

Wavelet Methods for Time Series Analysis

Part II: Wavelet-Based Statistical Analysis of Time Series

- topics to covered:
 - wavelet variance (analysis phase of MODWT)
 - wavelet-based signal extraction (synthesis phase of DWT)
 - wavelet-based decorrelation of time series (analysis phase of DWT, but synthesis phase plays a role also)

Wavelet Variance: Overview

- review of decomposition of sample variance using wavelets
- theoretical wavelet variance for stochastic processes
 - stationary processes
 - nonstationary processes with stationary differences
- sampling theory for Gaussian processes
- real-world examples
- extensions and summary

Decomposing Sample Variance of Time Series

- let X_0, X_1, \dots, X_{N-1} represent time series with N values
- let \bar{X} denote sample mean of X_t 's: $\bar{X} \equiv \frac{1}{N} \sum_{t=0}^{N-1} X_t$
- let $\hat{\sigma}_X^2$ denote sample variance of X_t 's:

$$\hat{\sigma}_X^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \bar{X})^2$$

- idea is to decompose (analyze, break up) $\hat{\sigma}_X^2$ into pieces that quantify how one time series might differ from another
- wavelet variance does analysis based upon differences between (possibly weighted) adjacent averages over scales

Empirical Wavelet Variance

- define empirical wavelet variance for scale $\tau_j \equiv 2^{j-1}$ as

$$\tilde{\nu}_X^2(\tau_j) \equiv \frac{1}{N} \sum_{t=0}^{N-1} \widetilde{W}_{j,t}^2, \quad \text{where } \widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}$$

- if $N = 2^J$, obtain analysis (decomposition) of sample variance:

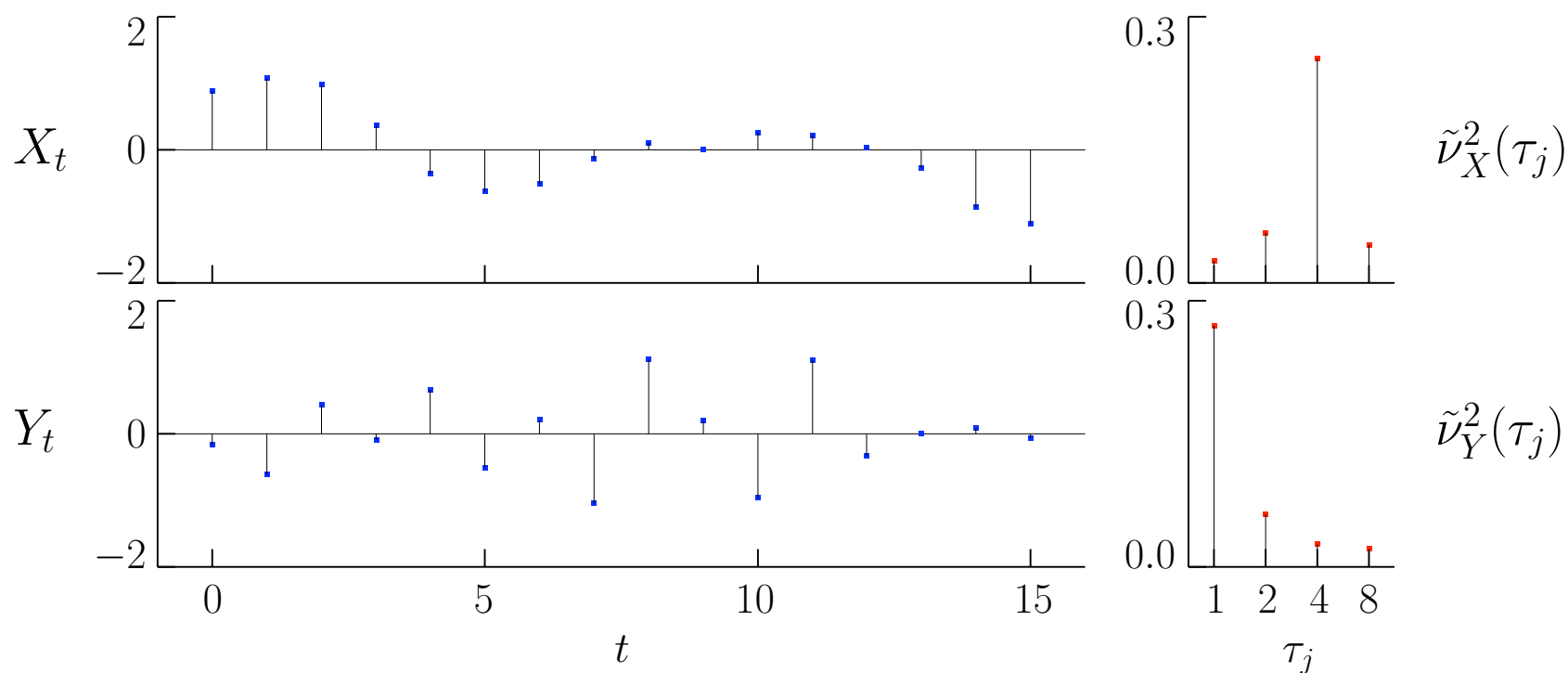
$$\hat{\sigma}_X^2 = \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \bar{X})^2 = \sum_{j=1}^J \tilde{\nu}_X^2(\tau_j)$$

(if N not a power of 2, can analyze variance to any level J_0 , but need additional component involving scaling coefficients)

- interpretation: $\tilde{\nu}_X^2(\tau_j)$ is portion of $\hat{\sigma}_X^2$ due to changes in averages over scale τ_j ; i.e., ‘scale by scale’ analysis of variance

Example of Empirical Wavelet Variance

- wavelet variances for time series X_t and Y_t of length $N = 16$, each with zero sample mean and same sample variance



Theoretical Wavelet Variance: I

- now assume X_t is a real-valued random variable (RV)
- let $\{X_t, t \in \mathbb{Z}\}$ denote a stochastic process, i.e., collection of RVs indexed by ‘time’ t (here \mathbb{Z} denotes the set of all integers)
- apply j th level equivalent MODWT filter $\{\tilde{h}_{j,l}\}$ to $\{X_t\}$ to create a new stochastic process:

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t \in \mathbb{Z},$$

which should be contrasted with

$$\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \dots, N - 1$$

Theoretical Wavelet Variance: II

- if Y is any RV, let $E\{Y\}$ denote its expectation
- let $\text{var}\{Y\}$ denote its variance: $\text{var}\{Y\} \equiv E\{(Y - E\{Y\})^2\}$
- definition of time dependent wavelet variance:

$$\nu_{X,t}^2(\tau_j) \equiv \text{var}\{\overline{W}_{j,t}\},$$

with conditions on X_t so that $\text{var}\{\overline{W}_{j,t}\}$ exists and is finite

- $\nu_{X,t}^2(\tau_j)$ depends on τ_j and t
- will focus on time independent wavelet variance

$$\nu_X^2(\tau_j) \equiv \text{var}\{\overline{W}_{j,t}\}$$

(can adapt theory to handle time varying situation)

- $\nu_X^2(\tau_j)$ well-defined for stationary processes and certain related processes, so let's review concept of stationarity

Definition of a Stationary Process

- if U and V are two RVs, denote their covariance by

$$\text{cov}\{U, V\} = E\{(U - E\{U\})(V - E\{V\})\}$$

- stochastic process X_t called stationary if
 - $E\{X_t\} = \mu_X$ for all t , i.e., constant independent of t
 - $\text{cov}\{X_t, X_{t+\tau}\} = s_{X,\tau}$, i.e., depends on lag τ , but not t
- $s_{X,\tau}$, $\tau \in \mathbb{Z}$, is autocovariance sequence (ACVS)
- $s_{X,0} = \text{cov}\{X_t, X_t\} = \text{var}\{X_t\}$; i.e., variance same for all t

Wavelet Variance for Stationary Processes

- for stationary processes, wavelet variance decomposes $\text{var} \{X_t\}$:

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var} \{X_t\},$$

which is similar to

$$\sum_{j=1}^J \tilde{\nu}_X^2(\tau_j) = \hat{\sigma}_X^2$$

- $\nu_X^2(\tau_j)$ is thus contribution to $\text{var} \{X_t\}$ due to scale τ_j
- note: $\nu_X^2(\tau_j)$ and X_t^2 have same units (can be important for interpretability)

White Noise Process

- simplest example of a stationary process is ‘white noise’
- process X_t said to be white noise if
 - it has a constant mean $E\{X_t\} = \mu_X$
 - it has a constant variance $\text{var}\{X_t\} = \sigma_X^2$
 - $\text{cov}\{X_t, X_{t+\tau}\} = 0$ for all t and nonzero τ ; i.e., distinct RVs in the process are uncorrelated
- ACVS for white noise takes a very simple form:

$$s_{X,\tau} = \text{cov}\{X_t, X_{t+\tau}\} = \begin{cases} \sigma_X^2, & \tau = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Wavelet Variance for White Noise Process: I

- for a white noise process, can show that

$$\nu_X^2(\tau_j) = \frac{\text{var} \{X_t\}}{2^j} \propto \tau_j^{-1} \text{ since } \tau_j = 2^{j-1}$$

- note that

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var} \{X_t\} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) = \text{var} \{X_t\},$$

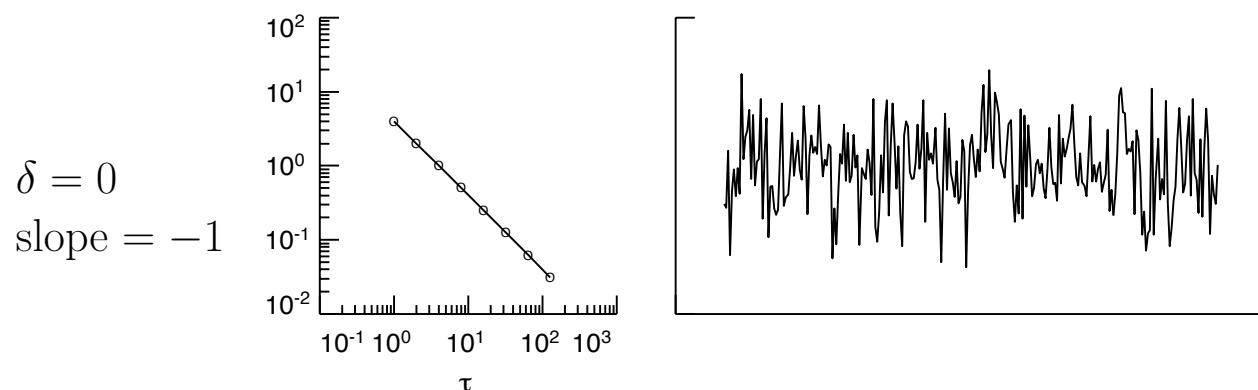
as required

- note also that

$$\log(\nu_X^2(\tau_j)) \propto -\log(\tau_j),$$

so plot of $\log(\nu_X^2(\tau_j))$ vs. $\log(\tau_j)$ is linear with a slope of -1

Wavelet Variance for White Noise Process: II



- $\nu_X^2(\tau_j)$ versus τ_j for $j = 1, \dots, 8$ (left-hand plot), along with sample of length $N = 256$ of Gaussian white noise
- largest contribution to $\text{var} \{X_t\}$ is at smallest scale τ_1
- note: later on, we will discuss fractionally differenced (FD) processes that are characterized by a parameter δ ; when $\delta = 0$, an FD process is the same as a white noise process

Generalization to Certain Nonstationary Processes

- if wavelet filter is properly chosen, $\nu_X^2(\tau_j)$ well-defined for certain processes with stationary backward differences (increments); these are also known as intrinsically stationary processes
- first order backward difference of X_t is process defined by

$$X_t^{(1)} = X_t - X_{t-1}$$

- second order backward difference of X_t is process defined by

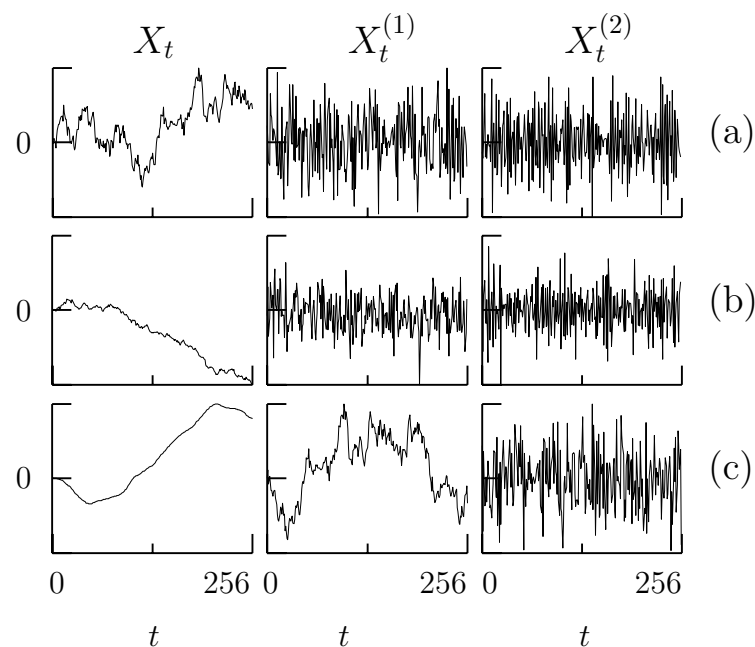
$$X_t^{(2)} = X_t^{(1)} - X_{t-1}^{(1)} = X_t - 2X_{t-1} + X_{t-2}$$

- X_t said to have d th order stationary backward differences if

$$Y_t \equiv \sum_{k=0}^d \binom{d}{k} (-1)^k X_{t-k}$$

forms a stationary process (d is a nonnegative integer)

Examples of Processes with Stationary Increments



- 1st column shows, from top to bottom, realizations from
 - (a) random walk: $X_t = \sum_{u=1}^t \epsilon_u$, & ϵ_t is zero mean white noise
 - (b) like (a), but now ϵ_t has mean of -0.2
 - (c) random run: $X_t = \sum_{u=1}^t Y_u$, where Y_t is a random walk
- 2nd & 3rd columns show 1st & 2nd differences $X_t^{(1)}$ and $X_t^{(2)}$

Wavelet Variance for Processes with Stationary Backward Differences: I

- let $\{X_t\}$ be nonstationary with d th order stationary differences
- if we use a Daubechies wavelet filter of width L satisfying $L \geq 2d$, then $\nu_X^2(\tau_j)$ is well-defined and finite for all τ_j , but now

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \infty$$

- works because there is a backward difference operator of order $d = L/2$ embedded within $\{\tilde{h}_{j,l}\}$, so this filter reduces X_t to

$$\sum_{k=0}^d \binom{d}{k} (-1)^k X_{t-k} = Y_t$$

and then creates localized weighted averages of Y_t 's

Wavelet Variance for Random Walk Process: I

- random walk process $X_t = \sum_{u=1}^t \epsilon_u$ has first order ($d = 1$) stationary differences since $X_t - X_{t-1} = \epsilon_t$ (i.e., white noise)
- $L \geq 2d$ holds for all wavelets when $d = 1$; for Haar ($L = 2$),

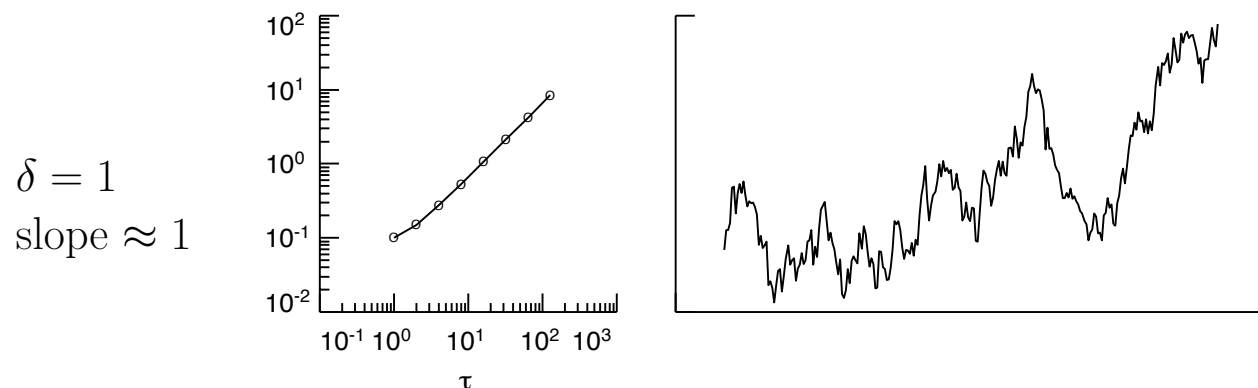
$$\nu_X^2(\tau_j) = \frac{\text{var}\{\epsilon_t\}}{6} \left(\tau_j + \frac{1}{2\tau_j} \right) \approx \frac{\text{var}\{\epsilon_t\}}{6} \tau_j,$$

with the approximation becoming better as τ_j increases

- note that $\nu_X^2(\tau_j)$ increases as τ_j increases
- $\log(\nu_X^2(\tau_j)) \propto \log(\tau_j)$ approximately, so plot of $\log(\nu_X^2(\tau_j))$ vs. $\log(\tau_j)$ is approximately linear with a slope of +1
- as required, also have

