Wavelet Analysis of Clock Noise

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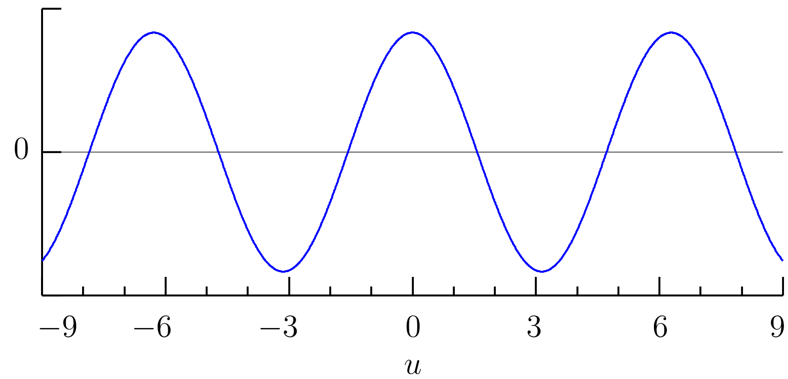
Overview

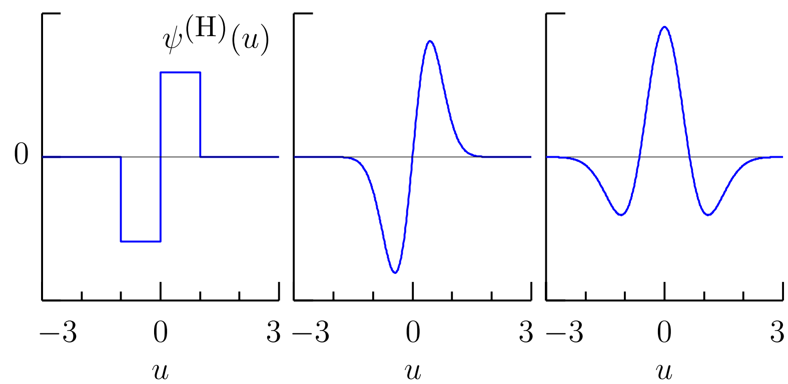
- as a subject, wavelets are
  - relatively new (1983 to present)
  - a synthesis of old/new ideas
  - a keyword in $20,652+$ articles since 1990 (a tidal wave!!!)
- wavelets decompose time series over time & different scales
  - time series = sequence of observations collected over time
  - scale = interval (span) of time (e.g., second, day, . . .)
- in PTTI applications, wavelets can help in
  - characterization of frequency instability
  - estimation of parameters for statistical models
  - potentially other areas

What is a Wavelet?

- sines & cosines are 'big waves'

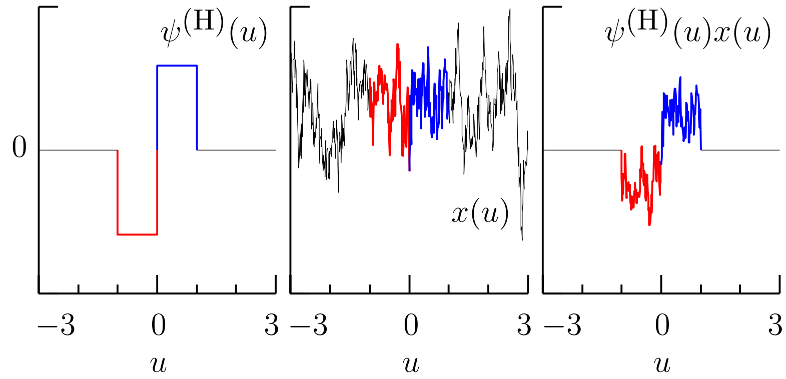


- wavelets are 'small waves' (left-hand is Haar wavelet $\psi^{(\mathrm{H})}(u)$)



What is Wavelet Analysis?: I

- multiply wavelet & time series $x(u)$ together & integrate:

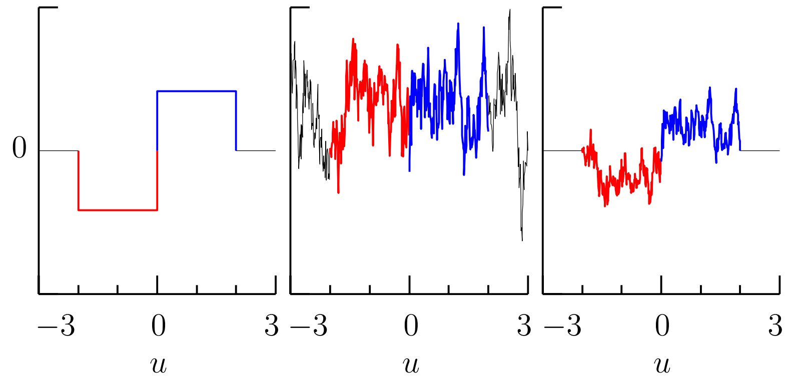


- $\int_{-\infty}^{\infty} \psi^{(\mathrm{H})}(u)x(u)\,du = W(1,0)$ is proportional to difference between averages of $x(u)$ over intervals $[-1,0]$ and $[0,1]$

