used residuals to estimate a CS of 20 months via involved procedure based on cumulative sums
- interpreted CS as being due to influence of El Niño/La Niña cycles on global temperatures
- 95% CI for wavelet-based CS traps 20 months, but no need for explicit detrending due to differencing operations embedded within wavelet filters (eliminates trends that can be modelled by low-order polynomials, as verified here by use of length $L_1 = 8$ wavelet filter)

Coherent Structures in River Flows: I

- first 5000 values from time series of length $N = 29972$ sampled every $\frac{1}{25}$ sec. from Univ. of WA velocity profiler set on bottom of river estuary downstream from sill pointing upwards

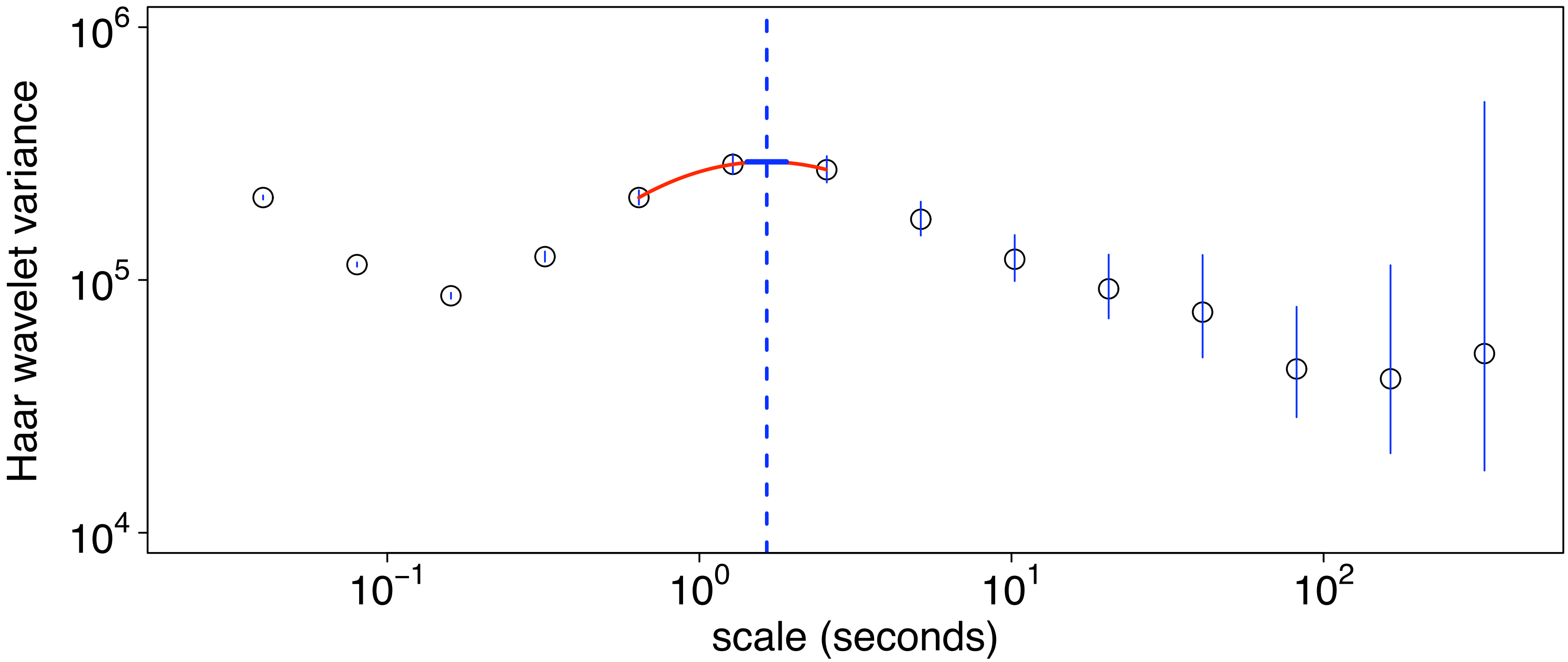


Coherent Structures in River Flows: II

- videos of river surface clearly show quasi-periodic upwellings from river appearing as temporary ‘blobs’ (coherent structures), with each blob dissipating within a second or so, with another one forming immediately afterwards
- quantifying this little-understood phenomenon using standard Fourier-based spectral analysis problematic because it appears as a small perturbation in a low-frequency rolloff
- by contrast, as shown on next overhead, WV clearly displays a peak, rendering phenomenon as interpretable in terms of a CS
- time-evolving properties of blobs can be studied by estimating CSs for portions of time series spanning successive 20-minute time intervals