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \frac{\text{var}\{\epsilon_t\}}{6} \left(1 + \frac{1}{2} + 2 + \frac{1}{4} + 4 + \frac{1}{8} + \dots \right) = \infty$$

Wavelet Variance for Random Walk Process: II



- $\nu_X^2(\tau_j)$ versus τ_j for $j = 1, \dots, 8$ (left-hand plot), along with sample of length $N = 256$ of a Gaussian random walk process
- smallest contribution to $\text{var} \{X_t\}$ is at smallest scale τ_1
- note: a fractionally differenced process with parameter $\delta = 1$ is the same as a random walk process

Fractionally Differenced (FD) Processes: I

- can create a continuum of processes that ‘interpolate’ between white noise and random walks and ‘extrapolate’ beyond them using notion of ‘fractional differencing’ (Granger and Joyeux, 1980; Hosking, 1981)
- FD(δ) process is determined by 2 parameters, namely, δ and σ_ϵ^2 , where $-\infty < \delta < \infty$ and $\sigma_\epsilon^2 > 0$ (σ_ϵ^2 is less important than δ)
- if $\delta < 1/2$, FD process $\{X_t\}$ is stationary, and, in particular,
 - reduces to white noise if $\delta = 0$
 - has ‘long memory’ or ‘long range dependence’ if $\delta > 0$
 - is ‘antipersistent’ if $\delta < 0$ (i.e., $\text{cov}\{X_t, X_{t+1}\} < 0$)

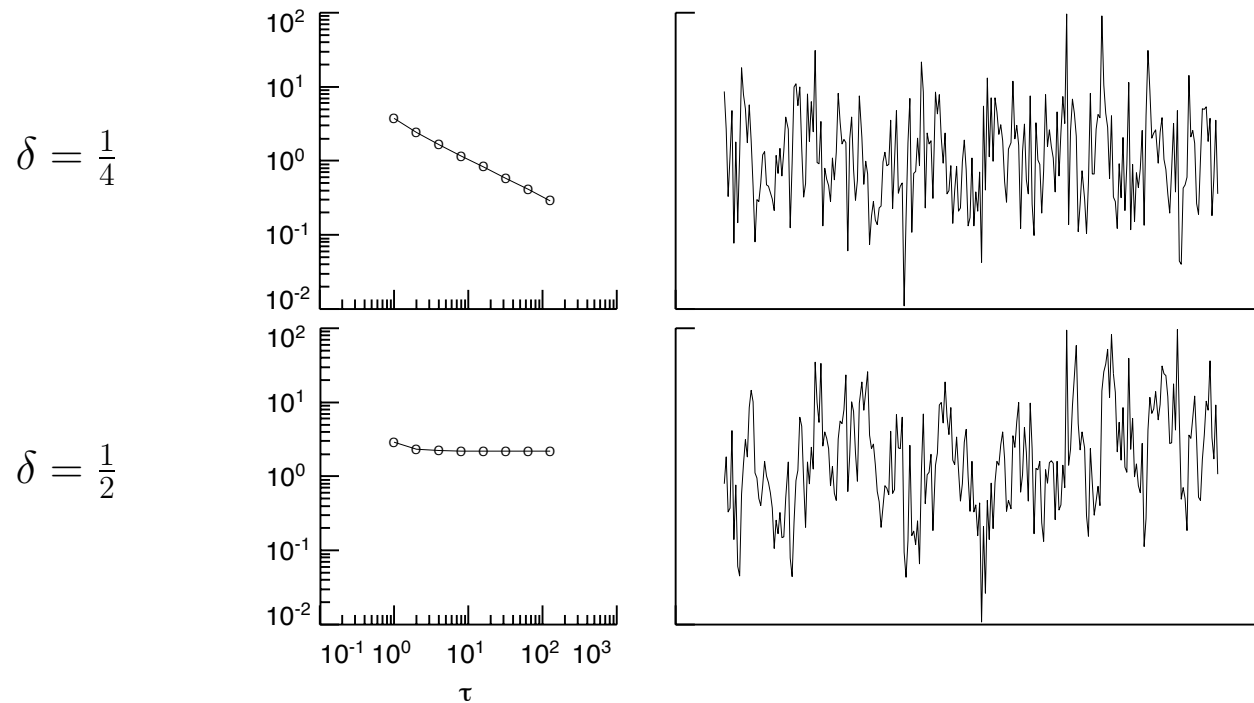
Fractionally Differenced (FD) Processes: II

- if $\delta \geq 1/2$, FD process $\{X_t\}$ is nonstationary with d th order stationary backward differences $\{Y_t\}$
 - here $d = \lfloor \delta + 1/2 \rfloor$, where $\lfloor x \rfloor$ is integer part of x
 - $\{Y_t\}$ is stationary FD($\delta - d$) process
- if $\delta = 1$, FD process is the same as a random walk process
- except possibly for two or three smallest scales, have

$$\nu_X^2(\tau_j) \approx C\tau_j^{2\delta-1}$$

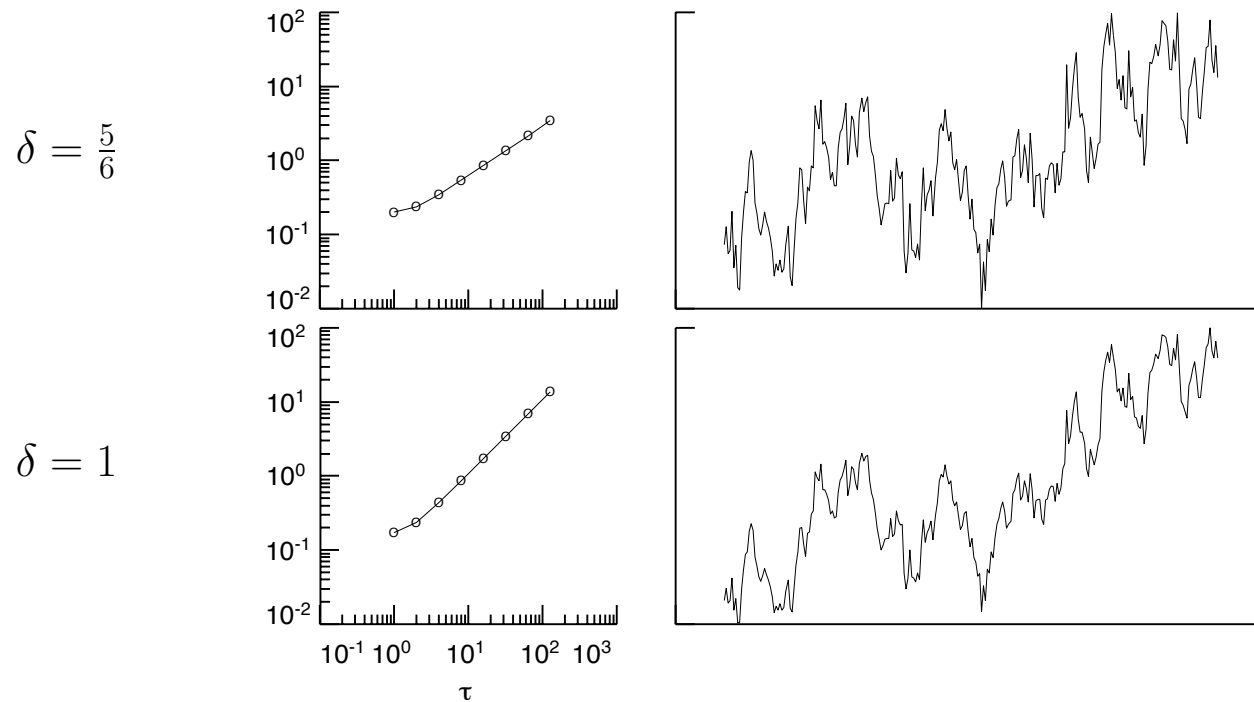
- thus $\log(\nu_X^2(\tau_j)) \approx \log(C) + (2\delta - 1)\log(\tau_j)$, so a log/log plot of $\nu_X^2(\tau_j)$ vs. τ_j looks approximately linear with slope $2\delta - 1$ for τ_j large enough

LA(8) Wavelet Variance for 2 FD Processes



- see overhead 12 for $\delta = 0$ (white noise), which has slope = -1
- $\delta = \frac{1}{4}$ has slope $-\frac{1}{2}$
- $\delta = \frac{1}{2}$ has slope 0 (related to so-called ‘pink noise’)

LA(8) Wavelet Variance for 2 More FD Processes



- $\delta = \frac{5}{6}$ has slope $\frac{2}{3}$ (related to Kolmogorov turbulence)
- $\delta = 1$ has slope 1 (random walk)
- nonnegative slopes indicate nonstationarity, while negative slopes indicate stationarity

Wavelet Variance for Processes with Stationary Backward Differences: II

- summary: $\nu_X^2(\tau_j)$ well-defined for process $\{X_t\}$ that is
 - stationary
 - nonstationary with d th order stationary increments, but width of wavelet filter must satisfy $L \geq 2d$
- if $\{X_t\}$ is stationary, then

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var} \{X_t\} < \infty$$

(recall that each RV in a stationary process must have the same finite variance)

Wavelet Variance for Processes with Stationary Backward Differences: III

- if $\{X_t\}$ is nonstationary, then

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \infty$$

- with a suitable construction, we can take variance of nonstationary process with d th order stationary increments to be ∞
- using this construction, we have

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var} \{X_t\}$$

for both the stationary and nonstationary cases

Background on Gaussian Random Variables

- $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian (normal) RV with mean μ and variance σ^2
- will write

$$X \stackrel{d}{=} \mathcal{N}(\mu, \sigma^2)$$

to mean ‘RV X has same distribution as Gaussian RV’

- RV $\mathcal{N}(0, 1)$ often written as Z (called standard Gaussian or standard normal)
- let $\Phi(\cdot)$ be Gaussian cumulative distribution function

$$\Phi(z) \equiv \mathbf{P}[Z \leq z] = \int_{-\infty}^z \frac{1}{\sqrt{(2\pi)}} e^{-x^2/2} dx$$

- inverse $\Phi^{-1}(\cdot)$ of $\Phi(\cdot)$ is such that $\mathbf{P}[Z \leq \Phi^{-1}(p)] = p$
- $\Phi^{-1}(p)$ called $p \times 100\%$ percentage point

Background on Chi-Square Random Variables

- X said to be a chi-square RV with η degrees of freedom if its probability density function (PDF) is given by

$$f_X(x; \eta) = \frac{1}{2^{\eta/2} \Gamma(\eta/2)} x^{(\eta/2)-1} e^{-x/2}, \quad x \geq 0, \quad \eta > 0$$

- χ_η^2 denotes RV with above PDF
- if Z_1, Z_2, \dots, Z_η are independent standard Gaussian RVs, then

$$Z_1^2 + Z_2^2 + \dots + Z_\eta^2 \stackrel{d}{=} \chi_\eta^2$$

- two important facts: $E\{\chi_\eta^2\} = \eta$ and $\text{var}\{\chi_\eta^2\} = 2\eta$
- let $Q_\eta(p)$ denote the p th percentage point for the RV χ_η^2 :

$$\mathbf{P}[\chi_\eta^2 \leq Q_\eta(p)] = p$$

Expected Value of Wavelet Coefficients

- in preparation for considering problem of estimating $\nu_X^2(\tau_j)$ given an observed time series, need to consider $E\{\overline{W}_{j,t}\}$
- if $\{X_t\}$ is nonstationary but has d th order stationary increments, let $\{Y_t\}$ be stationary process obtained by differencing $\{X_t\}$ d times; if $\{X_t\}$ is stationary ($d = 0$ case), let $Y_t = X_t$
- with $\mu_Y \equiv E\{Y_t\}$, have
 - $E\{\overline{W}_{j,t}\} = 0$ if either (i) $L > 2d$ or (ii) $L = 2d$ and $\mu_Y = 0$
 - $E\{\overline{W}_{j,t}\} \neq 0$ if $\mu_Y \neq 0$ and $L = 2d$
- thus have $E\{\overline{W}_{j,t}\} = 0$ if L is picked large enough ($L > 2d$ is sufficient, but might not be necessary)
- knowing $E\{\overline{W}_{j,t}\} = 0$ eases job of estimating $\nu_X^2(\tau_j)$ considerably

Unbiased Estimator of Wavelet Variance: I

- given a realization of X_0, X_1, \dots, X_{N-1} from a process with d th order stationary differences, want to estimate $\nu_X^2(\tau_j)$
- for wavelet filter such that $L \geq 2d$ and $E\{\overline{W}_{j,t}\} = 0$, have

$$\nu_X^2(\tau_j) = \text{var}\{\overline{W}_{j,t}\} = E\{\overline{W}_{j,t}^2\}$$

- can base estimator on squares of

$$\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \dots, N-1$$

- recall that

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t \in \mathbb{Z}$$

Unbiased Estimator of Wavelet Variance: II

- comparing

$$\widetilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N} \quad \text{with} \quad \overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}$$

says that $\widetilde{W}_{j,t} = \overline{W}_{j,t}$ if ‘mod N ’ not needed; this happens when $L_j - 1 \leq t < N$ (recall that $L_j = (2^j - 1)(L - 1) + 1$)

- if $N - L_j \geq 0$, unbiased estimator of $\nu_X^2(\tau_j)$ is

$$\hat{\nu}_X^2(\tau_j) \equiv \frac{1}{N - L_j + 1} \sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2 = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \overline{W}_{j,t}^2,$$

where $M_j \equiv N - L_j + 1$

Statistical Properties of $\hat{\nu}_X^2(\tau_j)$

- assume that $\{\overline{W}_{j,t}\}$ is Gaussian stationary process with mean zero and ACVS $\{s_{j,\tau}\}$
- suppose $\{s_{j,\tau}\}$ is such that

$$A_j \equiv \sum_{\tau=-\infty}^{\infty} s_{j,\tau}^2 < \infty$$

(if $A_j = \infty$, can make it finite usually by just increasing L)

- can show that $\hat{\nu}_X^2(\tau_j)$ is asymptotically Gaussian with mean $\nu_X^2(\tau_j)$ and large sample variance $2A_j/M_j$; i.e.,

$$\frac{\hat{\nu}_X^2(\tau_j) - \nu_X^2(\tau_j)}{(2A_j/M_j)^{1/2}} = \frac{M_j^{1/2}(\hat{\nu}_X^2(\tau_j) - \nu_X^2(\tau_j))}{(2A_j)^{1/2}} \stackrel{d}{=} \mathcal{N}(0, 1)$$

approximately for large $M_j \equiv N - L_j + 1$

Estimation of A_j

- in practical applications, need to estimate $A_j = \sum_{\tau} s_{j,\tau}^2$
- can argue that, for large M_j , the estimator

$$\hat{A}_j \equiv \frac{\left(\hat{s}_{j,0}^{(p)}\right)^2}{2} + \sum_{\tau=1}^{M_j-1} \left(\hat{s}_{j,\tau}^{(p)}\right)^2,$$

is approximately unbiased, where

$$\hat{s}_{j,\tau}^{(p)} \equiv \frac{1}{M_j} \sum_{t=L_j-1}^{N-1-|\tau|} \widetilde{W}_{j,t} \widetilde{W}_{j,t+|\tau|}, \quad 0 \leq |\tau| \leq M_j - 1$$