- defines wavelet coefficient $W(1,0)$ for
  - scale 1 (width of each interval)
  - time 0 (center of combined intervals)

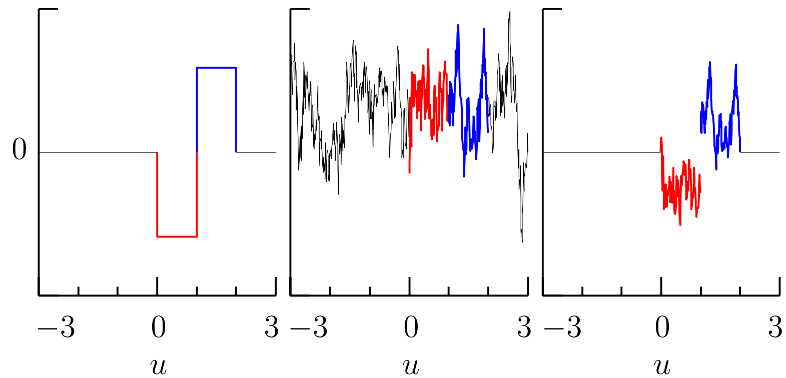
What is Wavelet Analysis?: II

- stretch or shrink wavelet to define $W(\tau, 0)$ for other scales τ:



yields $W(2, 0)$

- relocate to define $W(\tau, t)$ for other times t:



yields $W(1, 1)$

What is Wavelet Analysis?: III

- $W(\tau, t)$ over all scales $\tau > 0$ and all times t called continuous wavelet transform (CWT) for $x(u)$
- CWT analyzes $x(u)$ into components that are
  - associated with a scale and a time
  - physically related to a difference of averages
- similar interpretation for other wavelets $\psi(u)$
- $W(\tau, t)$ equivalent to $x(u)$ since, given CWT, can recover $x(u)$:

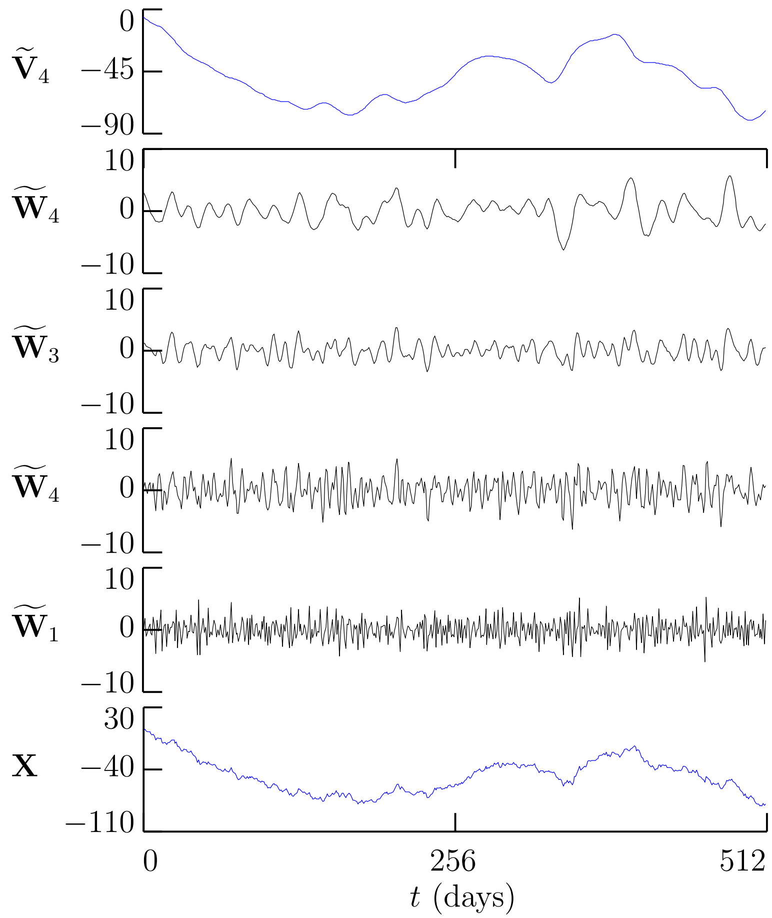
$$x(u) = \frac{1}{C_\psi} \int_0^\infty \frac{1}{\tau^2} \left[\int_{-\infty}^\infty W(\tau, t) \frac{1}{\sqrt{\tau}} \psi\left(\frac{u-t}{\tau}\right) dt \right] d\tau,$$