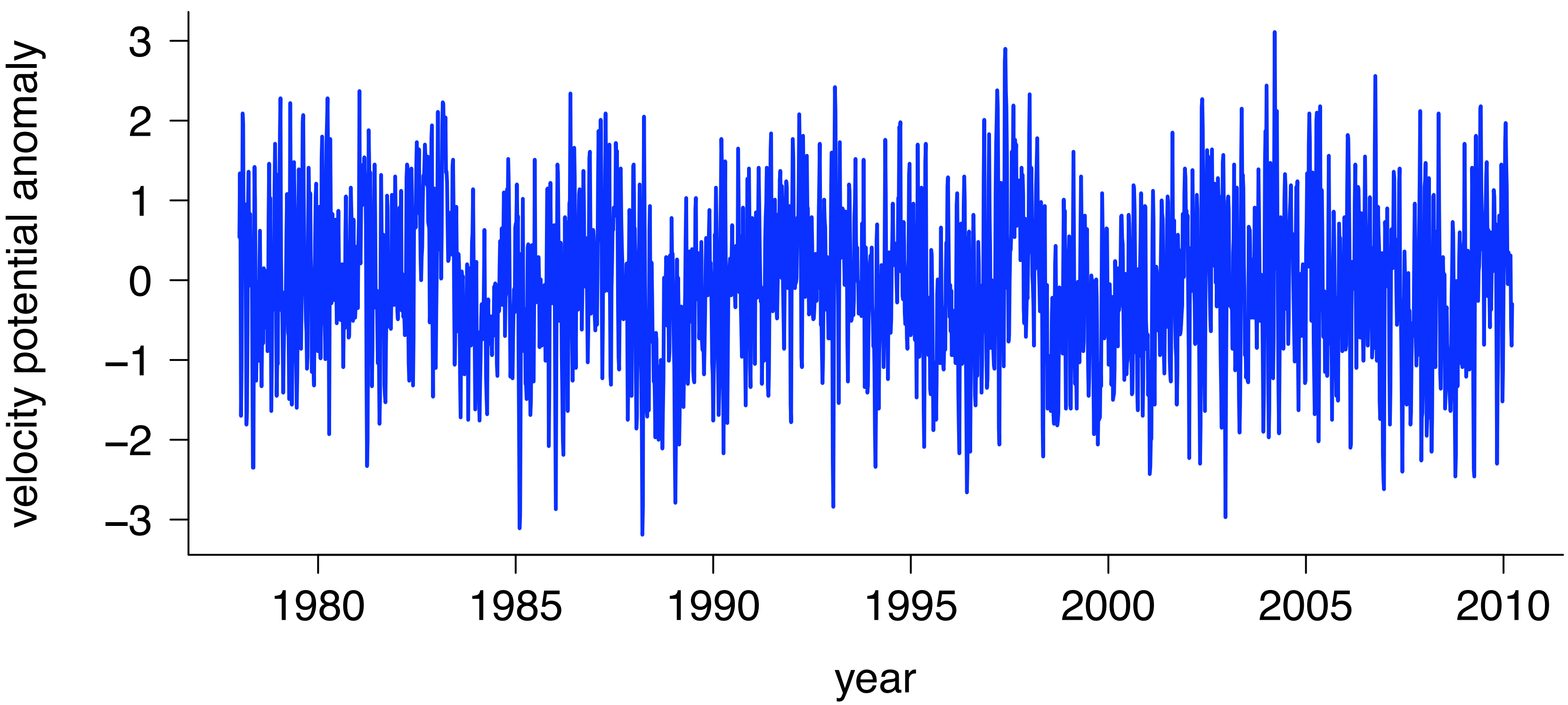
Coherent Structures in River Flows: III

- $\hat{\tau}_{c,6} = 1.6$ seconds with associated 95% CI of $[1.4, 1.9]$



Madden–Julian Atmospheric Oscillation: I

- 200 hectopascal velocity potential anomalies, a Madden–Julian oscillation (MJO) index from NOAA Web site ($N = 2354$ values covering 32+ years with sampling interval of 5 days)