- Monte Carlo results: \hat{A}_j reasonably good for $M_j \geq 128$

Confidence Intervals for $\nu_X^2(\tau_j)$: I

- based upon large sample theory, can form a $100(1 - 2p)\%$ confidence interval (CI) for $\nu_X^2(\tau_j)$:

$$\left[\hat{\nu}_X^2(\tau_j) - \Phi^{-1}(1 - p) \frac{\sqrt{2A_j}}{\sqrt{M_j}}, \hat{\nu}_X^2(\tau_j) + \Phi^{-1}(1 - p) \frac{\sqrt{2A_j}}{\sqrt{M_j}} \right];$$

i.e., random interval traps unknown $\nu_X^2(\tau_j)$ with probability $1 - 2p$

- if A_j replaced by \hat{A}_j , get approximate $100(1 - 2p)\%$ CI
- critique: lower limit of CI can very well be negative even though $\nu_X^2(\tau_j) \geq 0$ always
- can avoid this problem by using a χ^2 approximation

Confidence Intervals for $\nu_X^2(\tau_j)$: II

- χ_η^2 useful for approximating distribution of sum of squared Gaussian RVs, which is what we are dealing with here:

$$\hat{\nu}_X^2(\tau_j) = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \bar{W}_{j,t}^2$$

- idea is to assume $\hat{\nu}_X^2(\tau_j) \stackrel{d}{=} a\chi_\eta^2$, where a and η are constants to be set via moment matching
- because $E\{\chi_\eta^2\} = \eta$ and $\text{var}\{\chi_\eta^2\} = 2\eta$, we have $E\{a\chi_\eta^2\} = a\eta$ and $\text{var}\{a\chi_\eta^2\} = 2a^2\eta$
- can equate $E\{\hat{\nu}_X^2(\tau_j)\}$ & $\text{var}\{\hat{\nu}_X^2(\tau_j)\}$ to $a\eta$ & $2a^2\eta$ to determine a & η

Confidence Intervals for $\nu_X^2(\tau_j)$: III

- obtain

$$\eta = \frac{2 \left(E\{\hat{\nu}_X^2(\tau_j)\} \right)^2}{\text{var}\{\hat{\nu}_X^2(\tau_j)\}} = \frac{2\nu_X^4(\tau_j)}{\text{var}\{\hat{\nu}_X^2(\tau_j)\}} \quad \text{and} \quad a = \frac{\nu_X^2(\tau_j)}{\eta}$$

- after η has been determined, can obtain a CI for $\nu_X^2(\tau_j)$: with probability $1 - 2p$, the random interval

$$\left[\frac{\eta \hat{\nu}_X^2(\tau_j)}{Q_\eta(1-p)}, \frac{\eta \hat{\nu}_X^2(\tau_j)}{Q_\eta(p)} \right]$$

traps the true unknown $\nu_X^2(\tau_j)$

- lower limit is now nonnegative
- as $N \rightarrow \infty$, above CI and Gaussian-based CI converge

Three Ways to Set η

1. use large sample theory with appropriate estimates:

$$\eta = \frac{2\nu_X^4(\tau_j)}{\text{var}\{\hat{\nu}_X^2(\tau_j)\}} \approx \frac{2\nu_X^4(\tau_j)}{2A_j/M_j} \text{ suggests } \hat{\eta}_1 = \frac{M_j \hat{\nu}_X^4(\tau_j)}{\hat{A}_j}$$

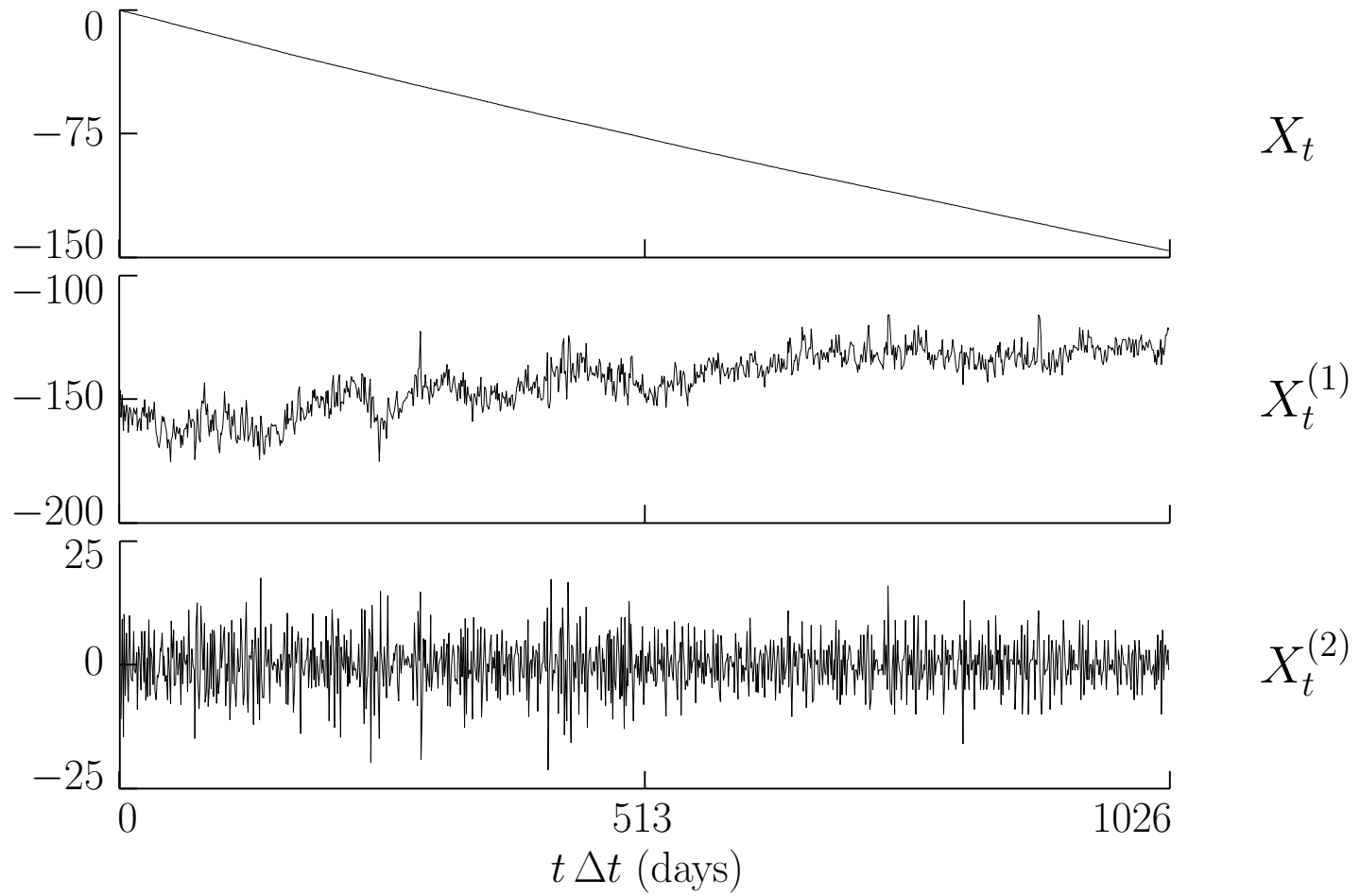
2. make an assumption about the effect of wavelet filter on $\{X_t\}$ to obtain simple approximation

$$\eta_3 = \max\{M_j/2^j, 1\}$$

(this effective – but conservative – approach is valuable if there are insufficient data to reliably estimate A_j)

3. third way requires assuming shape of spectral density function associated with $\{X_t\}$ (questionable assumption, but common practice in, e.g., atomic clock literature)

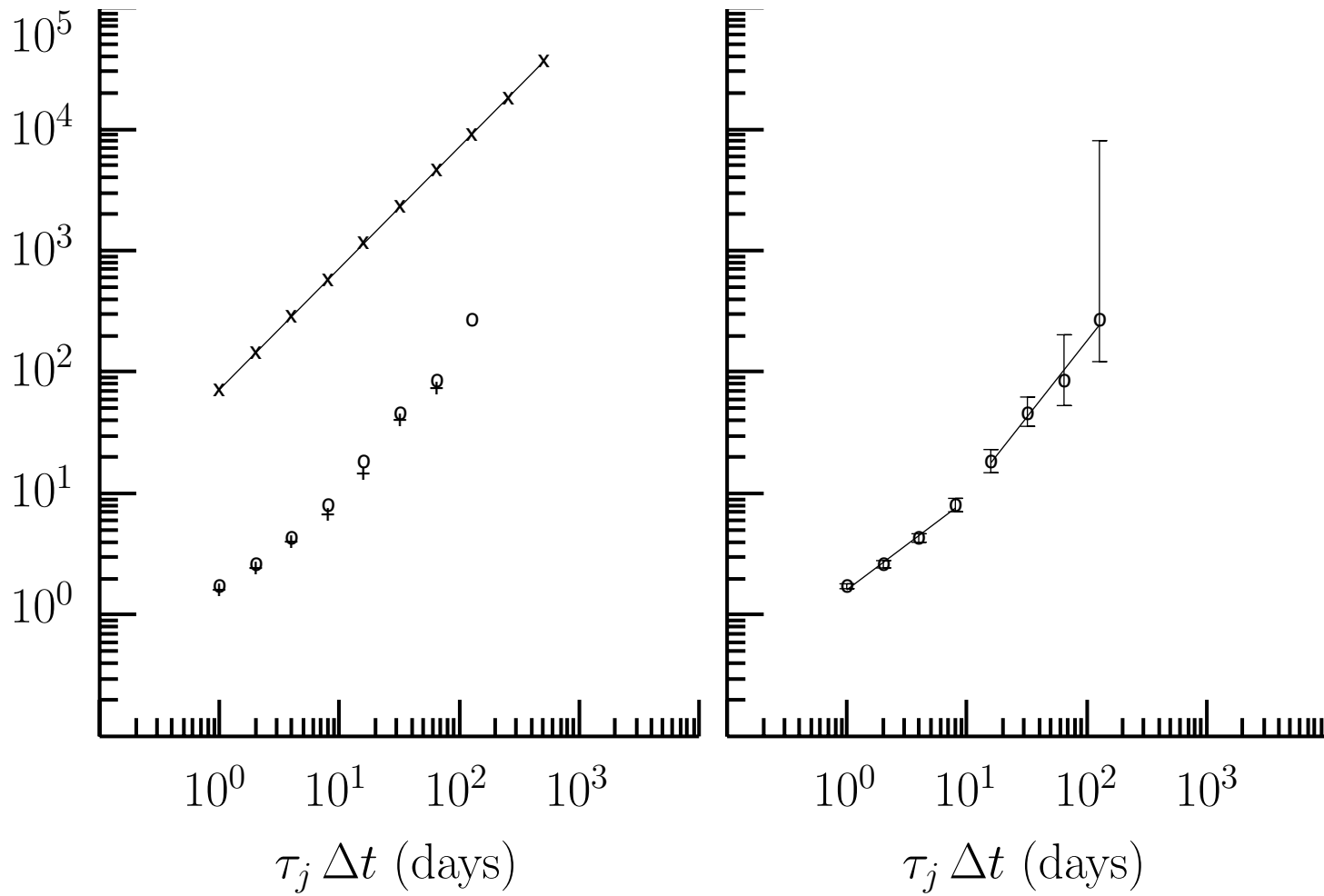
Atomic Clock Deviates: I



Atomic Clock Deviates: II

- top plot: errors $\{X_t\}$ in time kept by atomic clock 571 (measured in microseconds: 1,000,000 microseconds = 1 second)
- middle: 1st backward differences $\{X_t^{(1)}\}$ in nanoseconds (1000 nanoseconds = 1 microsecond)
- bottom: 2nd backward differences $\{X_t^{(2)}\}$, also in nanoseconds
- if $\{X_t\}$ nonstationary with d th order stationary increments, need $L \geq 2d$, but might need $L > 2d$ to get $E\{\overline{W}_{j,t}\} = 0$
- might regard $\{X_t^{(1)}\}$ as realization of stationary process, but, if so, with a mean value far from 0; $\{X_t^{(2)}\}$ resembles realization of stationary process, but mean value still might not be 0 if we believe there is a linear trend in $\{X_t^{(1)}\}$; thus might need $L \geq 6$, but could get away with $L \geq 4$

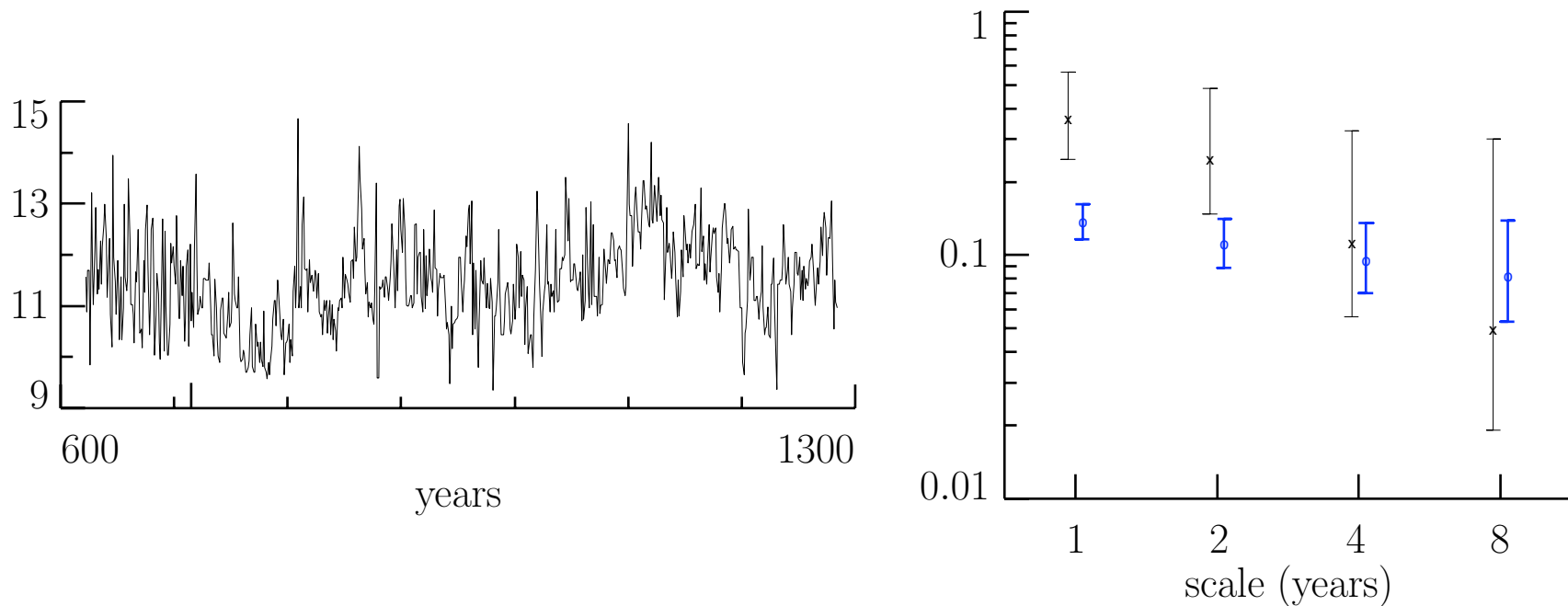
Atomic Clock Deviates: III



Atomic Clock Deviates: IV

- square roots of wavelet variance estimates for atomic clock time errors $\{X_t\}$ based upon unbiased MODWT estimator with
 - Haar wavelet (\mathbf{x} 's in left-hand plot, with linear fit)
 - D(4) wavelet (circles in left- and right-hand plots)
 - D(6) wavelet (pluses in left-hand plot).
- Haar wavelet inappropriate
 - need $\{X_t^{(1)}\}$ to be a realization of a stationary process with mean 0 (stationarity might be OK, but mean 0 is way off)
 - linear appearance can be explained in terms of nonzero mean
- 95% confidence intervals in the right-hand plot are the square roots of intervals computed using the chi-square approximation with η given by $\hat{\eta}_1$ for $j = 1, \dots, 6$ and by η_3 for $j = 7 \& 8$