where C_ψ is a constant depending on specific wavelet $\psi(u)$

Maximal Overlap Discrete Wavelet Transform

- let $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ be observed time series
- can formulate MODWT of $\mathbf{X}$ as vectors $\widetilde{\mathbf{W}}_1, \ldots, \widetilde{\mathbf{W}}_{J_0}$ & $\widetilde{\mathbf{V}}_{J_0}$, each of dimension N (number of levels J_0 chosen by user)
- $\widetilde{\mathbf{W}}_j$ contains wavelet coefficients, $j = 1, \ldots, J_0$
  - associated with differences in averages over scale $\tau_j = 2^{j-1}$
  - closely related to $W(\tau_j, t)$ over restricted set of times
- $\widetilde{\mathbf{V}}_{J_0}$ contains scaling coefficients
  - associated with averages over scale $2\tau_{J_0} = 2^{J_0}$
  - summarizes $W(\tau, t)$ over scales $\tau > \tau_{J_0}$
- $\mathbf{X}$ & MODWT equivalent: given MODWT, can recover $\mathbf{X}$

Example: MODWT Coefficients for Clock 55



- can use to track variations across time at a given scale

Wavelet-Based Analysis of Variance: I

- consider 'energy' in time series: $\|\mathbf{X}\|^2 = \mathbf{X}^T\mathbf{X} = \sum_{t=0}^{N-1} X_t^2$
- energy preserved in MODWT coefficients:

$$\|\mathbf{X}\|^2 = \sum_{j=1}^{J_0} \|\widetilde{\mathbf{W}}_j\|^2 + \|\widetilde{\mathbf{V}}_{J_0}\|^2$$

- leads to analysis of sample variance:

$$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{t=0}^{N-1}(X_t - \hat{\mu}_X)^2 = \frac{1}{N}\left(\sum_{j=1}^{J_0}\|\widetilde{\mathbf{W}}_j\|^2 + \|\widetilde{\mathbf{V}}_{J_0}\|^2\right) - \hat{\mu}_X^2,$$

where $\hat{\mu}_X = \frac{1}{N}\sum_{t=0}^{N-1} X_t$ is sample mean

Wavelet-Based Analysis of Variance: II

- if $\mathbf{X}$ realization of process with stationary increments, $\|\widetilde{\mathbf{W}}_j\|^2/N$ is estimator of wavelet variance $\nu_X^2(\tau_j)$
- wavelet variance analyzes process variance σ_X^2 across scales τ_j:

$$\sigma_X^2 = \text{var}\,\{X_t\} = \sum_{j=1}^{\infty} \nu_X^2(\tau_j)$$

(note: σ_X^2 can be infinite for certain processes)

- special case: Haar wavelet variance with fractional frequency deviates $\overline{Y}_t$ essentially same as Allan variance $\sigma_{\overline{Y}}^2(2,\tau_j)$ since

$$\nu_{\overline{Y}}^2(\tau_j) = \tfrac{1}{2}\sigma_{\overline{Y}}^2(2,\tau_j)$$

- Q: 'old wine in a new bottle,' or something new?