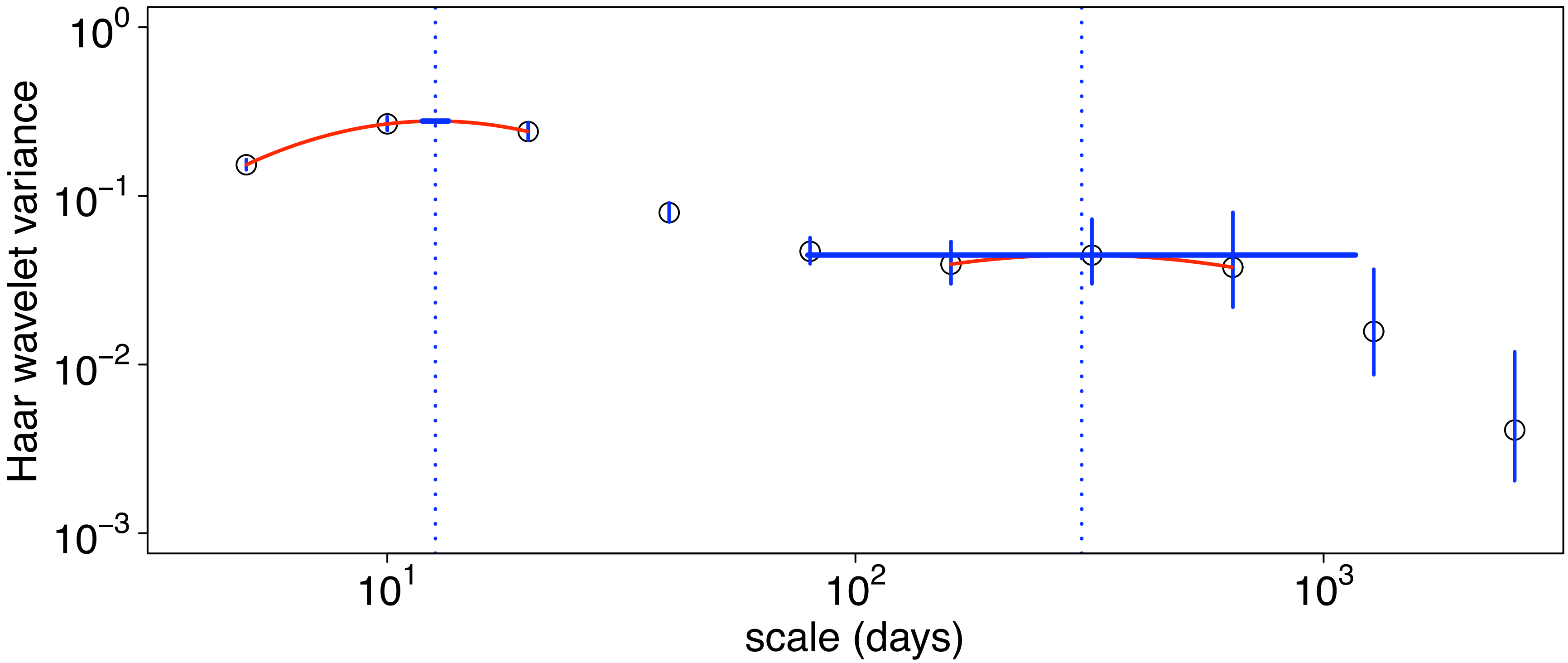


Madden–Julian Atmospheric Oscillation: II

- Madden and Julian (1994) originally discussed 40–50 day oscillation appearing in various atmospheric time series collected in tropics
- MJO now called 30–60 day or intraseasonal oscillation