Annual Minima of Nile River

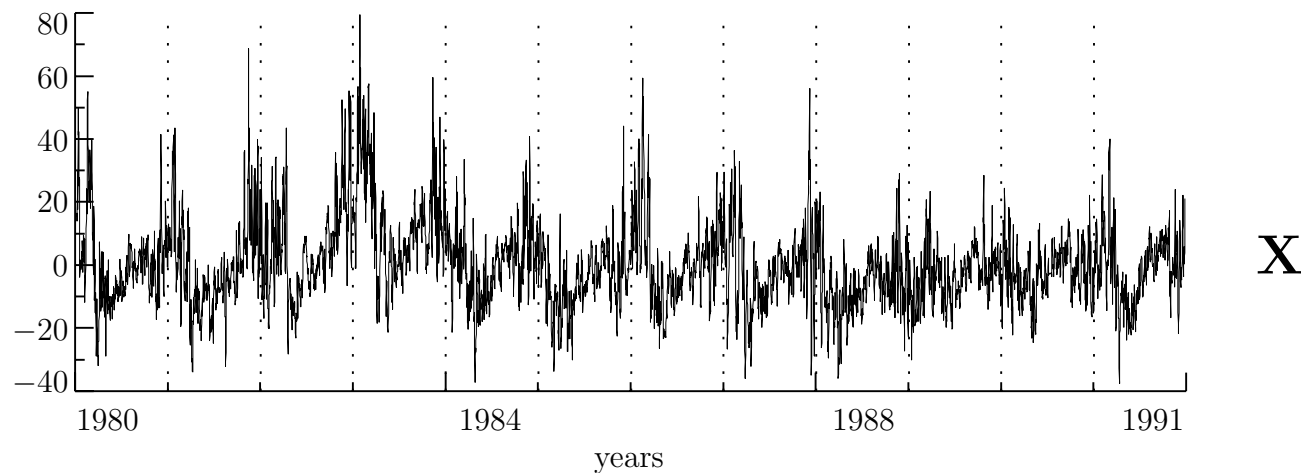


- left-hand plot: annual minima of Nile River
- right: Haar $\hat{\nu}_X^2(\tau_j)$ before (**x**'s) and after (**o**'s) year 715.5, with 95% confidence intervals based upon $\chi_{\eta_3}^2$ approximation

Wavelet Variance Analysis of Time Series with Time-Varying Statistical Properties

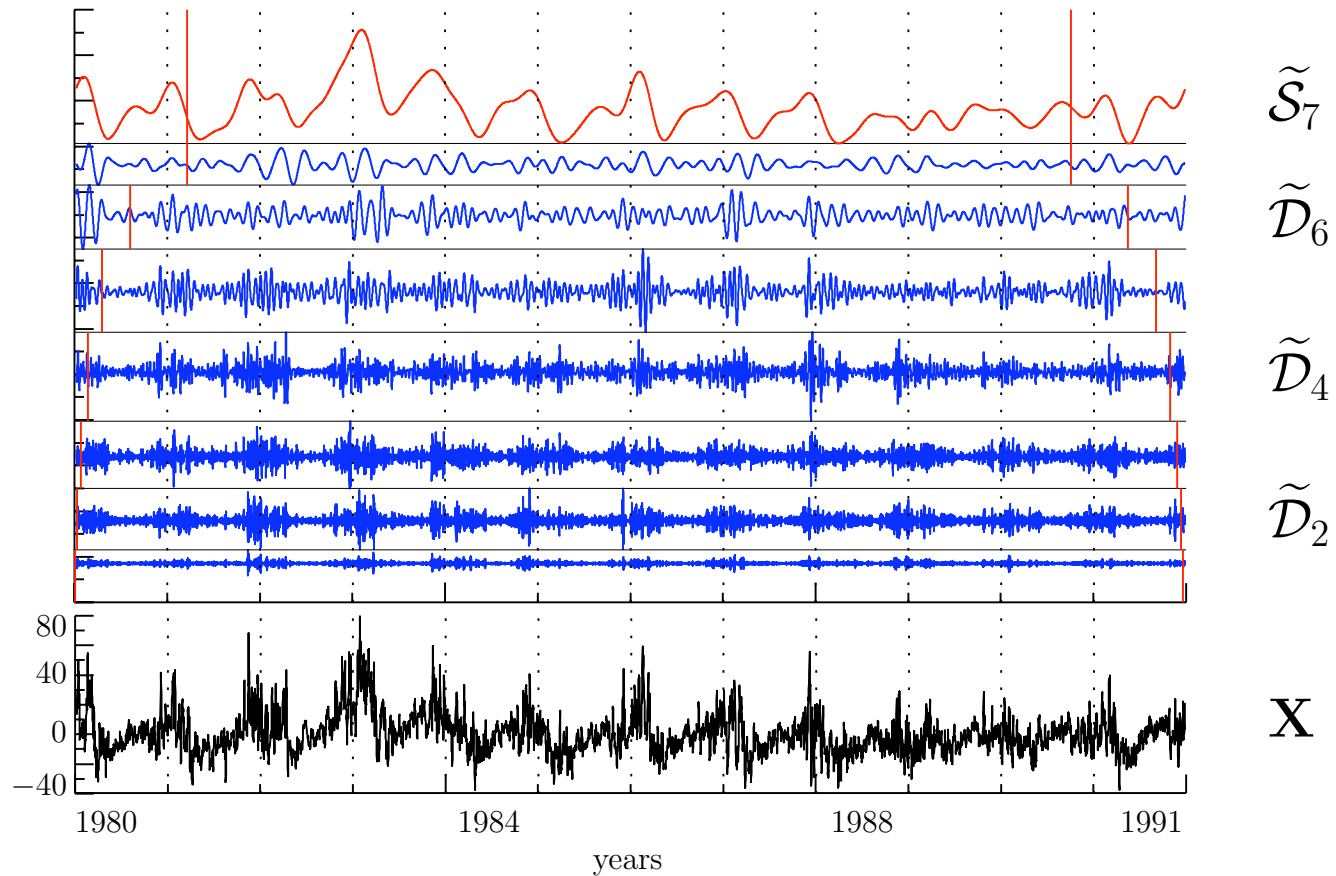
- each wavelet coefficient $\widetilde{W}_{j,t}$ formed using portion of X_t
- suppose X_t associated with actual time $t_0 + t \Delta t$
 - * t_0 is actual time of first observation X_0
 - * Δt is spacing between adjacent observations
- suppose $\tilde{h}_{j,l}$ is least asymmetric Daubechies wavelet
- can associate $\widetilde{W}_{j,t}$ with an interval of width $2\tau_j \Delta t$ centered at
$$t_0 + (2^j(t+1) - 1 - |\nu_j^{(H)}| \bmod N) \Delta t,$$
where, e.g., $|\nu_j^{(H)}| = [7(2^j - 1) + 1]/2$ for LA(8) wavelet
- can thus form ‘localized’ wavelet variance analysis (implicitly assumes stationarity or stationary increments locally)

Subtidal Sea Level Fluctuations: I



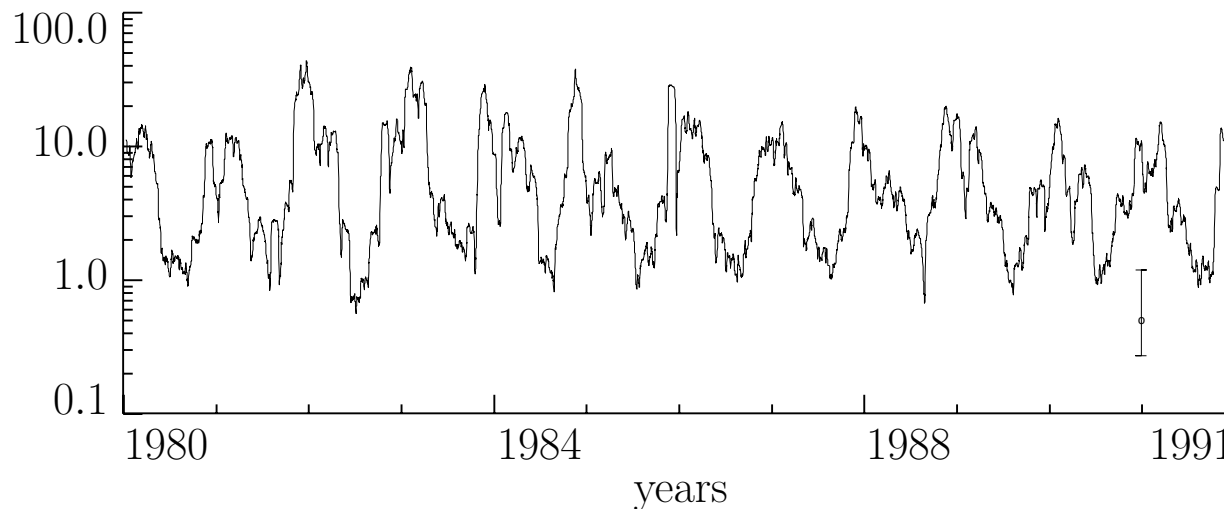
- subtidal sea level fluctuations \mathbf{X} for Crescent City, CA, collected by National Ocean Service with permanent tidal gauge
- $N = 8746$ values from Jan 1980 to Dec 1991 (almost 12 years)
- one value every 12 hours, so $\Delta t = 1/2$ day
- ‘subtidal’ is what remains after diurnal & semidiurnal tides are removed by low-pass filter (filter seriously distorts frequency band corresponding to first physical scale $\tau_1 \Delta t = 1/2$ day)

Subtidal Sea Level Fluctuations: II



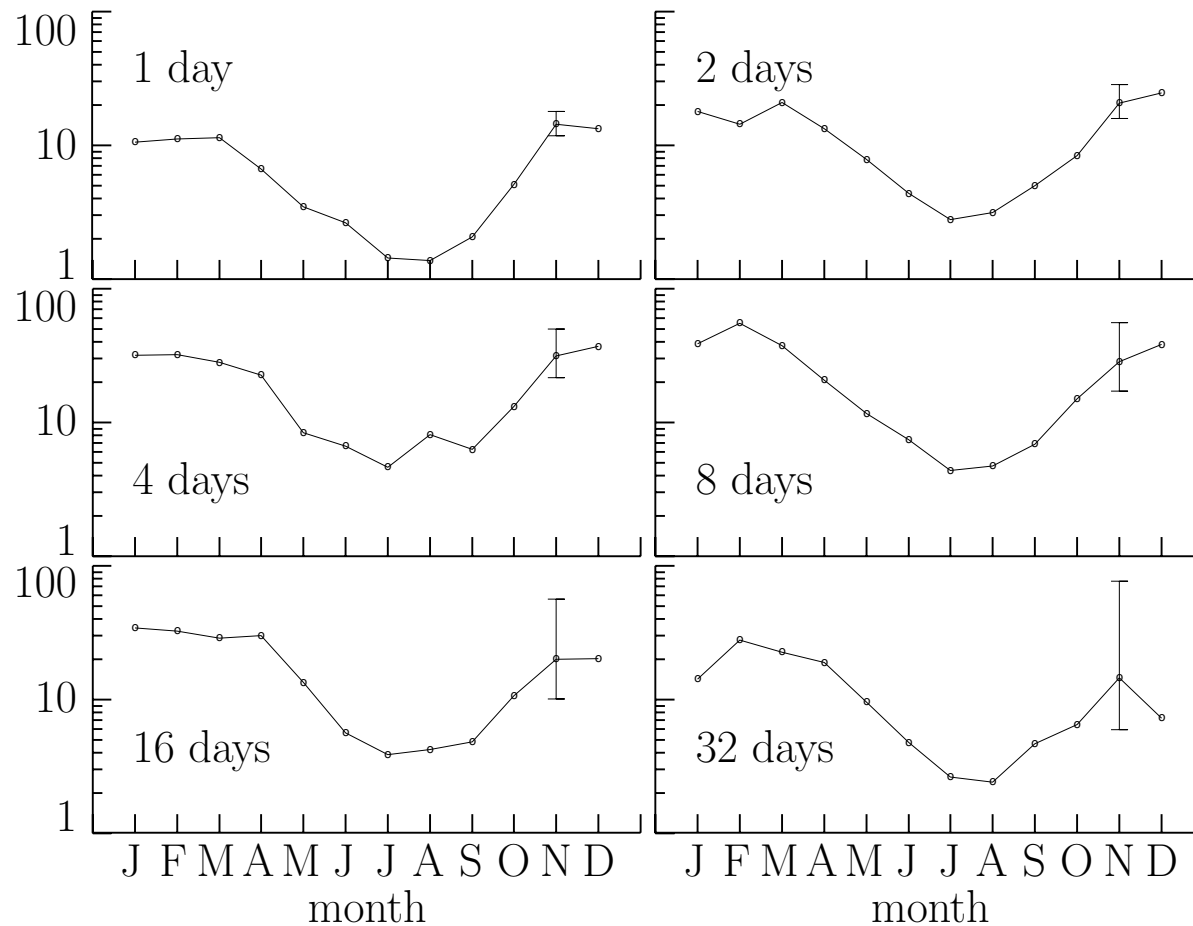
- level $J_0 = 7$ LA(8) MODWT multiresolution analysis

Subtidal Sea Level Fluctuations: III



- estimated time-dependent $LA(8)$ wavelet variances for physical scale $\tau_2 \Delta t = 1$ day based upon averages over monthly blocks (30.5 days, i.e., 61 data points)
- plot also shows a representative 95% confidence interval based upon a hypothetical wavelet variance estimate of $1/2$ and a chi-square distribution with $\nu = 15.25$

Subtidal Sea Level Fluctuations: IV



- estimated LA(8) wavelet variances for physical scales $\tau_j \Delta t = 2^{j-2}$ days, $j = 2, \dots, 7$, grouped by calendar month

Some Extensions

- wavelet cross-covariance and cross-correlation (Whitcher, Guttorp and Percival, 2000; Serroukh and Walden, 2000a, 2000b)
- asymptotic theory for non-Gaussian processes satisfying a certain ‘mixing’ condition (Serroukh, Walden and Percival, 2000)
- biased estimators of wavelet variance (Aldrich, 2005)
- unbiased estimator of wavelet variance for ‘gappy’ time series (Mondal and Percival, 2010a)
- robust estimation (Mondal and Percival, 2010b)
- wavelet variance for random fields (Mondal and Percival, 2010c)
- wavelet-based characteristic scales (Keim and Percival, 2010)

Summary

- wavelet variance gives scale-based analysis of variance
- presented statistical theory for Gaussian processes with stationary increments
- in addition to the applications we have considered, the wavelet variance has been used to analyze
 - genome sequences
 - changes in variance of soil properties
 - canopy gaps in forests
 - accumulation of snow fields in polar regions
 - boundary layer atmospheric turbulence
 - regular and semiregular variable stars

Wavelet-Based Signal Extraction: Overview

- outline key ideas behind wavelet-based approach
- description of four basic models for signal estimation
- discussion of why wavelets can help estimate certain signals
- simple thresholding & shrinkage schemes for signal estimation
- wavelet-based thresholding and shrinkage
- discuss some extensions to basic approach

Wavelet-Based Signal Estimation: I

- DWT analysis of \mathbf{X} yields $\mathbf{W} = \mathcal{W}\mathbf{X}$
- DWT synthesis $\mathbf{X} = \mathcal{W}^T\mathbf{W}$ yields multiresolution analysis by splitting $\mathcal{W}^T\mathbf{W}$ into pieces associated with different scales
- DWT synthesis can also estimate ‘signal’ hidden in \mathbf{X} if we can modify \mathbf{W} to get rid of noise in the wavelet domain
- if \mathbf{W}' is a ‘noise reduced’ version of \mathbf{W} , can form signal estimate via $\mathcal{W}^T\mathbf{W}'$

Wavelet-Based Signal Estimation: II

- key ideas behind simple wavelet-based signal estimation
 - certain signals can be efficiently described by the DWT using
 - * all of the scaling coefficients
 - * a small number of ‘large’ wavelet coefficients
 - noise is manifested in a large number of ‘small’ wavelet coefficients
 - can either ‘threshold’ or ‘shrink’ wavelet coefficients to eliminate noise in the wavelet domain
- key ideas led to wavelet thresholding and shrinkage proposed by Donoho, Johnstone and coworkers in 1990s

Models for Signal Estimation: I

- will consider two types of signals:
 1. \mathbf{D} , an N dimensional deterministic signal
 2. \mathbf{C} , an N dimensional stochastic signal; i.e., a vector of random variables (RVs) with covariance matrix $\Sigma_{\mathbf{C}}$
- will consider two types of noise:
 1. $\boldsymbol{\epsilon}$, an N dimensional vector of independent and identically distributed (IID) RVs with mean 0 and covariance matrix $\Sigma_{\boldsymbol{\epsilon}} = \sigma_{\boldsymbol{\epsilon}}^2 I_N$
 2. $\boldsymbol{\eta}$, an N dimensional vector of non-IID RVs with mean 0 and covariance matrix $\Sigma_{\boldsymbol{\eta}}$
 - * one form: RVs independent, but have different variances
 - * another form of non-IID: RVs are correlated

Models for Signal Estimation: II

- leads to four basic ‘signal + noise’ models for \mathbf{X}
 1. $\mathbf{X} = \mathbf{D} + \boldsymbol{\epsilon}$
 2. $\mathbf{X} = \mathbf{D} + \boldsymbol{\eta}$
 3. $\mathbf{X} = \mathbf{C} + \boldsymbol{\epsilon}$
 4. $\mathbf{X} = \mathbf{C} + \boldsymbol{\eta}$
- in the latter two cases, the stochastic signal \mathbf{C} is assumed to be independent of the associated noise

Signal Representation via Wavelets: I

- consider $\mathbf{X} = \mathbf{D} + \boldsymbol{\epsilon}$ first, and concentrate on signal \mathbf{D}
- signal estimation problem is simplified if we can assume that the important part of \mathbf{D} is in its large values
- assumption is not usually viable in the original (i.e., time domain) representation \mathbf{D} , but might be true in another domain
- an orthonormal transform \mathcal{O} might be useful because
 - $\mathbf{d} = \mathcal{O}\mathbf{D}$ is equivalent to \mathbf{D} (since $\mathbf{D} = \mathcal{O}^T\mathbf{d}$)
 - we might be able to find \mathcal{O} such that the signal is isolated in $M \ll N$ large transform coefficients
- Q: how can we judge whether a particular \mathcal{O} might be useful for representing \mathbf{D} ?