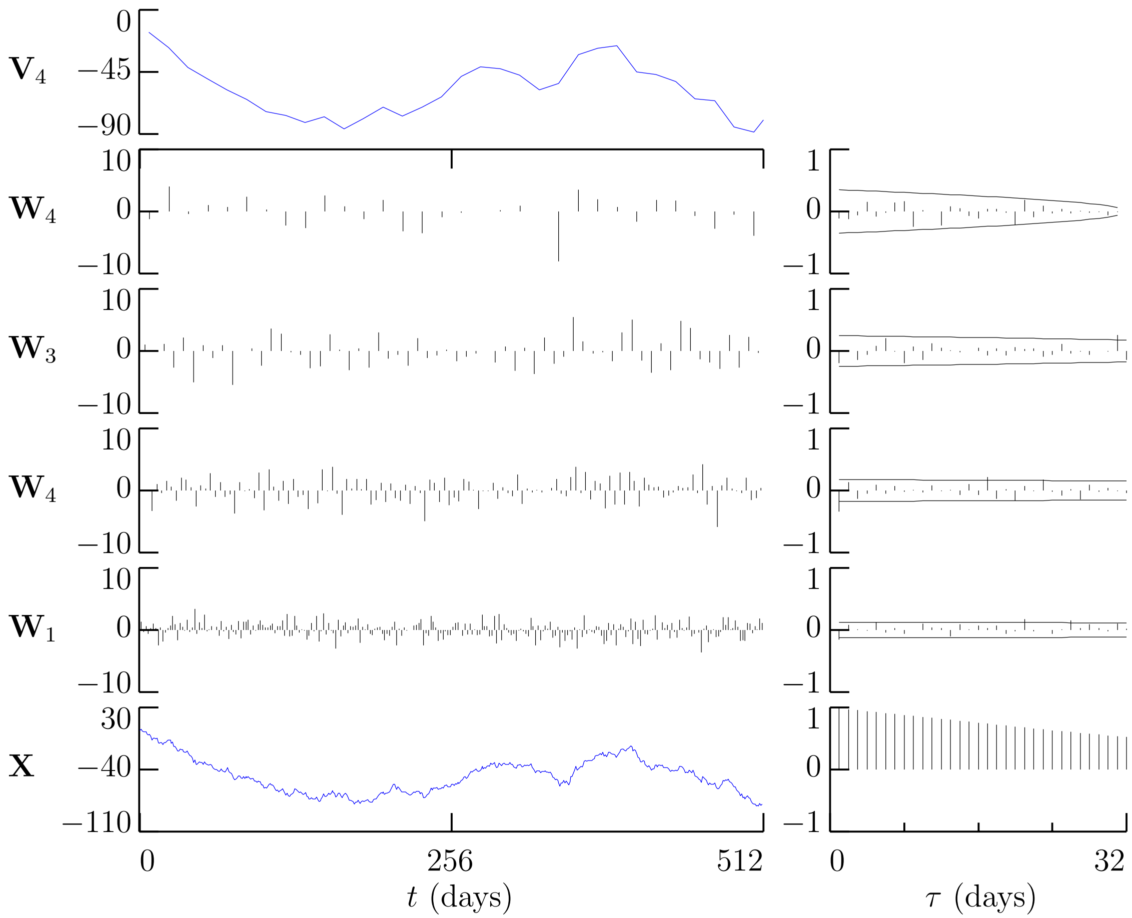
What Wavelets Bring to the Table

- $2\|\widetilde{\mathbf{W}}_j\|^2/N$ gives a previously unknown estimator for $\sigma^2_{\overline{Y}}(2, \tau_j)$
- with addition of 'reflection' boundary conditions, estimator is an improvement over existing estimators (smaller mean square error; Greenhall, Howe & Percival, 1999)
- non-Haar wavelets provide interesting generalizations
  - still provide exact decompositions of sample variance
  - can handle wider range of power laws
  - can handle polynomial trends of certain orders
  - competitive with modified Allan variance
- unified theory provides methods for getting confidence intervals that do not require *a prior* assumption of noise type

Discrete Wavelet Transform (DWT)

- obtain by subsampling and rescaling MODWT
- yields vectors $\mathbf{W}_1, \ldots, \mathbf{W}_{J_0}$ & $\mathbf{V}_{J_0}$
  - $\mathbf{W}_j$ has $N/2^j$ wavelet coefficients
  - $\mathbf{V}_{J_0}$ has $N/2^{J_0}$ scaling coefficients
- total # of DWT coefficients is N, i.e., dimension of $\mathbf{X}$
- $\mathbf{X}$ & DWT equivalent: given DWT, can recover $\mathbf{X}$
- DWT acts as a decorrelating transform

Example: DWT Coefficients for Clock 55 X_t



- have approximate within-scale & between-scale decorrelation (non-Haar wavelets offer better between-scale approximation)

Uses for DWT Decorrelating Property

- estimation of parameters for statistical models
  - consider modeling $\mathbf{X}$ as process with spectrum
  $$S_X(f) = C|f|^{\alpha}$$
  - power-law model depends on parameters C and α
  - consider estimating C and α via maximum likelihood (ML)
  - exact ML estimators difficult to obtain
  - DWT yields simple, but effective, approximate ML estimator
- testing for homogeneity of $\mathbf{X}$ at scale τ_j across time
- assessing variability in certain statistics via bootstrapping
- fast simulation of time series

Other Potential Uses for Wavelets in PTTI

- multiresolution analysis (based on wavelet synthesis of $\mathbf{X}$)
- detection of singularities (maximum modulus of CWT)
- data compression
- signal extraction (wavelet shrinkage)

Summary

- wavelets give insight into frequency instability characterization
  - emphasize role of exact analysis of process/sample variance
  - provide estimators with reduced mean square error
- wavelets lead to easily computed approximate maximum likelihood estimators for parameters of power-law processes
- many other potential uses
  - article #20,654 is waiting to be written!
- thanks to conference organizers for invitation to speak!

References

1. Greenhall, C. A., Howe, D. A., Percival, D. B., *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, 1999, **46**, 1183–1191.

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3. Percival, D. B., Walden, A. T., *Wavelet Methods for Time Series Analysis*, Cambridge, UK, Cambridge University Press, 2000,