Madden–Julian Atmospheric Oscillation: III

- $\hat{\tau}_{c,2} = 12.7$ days with associated 95% CI of $[11.9, 13.5]$

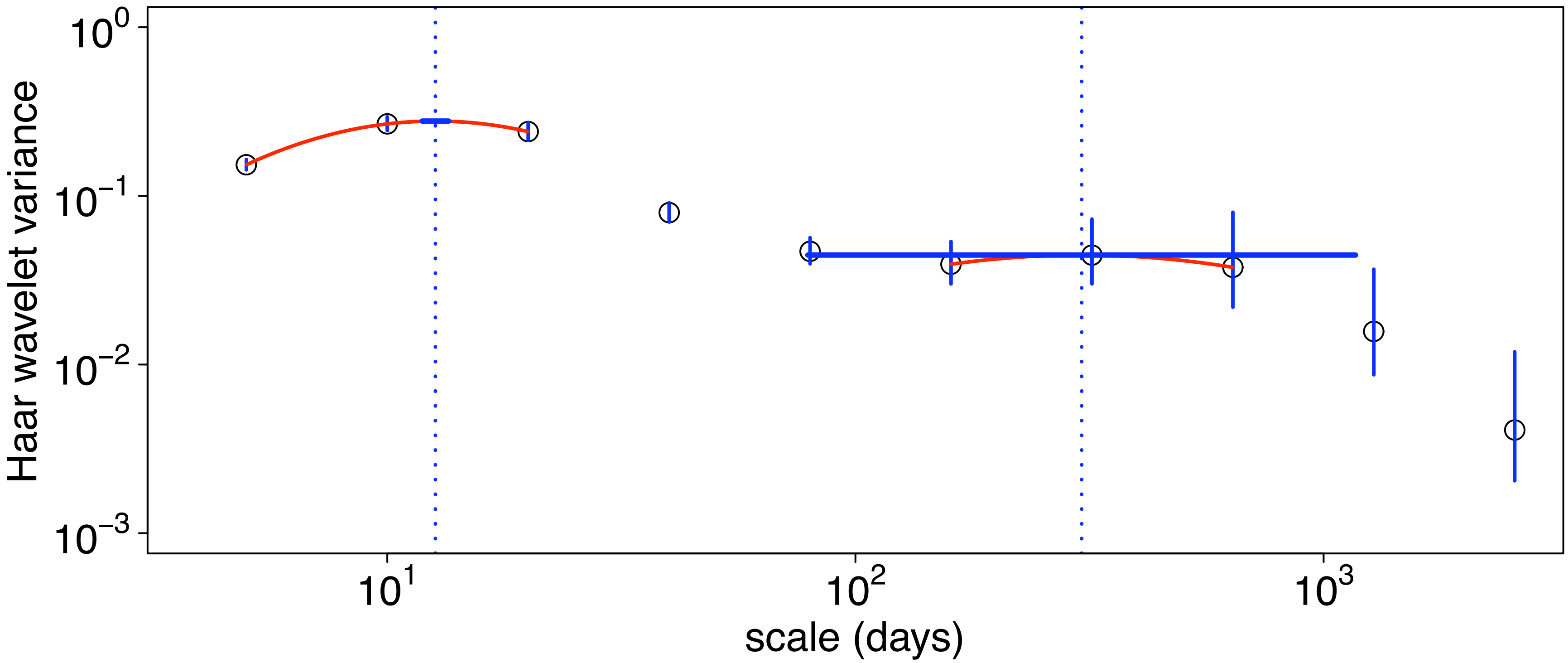


Madden–Julian Atmospheric Oscillation: IV

- since scale of τ is associated with periods in interval $[2\tau, 4\tau]$, CS point estimate of 12.7 days matches up with 25–51 day oscillations and hence with description of MJO as a 30–60 day oscillation
- deduction of MJO from Fourier-based spectral analysis problematic due to lack of standard way to determine beginning/end of frequency interval associated with broadband oscillations
- notion of CS bypasses this difficulty, opening up means of objectively tracking how MJO varies across time and over different time series