Signal Representation via Wavelets: II

- let d_j be the j th transform coefficient in $\mathbf{d} = \mathcal{O}\mathbf{D}$
- let $d_{(0)}, d_{(1)}, \dots, d_{(N-1)}$ be the d_j 's reordered by magnitude:

$$|d_{(0)}| \geq |d_{(1)}| \geq \dots \geq |d_{(N-1)}|$$

- example: if $\mathbf{d} = [-3, 1, 4, -7, 2, -1]^T$, then
 $d_{(0)} = d_3 = -7$, $d_{(1)} = d_2 = 4$, $d_{(2)} = d_0 = -3$ etc.
- define a normalized partial energy sequence (NPES):

$$C_{M-1} \equiv \frac{\sum_{j=0}^{M-1} |d_{(j)}|^2}{\sum_{j=0}^{N-1} |d_{(j)}|^2} = \frac{\text{energy in largest } M \text{ terms}}{\text{total energy in signal}}$$

- let \mathcal{I}_M be $N \times N$ diagonal matrix whose j th diagonal term is 1 if $|d_j|$ is one of the M largest magnitudes and is 0 otherwise

Signal Representation via Wavelets: III

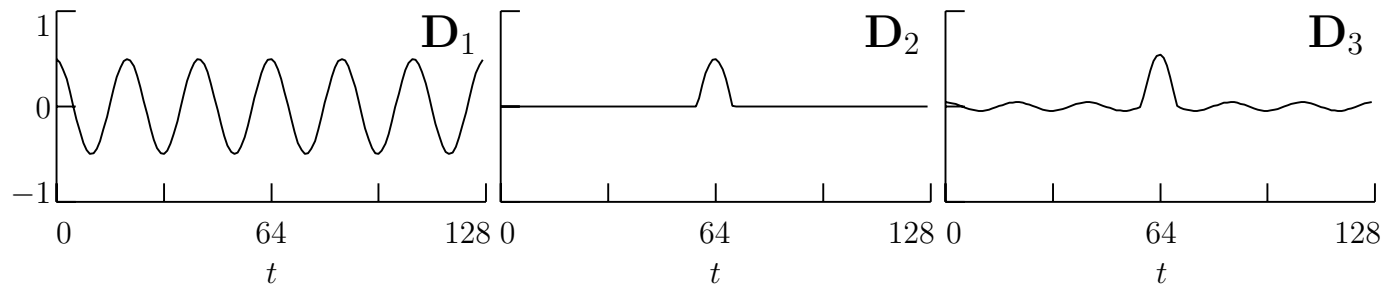
- form $\hat{\mathbf{D}}_M \equiv \mathcal{O}^T \mathcal{I}_M \mathbf{d}$, which is an approximation to \mathbf{D}
- when $\mathbf{d} = [-3, 1, 4, -7, 2, -1]^T$ and $M = 3$, we have

$$\mathcal{I}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and thus } \hat{\mathbf{D}}_M = \mathcal{O}^T \begin{bmatrix} -3 \\ 0 \\ 4 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

- one interpretation for NPES:

$$C_{M-1} = 1 - \frac{\|\mathbf{D} - \hat{\mathbf{D}}_M\|^2}{\|\mathbf{D}\|^2} = 1 - \text{relative approximation error}$$

Signal Representation via Wavelets: IV



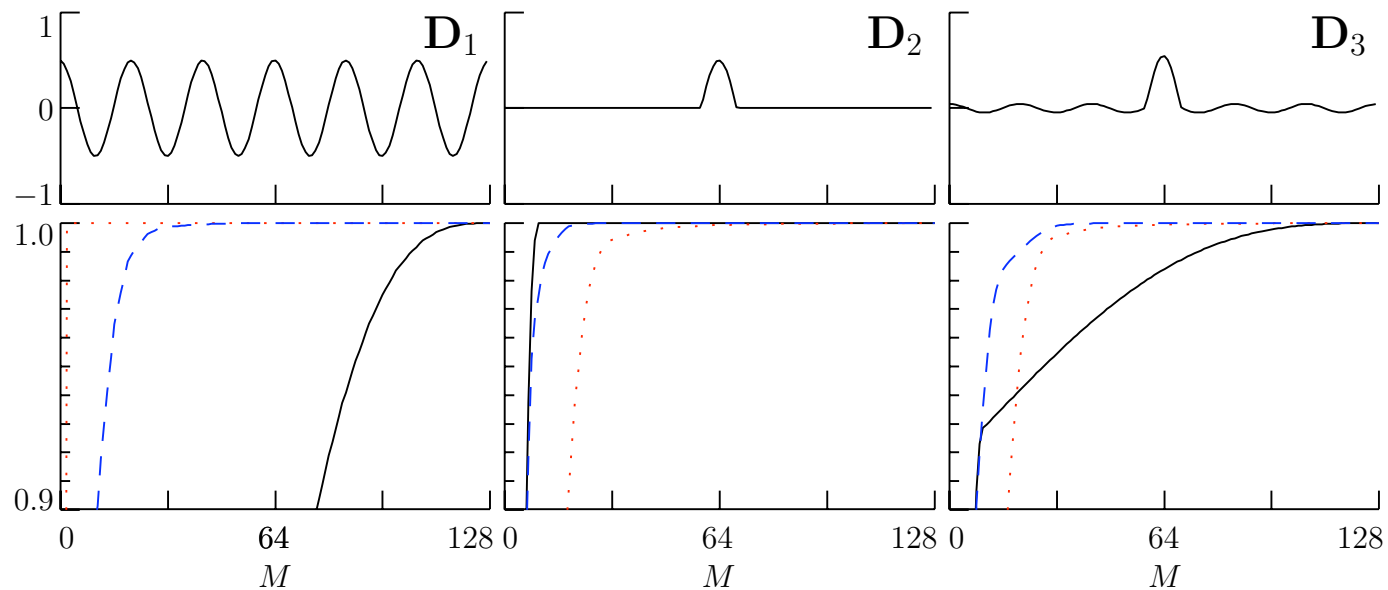
- consider three signals plotted above
- \mathbf{D}_1 is a sinusoid, which can be represented succinctly by the discrete Fourier transform (DFT)
- \mathbf{D}_2 is a bump (only a few nonzero values in the time domain)
- \mathbf{D}_3 is a linear combination of \mathbf{D}_1 and \mathbf{D}_2

Signal Representation via Wavelets: V

- consider three different orthonormal transforms
 - identity transform I (time)
 - the orthonormal DFT \mathcal{F} (frequency), where \mathcal{F} has (k, t) th element $\exp(-i2\pi tk/N)/\sqrt{N}$ for $0 \leq k, t \leq N - 1$
 - the LA(8) DWT \mathcal{W} (wavelet)
- # of terms M needed to achieve relative error $< 1\%$:

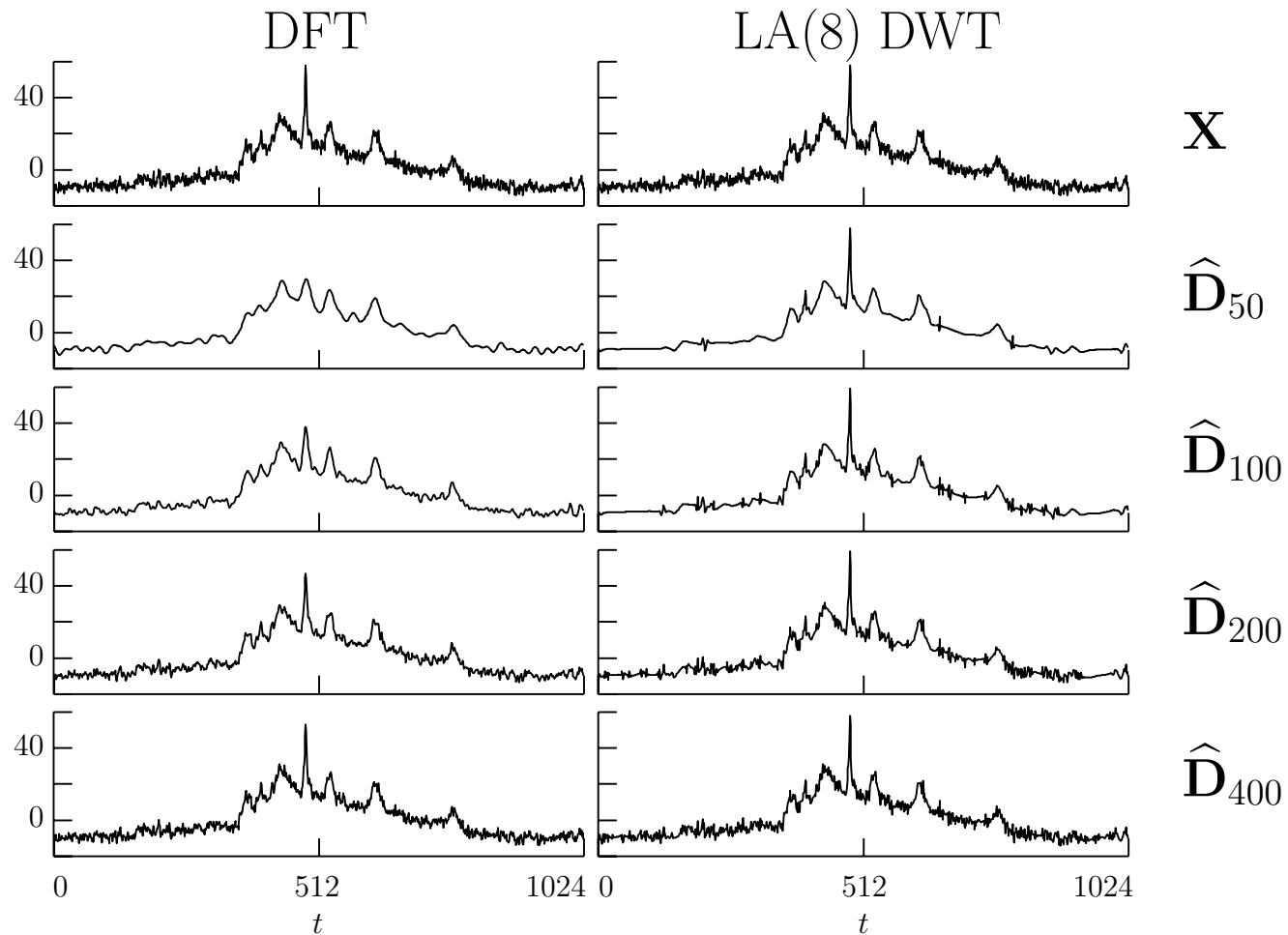
	D₁	D₂	D₃
DFT	2	29	28
identity	105	9	75
LA(8) wavelet	22	14	21

Signal Representation via Wavelets: VI



- use NPESs to see how well these three signals are represented in the time, frequency (DFT) and wavelet (LA(8)) domains
- time (solid curves), frequency (dotted) and wavelet (dashed)

Signal Representation via Wavelets: IX



- example: DFT $\hat{\mathbf{D}}_M$ (left-hand column) & $J_0 = 6$ LA(8) DWT $\hat{\mathbf{D}}_M$ (right) for NMR series \mathbf{X} (A. Maudsley, UCSF)

Signal Estimation via Thresholding: I

- thresholding schemes involve

1. computing $\mathbf{O} \equiv \mathcal{O}\mathbf{X}$

2. defining $\mathbf{O}^{(t)}$ as vector with l th element

$$O_l^{(t)} = \begin{cases} 0, & \text{if } |O_l| \leq \delta; \\ \text{some nonzero value,} & \text{otherwise,} \end{cases}$$

where nonzero values are yet to be defined

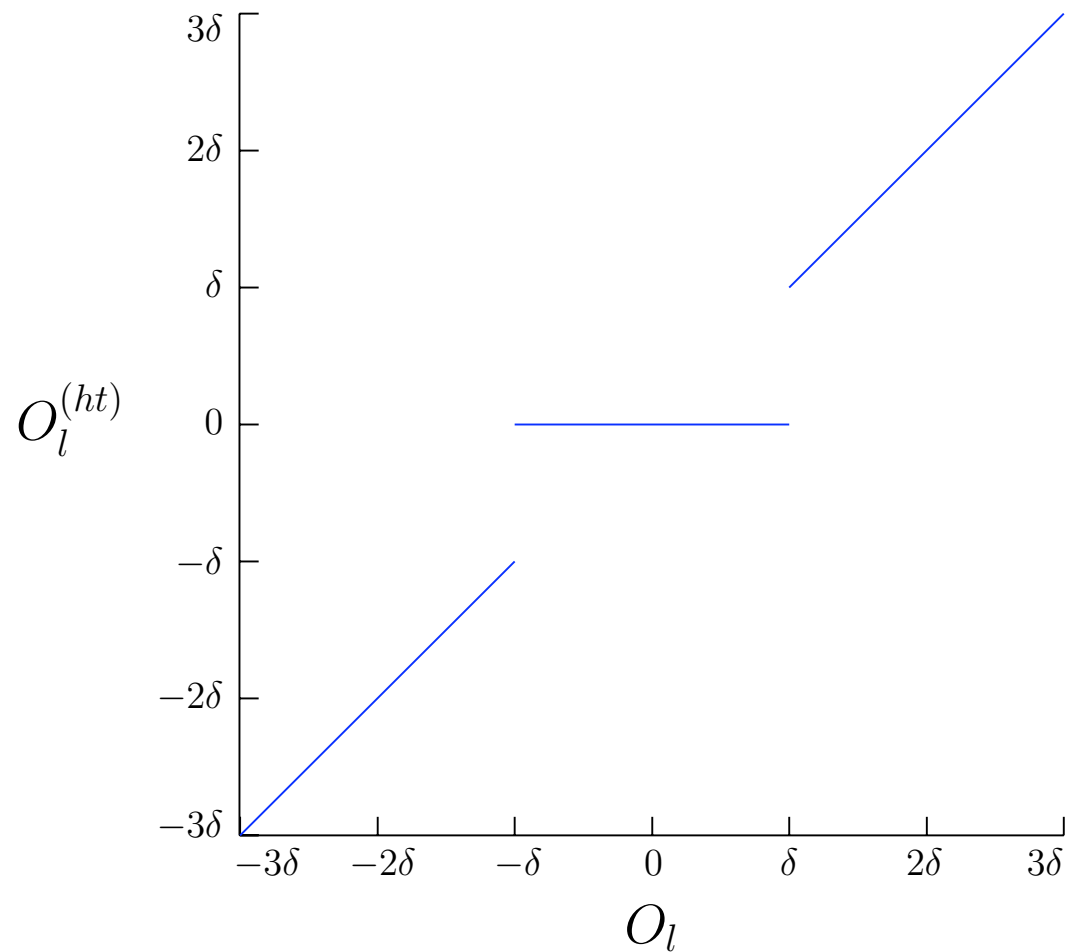
3. estimating \mathbf{D} via $\hat{\mathbf{D}}^{(t)} \equiv \mathcal{O}^T \mathbf{O}^{(t)}$

- simplest scheme is ‘hard thresholding’ (‘kill/keep’ strategy):

$$O_l^{(ht)} = \begin{cases} 0, & \text{if } |O_l| \leq \delta; \\ O_l, & \text{otherwise.} \end{cases}$$

Hard Thresholding Function

- plot shows mapping from O_l to $O_l^{(ht)}$



Signal Estimation via Thresholding: II

- hard thresholding is strategy that arises from solution to simple optimization problem, namely, find $\hat{\mathbf{D}}_M$ such that

$$\gamma_m \equiv \|\mathbf{X} - \hat{\mathbf{D}}_m\|^2 + m\delta^2$$

is minimized over all possible $\hat{\mathbf{D}}_m = \mathcal{O}^T \mathcal{I}_m \mathbf{O}$, $m = 0, \dots, N$

- δ is a fixed parameter that is set *a priori* (we assume $\delta > 0$)
- $\|\mathbf{X} - \hat{\mathbf{D}}_m\|^2$ is a measure of ‘fidelity’
 - rationale for this term: $\hat{\mathbf{D}}_m$ shouldn’t stray too far from \mathbf{X} (particularly if signal-to-noise ratio is high)
 - fidelity increases (the measure decreases) as m increases
 - in minimizing γ_m , consideration of this term alone suggests that m should be large

Signal Estimation via Thresholding: III

- $m\delta^2$ is a penalty for too many terms
 - rationale: heuristic says $\mathbf{d} = \mathcal{O}\mathbf{D}$ consists of just a few large coefficients
 - penalty increases as m increases
 - in minimizing γ_m , consideration of this term alone suggests that m should be small
- optimization problem: balance off fidelity & parsimony
- can show that $\gamma_m = \|\mathbf{X} - \hat{\mathbf{D}}_m\|^2 + m\delta^2$ is minimized when m is set such that \mathcal{I}_m picks out all coefficients satisfying $O_j^2 > \delta^2$

Signal Estimation via Thresholding: IV

- alternative scheme is ‘soft thresholding:’

$$O_l^{(st)} = \text{sign} \{O_l\} (|O_l| - \delta)_+,$$

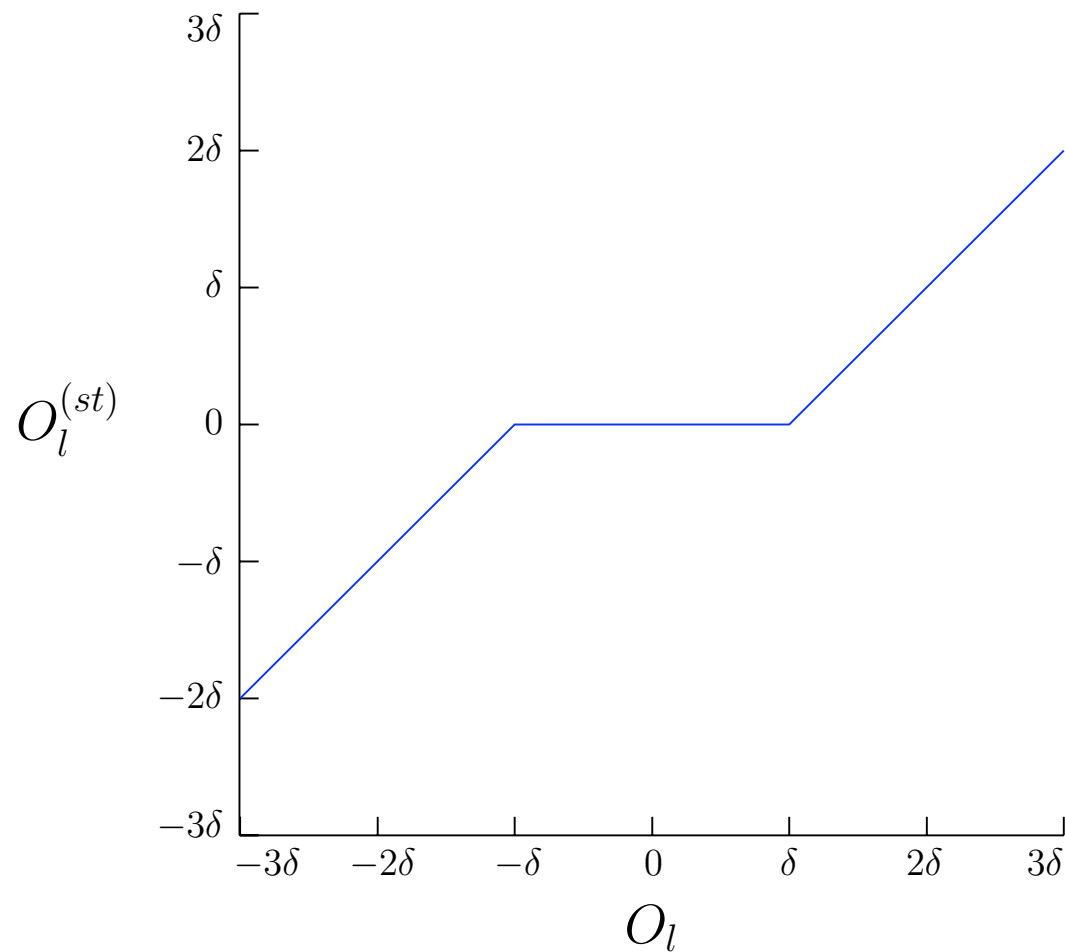
where

$$\text{sign} \{O_l\} \equiv \begin{cases} +1, & \text{if } O_l > 0; \\ 0, & \text{if } O_l = 0; \\ -1, & \text{if } O_l < 0. \end{cases} \quad \text{and} \quad (x)_+ \equiv \begin{cases} x, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

- one rationale for soft thresholding: fits into Stein’s class of estimators, for which unbiased estimation of risk is possible

Soft Thresholding Function

- here is the mapping from O_l to $O_l^{(st)}$



Signal Estimation via Thresholding: V

- third scheme is ‘mid thresholding:’

$$O_l^{(mt)} = \text{sign} \{O_l\} (|O_l| - \delta)_{++},$$

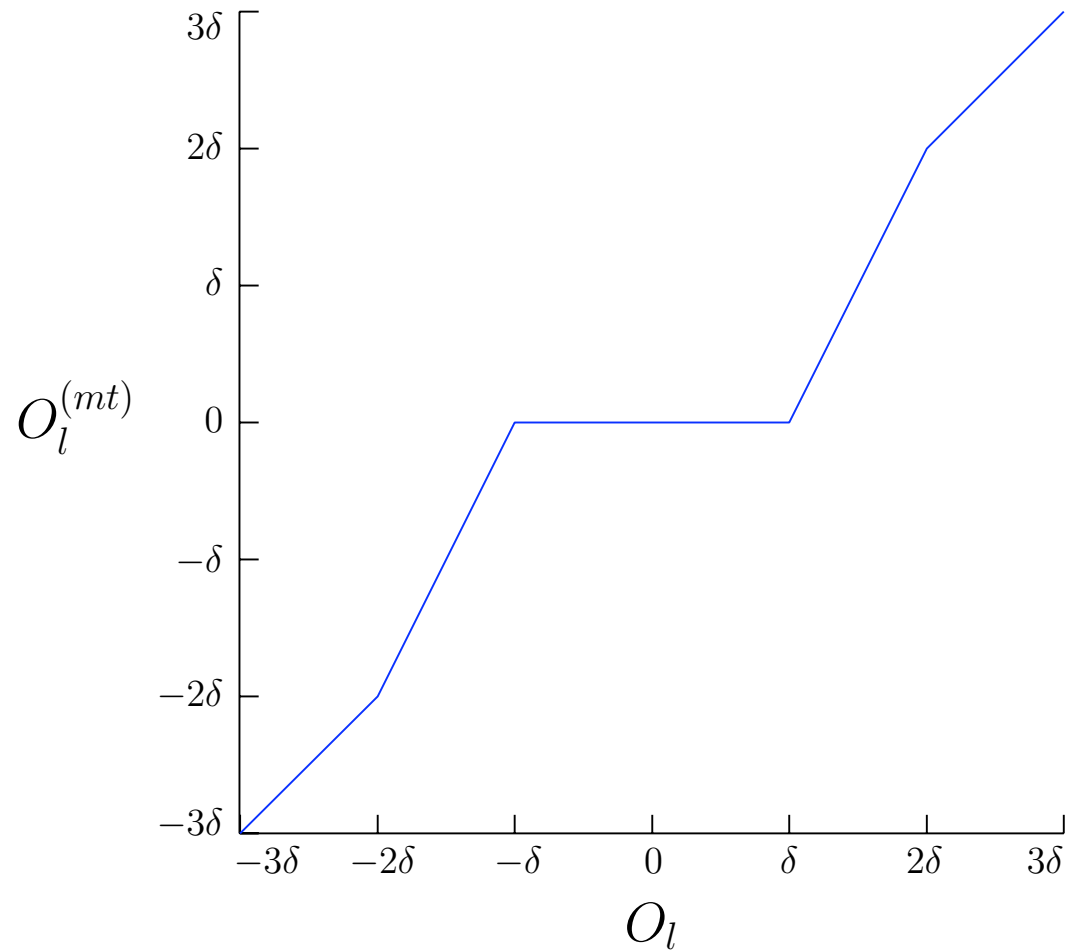
where

$$(|O_l| - \delta)_{++} \equiv \begin{cases} 2(|O_l| - \delta)_+, & \text{if } |O_l| < 2\delta; \\ |O_l|, & \text{otherwise} \end{cases}$$

- provides compromise between hard and soft thresholding

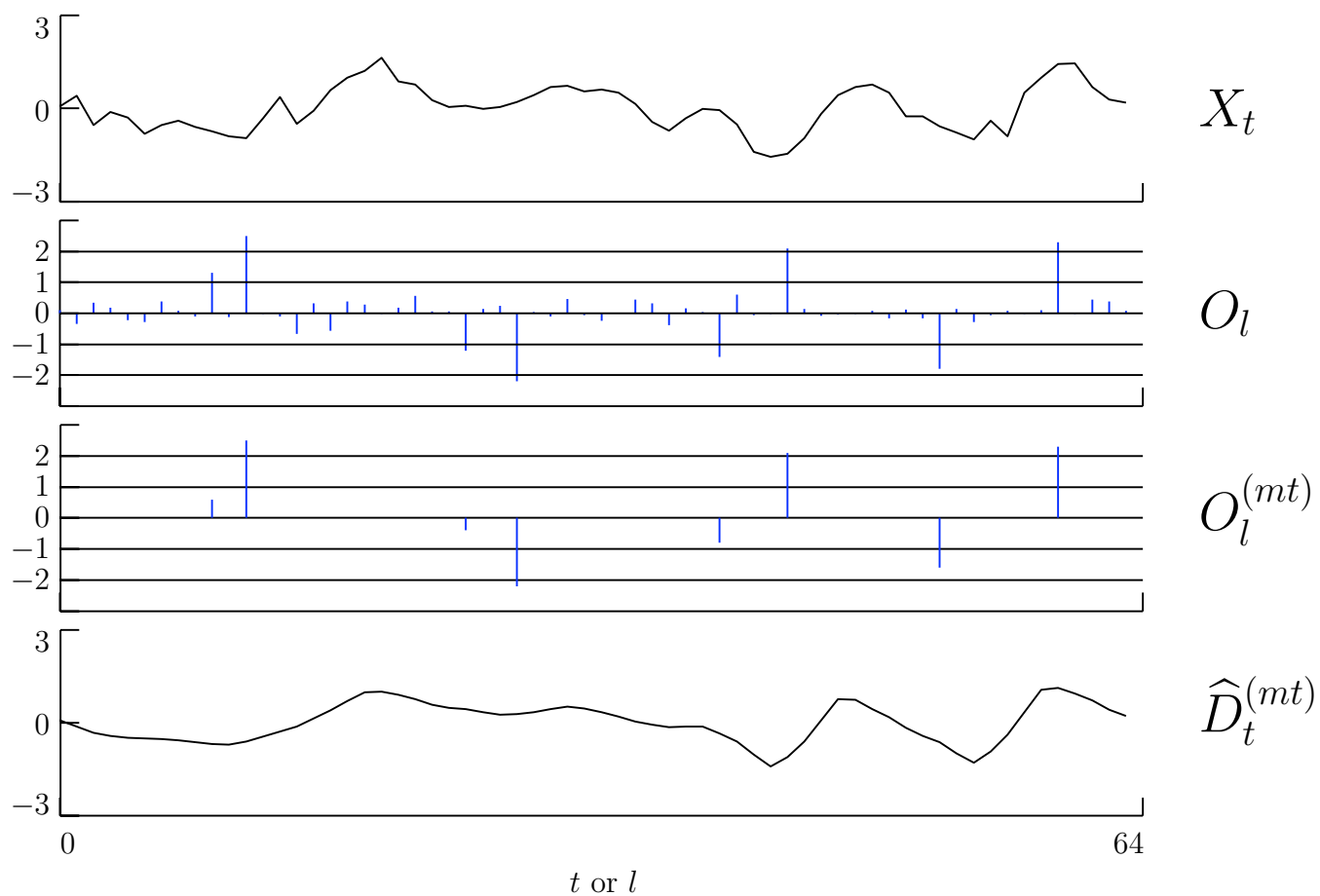
Mid Thresholding Function

- here is the mapping from O_l to $O_l^{(mt)}$



Signal Estimation via Thresholding: VI

- example of mid thresholding with $\delta = 1$



Universal Threshold

- Q: how do we go about setting δ ?
- specialize to IID Gaussian noise $\boldsymbol{\epsilon}$ with covariance $\sigma_\epsilon^2 I_N$
- can argue $\mathbf{e} \equiv \mathcal{O}\boldsymbol{\epsilon}$ is also IID Gaussian with covariance $\sigma_\epsilon^2 I_N$
- Donoho & Johnstone (1995) proposed $\delta^{(u)} \equiv \sqrt{[2\sigma_\epsilon^2 \log(N)]}$ ('log' here is 'log base e ')
- rationale for $\delta^{(u)}$: because of Gaussianity, can argue that

$$\mathbf{P} \left[\max_l \{|e_l|\} > \delta^{(u)} \right] \leq \frac{1}{\sqrt{[4\pi \log(N)]}} \rightarrow 0 \text{ as } N \rightarrow \infty$$

and hence $\mathbf{P} \left[\max_l \{|e_l|\} \leq \delta^{(u)} \right] \rightarrow 1$ as $N \rightarrow \infty$, so no noise will exceed threshold in the limit

Wavelet-Based Thresholding

- assume model of deterministic signal plus IID Gaussian noise with mean 0 and variance σ_ϵ^2 : $\mathbf{X} = \mathbf{D} + \boldsymbol{\epsilon}$
- using a DWT matrix \mathcal{W} , form $\mathbf{W} = \mathcal{W}\mathbf{X} = \mathcal{W}\mathbf{D} + \mathcal{W}\boldsymbol{\epsilon} \equiv \mathbf{d} + \mathbf{e}$
- because $\boldsymbol{\epsilon}$ IID Gaussian, so is \mathbf{e}
- Donoho & Johnstone (1994) advocate the following:
 - form partial DWT of level J_0 : $\mathbf{W}_1, \dots, \mathbf{W}_{J_0}$ and \mathbf{V}_{J_0}
 - threshold \mathbf{W}_j 's but leave \mathbf{V}_{J_0} alone (i.e., administratively, all $N/2^{J_0}$ scaling coefficients assumed to be part of \mathbf{d})
 - use universal threshold $\delta^{(u)} = \sqrt{[2\sigma_\epsilon^2 \log(N)]}$
 - use thresholding rule to form $\mathbf{W}_j^{(t)}$ (hard, etc.)
 - estimate \mathbf{D} by inverse transforming $\mathbf{W}_1^{(t)}, \dots, \mathbf{W}_{J_0}^{(t)}$ and \mathbf{V}_{J_0}