Madden–Julian Atmospheric Oscillation: V

- $\hat{\tau}_{c,7} \doteq 304$ days with associated 95% CI of $[79, 1170]$

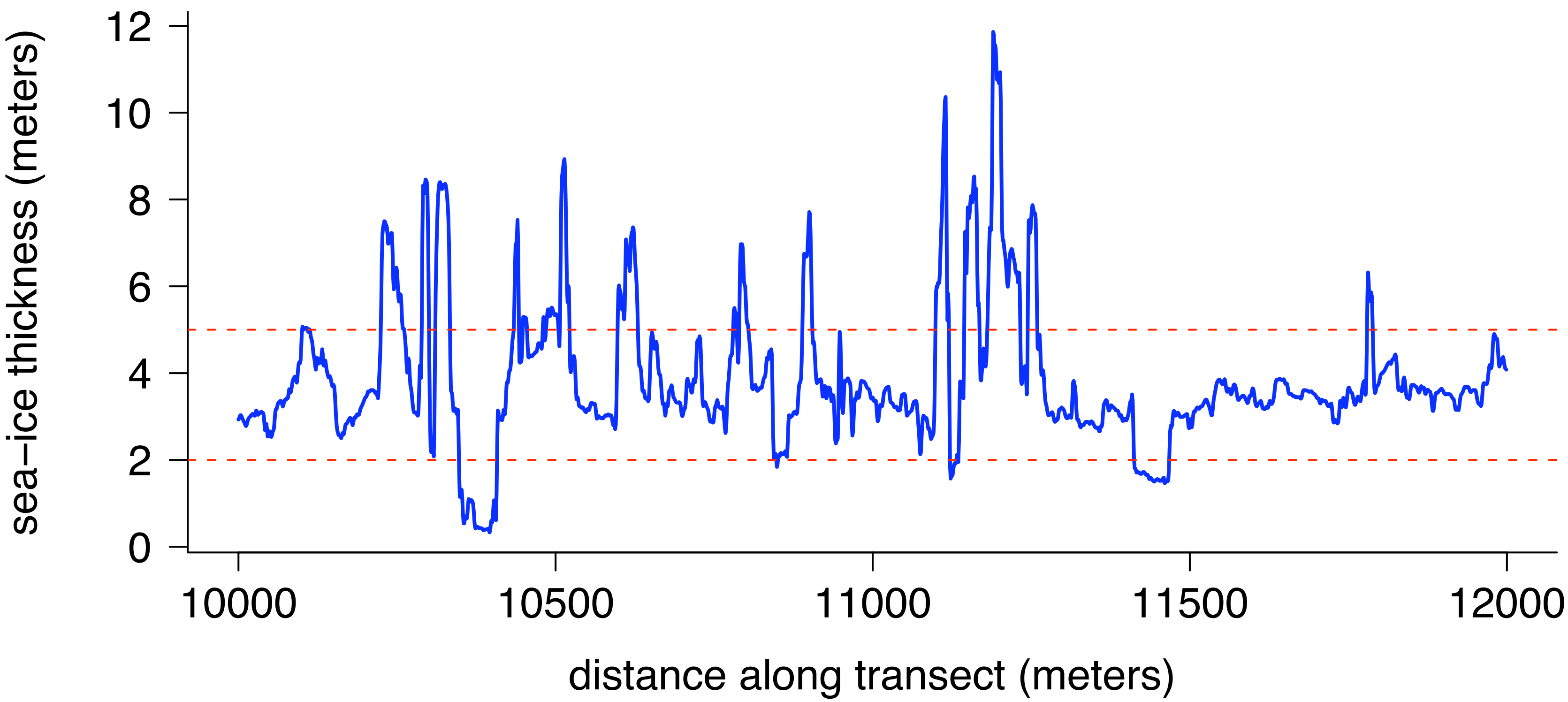


Madden–Julian Atmospheric Oscillation: VI

- interval of periods associated with second CS suggests an oscillation spanning two to three years that is about 5 times weaker than MJO
- presence of this weak CS is conditional upon peak pattern in WV being correct
- as example of use of sanity check, generated $100,000$ realizations of trivariate Gaussian with mean $[\hat{\nu}_6^2, \hat{\nu}_7^2, \hat{\nu}_8^2]^T$ & covariance matrix given by large sample theory
- of these realizations, 60% obeyed observed $\hat{\nu}_6^2 \leq \hat{\nu}_7^2 \geq \hat{\nu}_8^2$ pattern, but remaining 40% did not, casting considerable doubt on validity of observed peak pattern
- similar test on MJO-related CS yielded $99,916$ realizations with observed peak pattern and only 84 without