MAD Scale Estimator: I

- procedure assumes σ_ϵ is known, which is not usually the case
- if unknown, use median absolute deviation (MAD) scale estimator to estimate σ_ϵ using \mathbf{W}_1

$$\hat{\sigma}_{(\text{mad})} \equiv \frac{\text{median} \{ |W_{1,0}|, |W_{1,1}|, \dots, |W_{1, \frac{N}{2}-1}| \}}{0.6745}$$

- heuristic: bulk of $W_{1,t}$'s should be due to noise
- ‘0.6745’ yields estimator such that $E\{\hat{\sigma}_{(\text{mad})}\} = \sigma_\epsilon$ when $W_{1,t}$'s are IID Gaussian with mean 0 and variance σ_ϵ^2
- designed to be robust against large $W_{1,t}$'s due to signal

MAD Scale Estimator: II

- example: suppose \mathbf{W}_1 has 7 small ‘noise’ coefficients & 2 large ‘signal’ coefficients (say, a & b , with $2 \ll |a| < |b|$):

$$\mathbf{W}_1 = [1.23, -1.72, -0.80, -0.01, a, 0.30, 0.67, b, -1.33]^T$$

- ordering these by their magnitudes yields

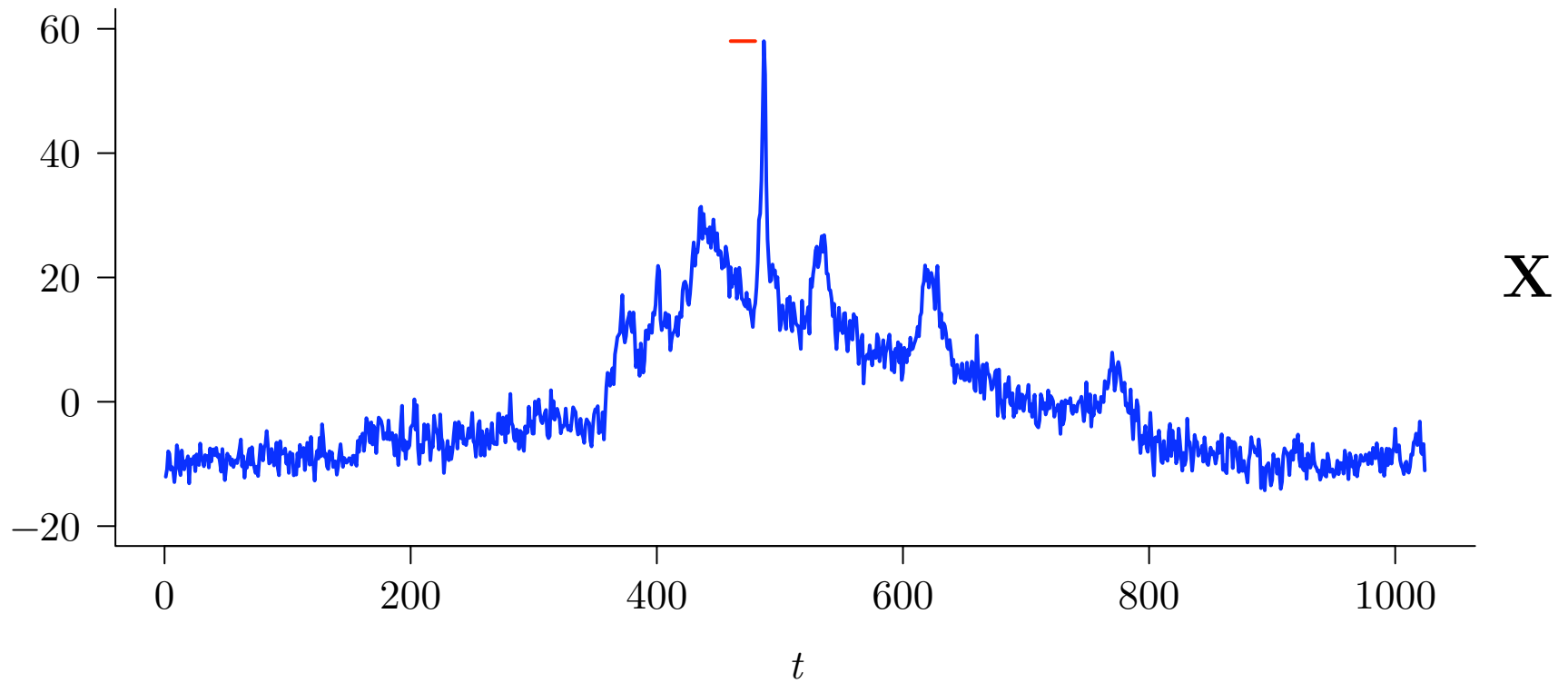
$$0.01, 0.30, 0.67, 0.80, 1.23, 1.33, 1.72, |a|, |b|$$

- median of these absolute deviations is 1.23, so

$$\hat{\sigma}_{(\text{mad})} = 1.23/0.6745 \doteq 1.82$$

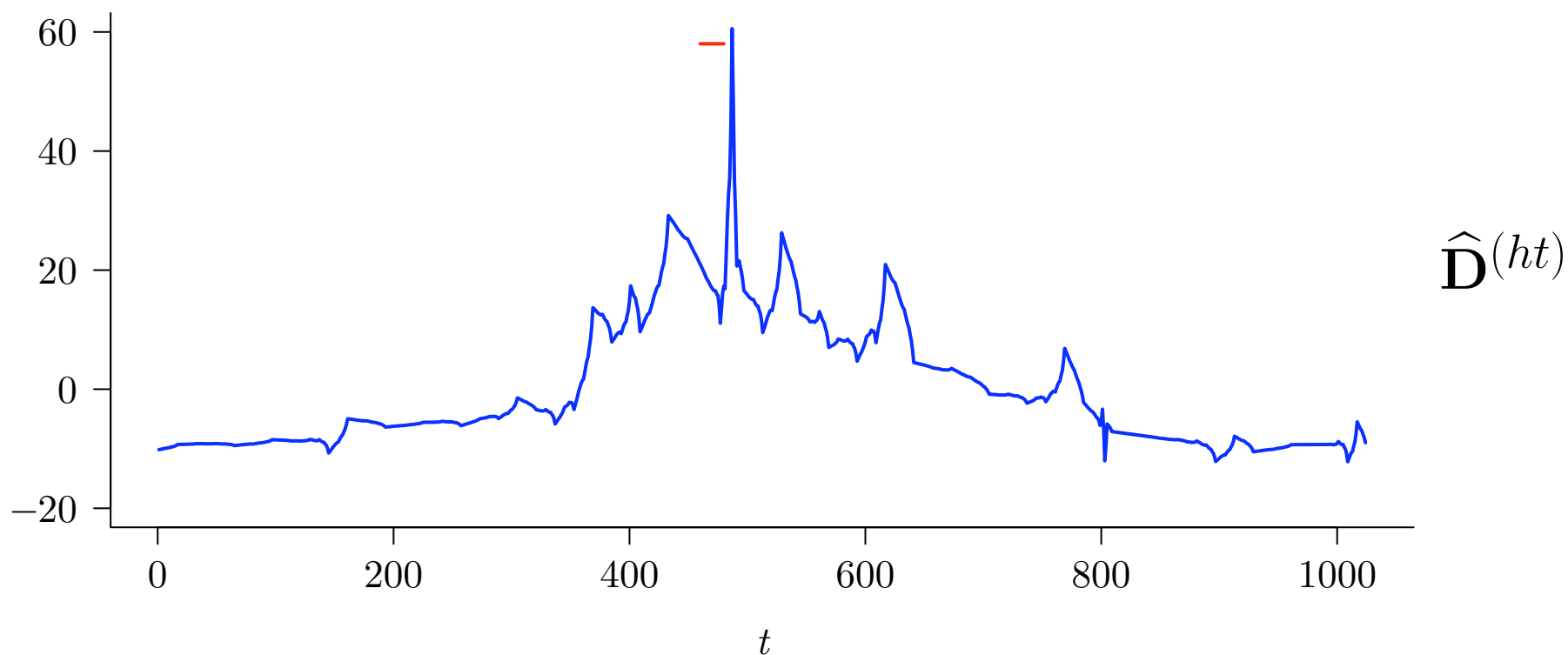
- $\hat{\sigma}_{(\text{mad})}$ not influenced adversely by a and b ; i.e., scale estimate depends largely on the many small coefficients due to noise

Examples of DWT-Based Thresholding: I



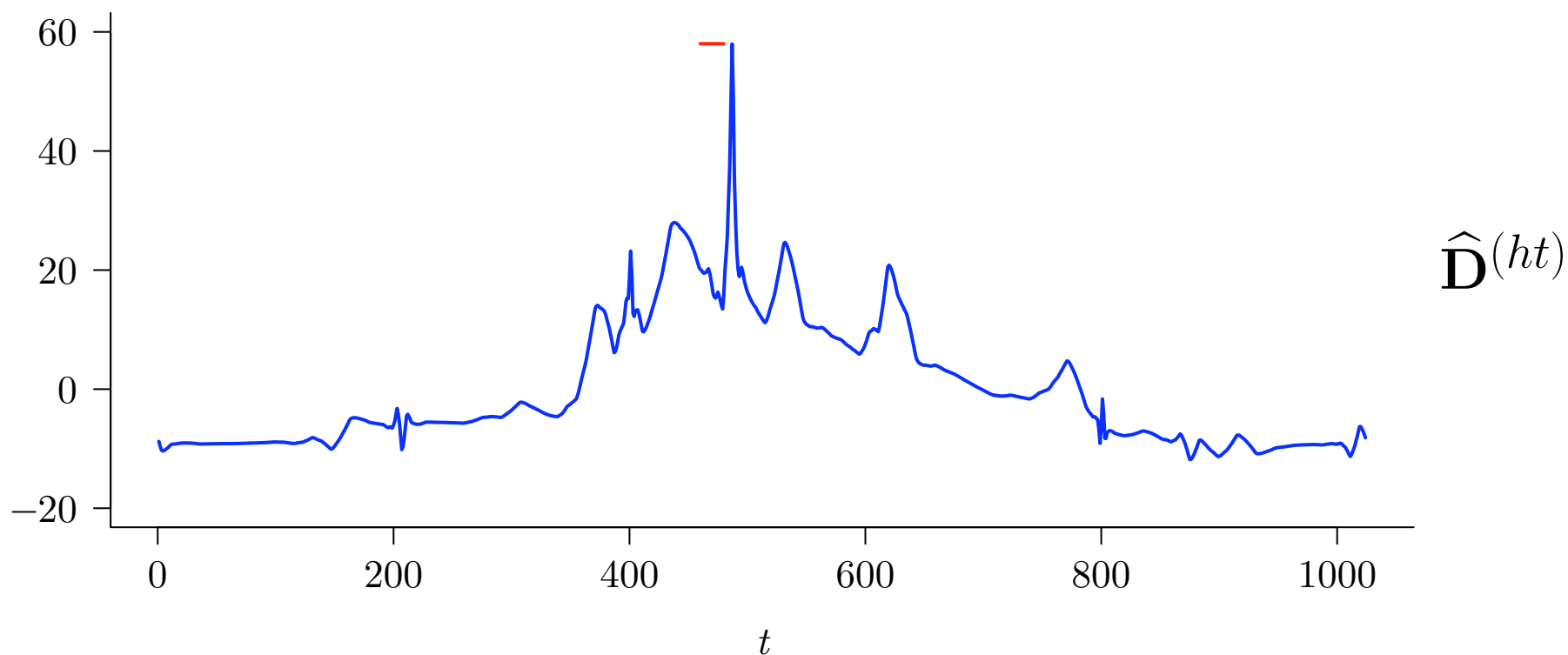
- NMR spectrum

Examples of DWT-Based Thresholding: II



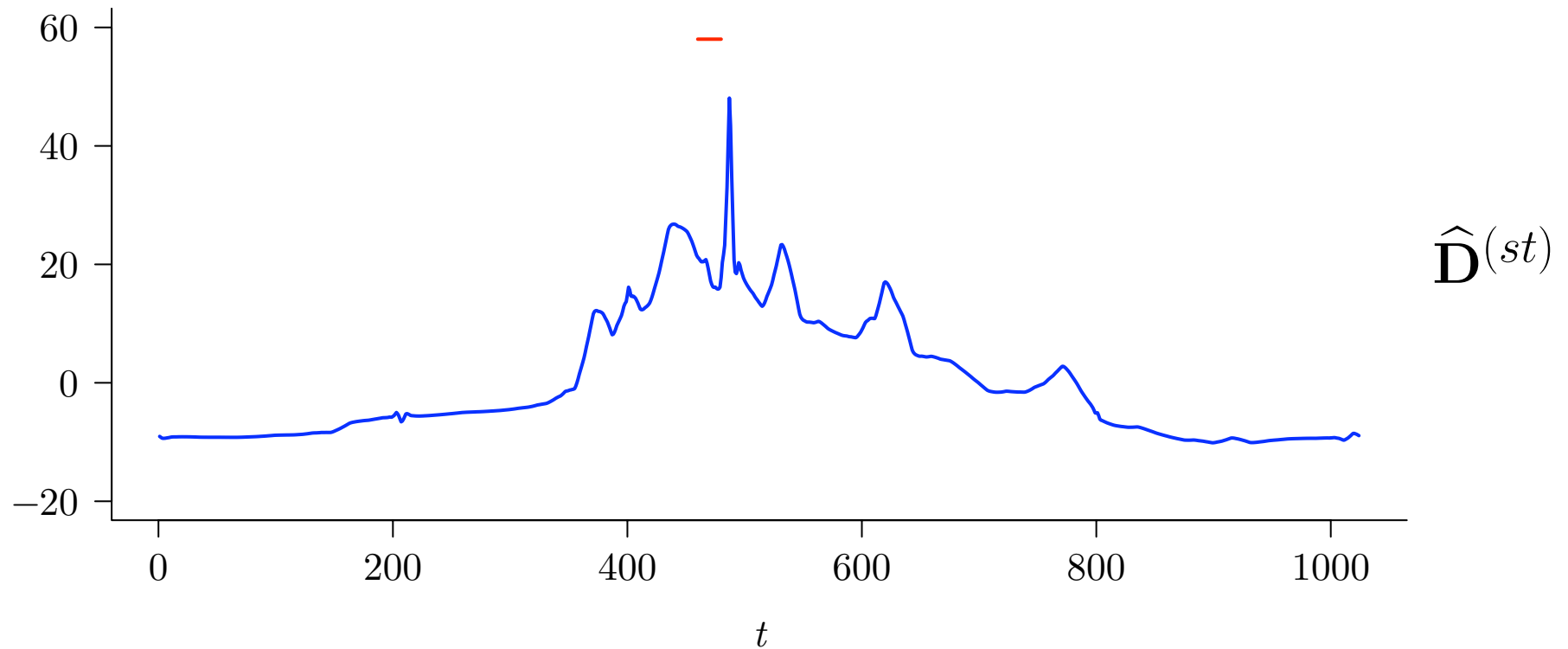
- signal estimate using $J_0 = 6$ partial D(4) DWT with hard thresholding and universal threshold level estimated by $\hat{\delta}^{(u)} = \sqrt{[2\hat{\sigma}_{(\text{mad})}^2 \log(N)]} \doteq 6.49$