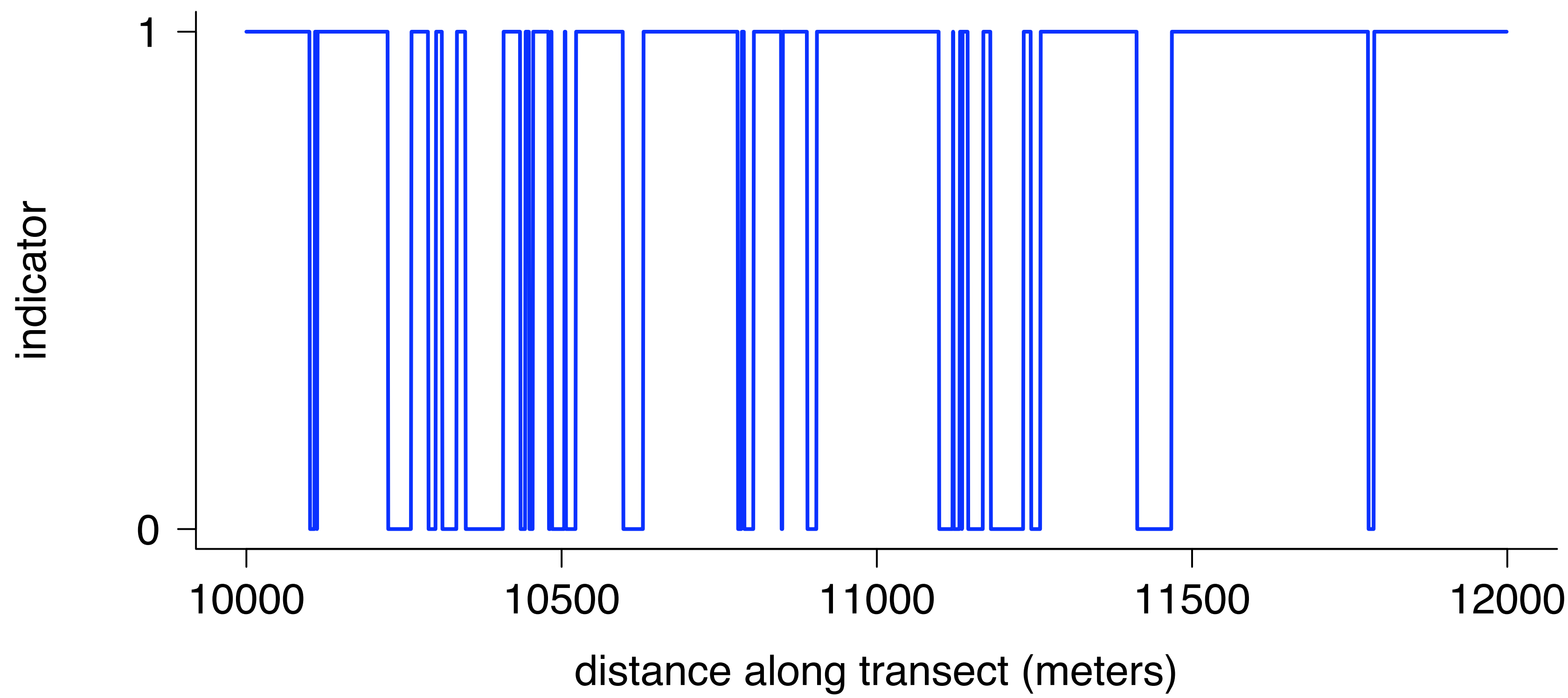
Medium Multiyear Arctic Sea Ice: I

- part of ice thickness measurements spaced a meter apart from a 50 km track collected using an upward-looking sonar on a submarine (data from U.S. National Snow and Ice Data Center)



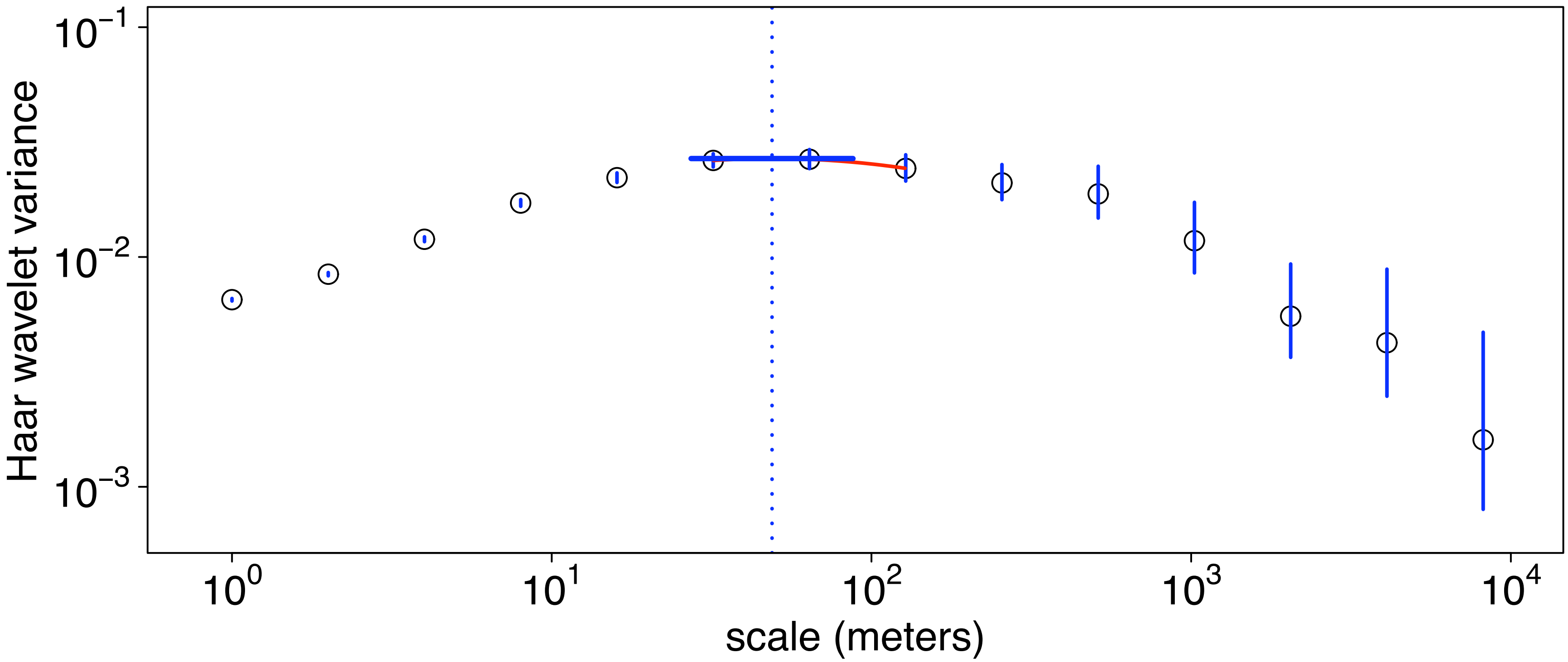
Medium Multiyear Arctic Sea Ice: II

- binary series indicating presence (1) or absence (0) of medium multiyear Arctic sea ice, which by definition has a thickness between 2 and 5 m



Medium Multiyear Arctic Sea Ice: III

- $\hat{\tau}_{c,7} \doteq 48.9$ m with associated 95% CI of $[27.3, 87.5]$



Medium Multiyear Arctic Sea Ice: IV

- CS is indicator of 'typical' extent of medium multiyear ice
- in face of other evidence that Arctic climate is dramatically changing, question of considerable interest is how stable CSs for different ice types are both spatially and temporally
- note: evidence of long-range dependence in ice thickness measurements implies long-range dependence in indicator series also, meaning that decorrelation time τ_D would be infinite for the medium multiyear ice indicators, rendering it useless as a measure of CS
- by contract, wavelet-based CS is finite and provides a useful summary of one aspect of indicator series

Concluding Remarks: I

- CS based upon peaks in WV curves has certain advantages over other definitions for CS, including
  - use of well-defined scale-based decomposition of variance afforded by DWT
  - ability to focus on localized properties of process rather than asymptotic decay rates of autocorrelation sequences
  - ability to handle certain nonstationary processes
  - ability to handle series with trends well approximated by low-order polynomials
  - availability of tractable large-sample theory that is applicable to time series of moderate size

Concluding Remarks: II

- avenues for future research
  - extension to non-Gaussian process (including better handling of indicator series)
  - handling irregularly sampled time series
  - extension to two-dimensional data, with potential applications in medical image processing (telemedicine?)

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