Examples of DWT-Based Thresholding: III



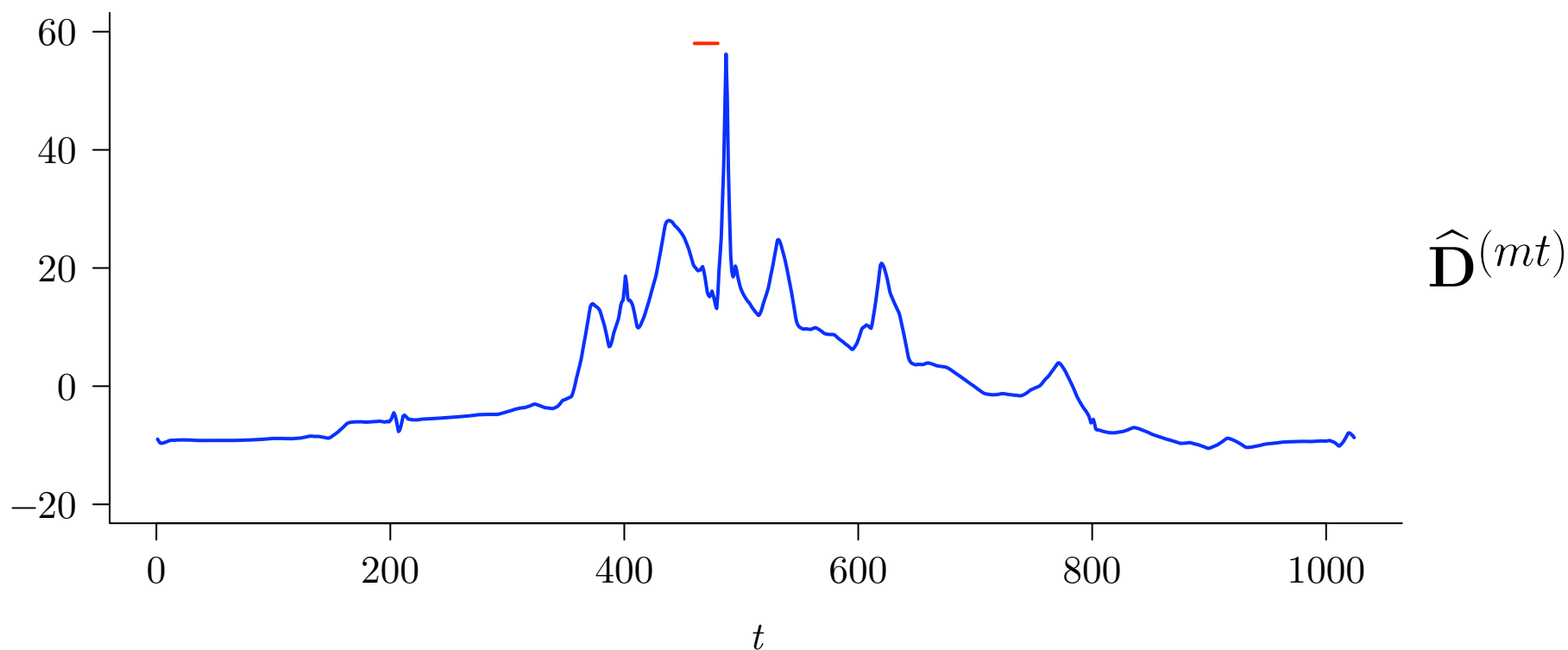
- same as before, but now using LA(8) DWT with $\hat{\delta}^{(u)} \doteq 6.13$

Examples of DWT-Based Thresholding: IV



- signal estimate using $J_0 = 6$ partial LA(8) DWT, but now with soft thresholding

Examples of DWT-Based Thresholding: V



- signal estimate using $J_0 = 6$ partial LA(8) DWT, but now with mid thresholding

MODWT-Based Thresholding

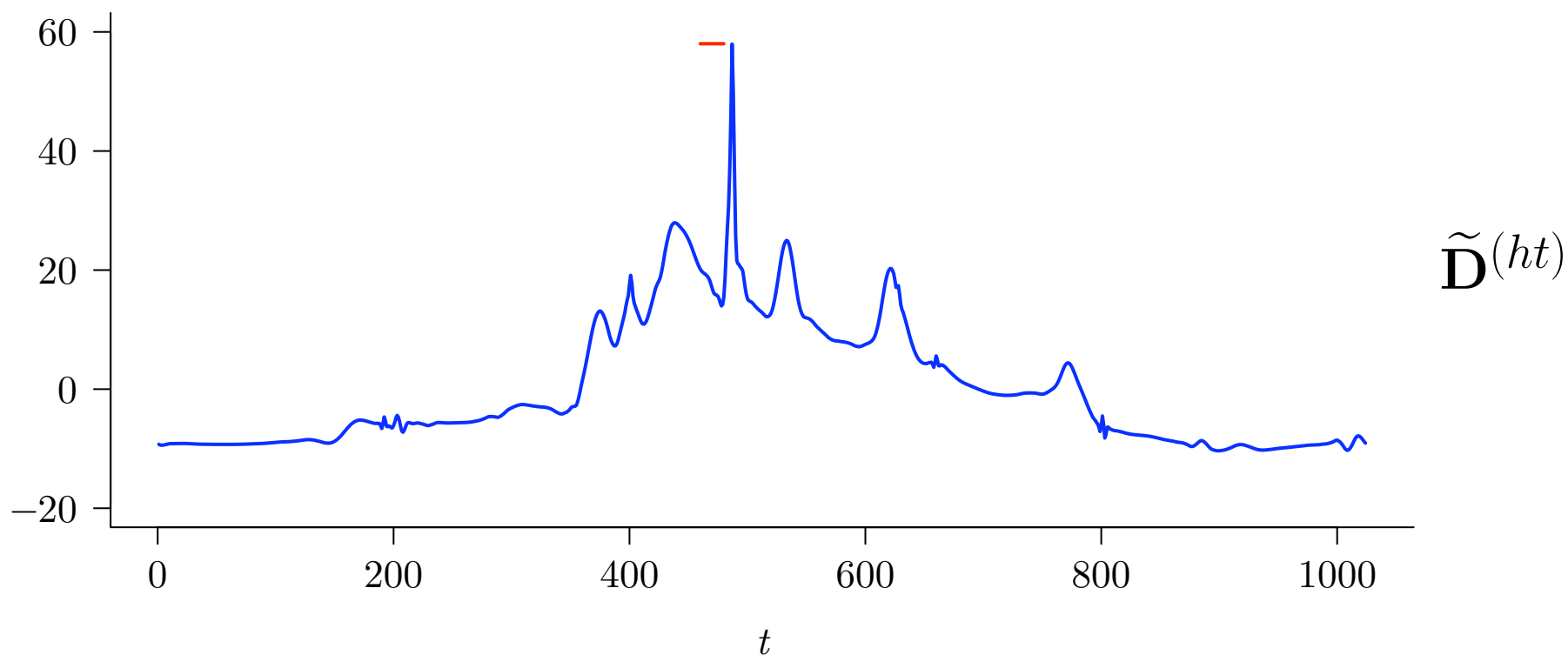
- can base thresholding procedure on MODWT rather than DWT, yielding signal estimators $\tilde{\mathbf{D}}^{(ht)}$, $\tilde{\mathbf{D}}^{(st)}$ and $\tilde{\mathbf{D}}^{(mt)}$
- because MODWT filters are normalized differently, universal threshold must be adjusted for each level:

$$\tilde{\delta}_j^{(u)} \equiv \sqrt{[2\tilde{\sigma}_{(\text{mad})}^2 \log(N)/2^j]},$$

where now MAD scale estimator is based on unit scale MODWT wavelet coefficients

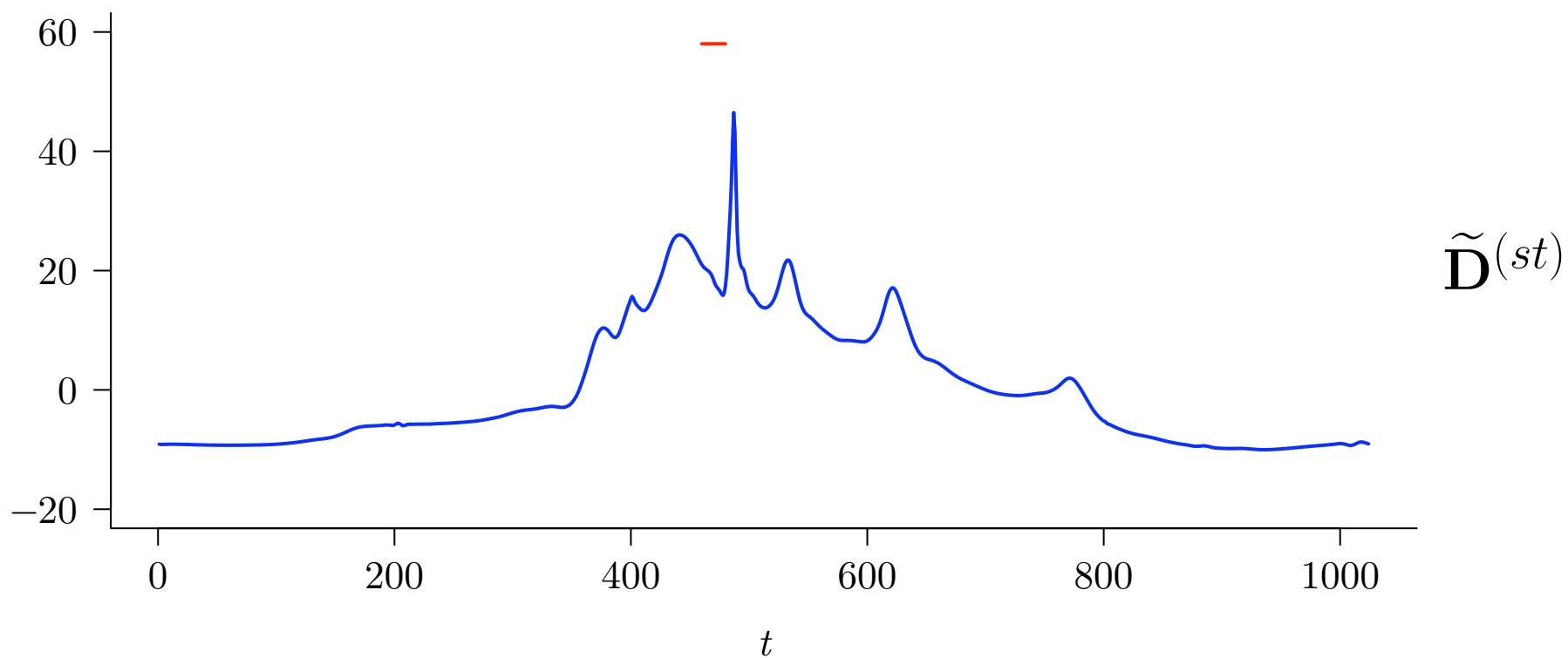
- results are identical to what ‘cycle spinning’ would yield

Examples of MODWT-Based Thresholding: I



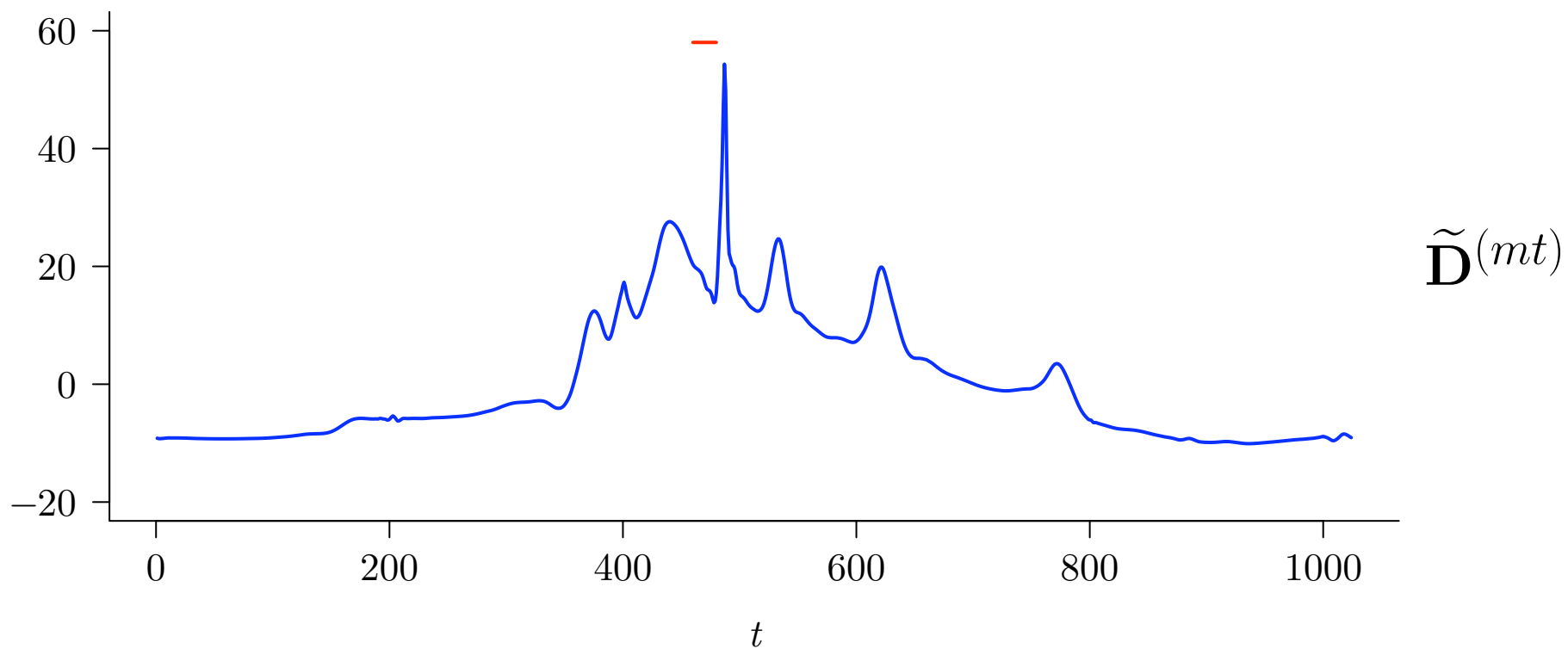
- signal estimate using $J_0 = 6$ LA(8) MODWT with hard thresholding

Examples of MODWT-Based Thresholding: II



- same as before, but now with soft thresholding

Examples of MODWT-Based Thresholding: III



- same as before, but now with mid thresholding

Signal Estimation via Shrinkage: I

- so far, we have only considered signal estimation via thresholding rules, which will map some O_l to zeros
- will now consider shrinkage rules, which differ from thresholding only in that nonzero coefficients are mapped to nonzero values rather than exactly zero (but values can be *very* close to zero!)
- several ways in which shrinkage rules arise – will consider a conditional mean approach (identical to a Bayesian approach)

Background on Conditional PDFs: I

- let X and Y be RVs with marginal probability density functions (PDFs) $f_X(\cdot)$ and $f_Y(\cdot)$
- let $f_{X,Y}(x, y)$ be their joint PDF at the point (x, y)
- conditional PDF of Y given $X = x$ is defined as

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

- $f_{Y|X=x}(\cdot)$ is a PDF, so its mean value is

$$E\{Y|X = x\} = \int_{-\infty}^{\infty} y f_{Y|X=x}(y) dy;$$

the above is called the conditional mean of Y , given X

Background on Conditional PDFs: II

- suppose RVs X and Y are related, but we can only observe X
- want to approximate unobservable Y based on function of X
- example: X represents a stochastic signal Y buried in noise
- suppose we want our approximation to be the function of X , say $U_2(X)$, such that the mean square difference between Y and $U_2(X)$ is as small as possible; i.e., we want

$$E\{(Y - U_2(X))^2\}$$

to be as small as possible

- solution is to use $U_2(X) = E\{Y|X\}$; i.e., the conditional mean of Y given X is our best guess at Y in the sense of minimizing the mean square error (related to fact that $E\{(Y - a)^2\}$ is smallest when $a = E\{Y\}$)

Conditional Mean Approach: I

- assume model of stochastic signal plus non-IID noise:
 $\mathbf{X} = \mathbf{C} + \boldsymbol{\eta}$ so that $\mathbf{O} = \mathcal{O}\mathbf{X} = \mathcal{O}\mathbf{C} + \mathcal{O}\boldsymbol{\eta} \equiv \mathbf{R} + \mathbf{n}$
- component-wise, have $O_l = R_l + n_l$
- because \mathbf{C} and $\boldsymbol{\eta}$ are independent, \mathbf{R} and \mathbf{n} must be also
- suppose we approximate R_l via $\hat{R}_l \equiv U_2(O_l)$, where $U_2(O_l)$ is selected to minimize $E\{(R_l - U_2(O_l))^2\}$
- solution is to set $U_2(O_l)$ equal to $E\{R_l|O_l\}$, so let's work out what form this conditional mean takes
- to get $E\{R_l|O_l\}$, need the PDF of R_l given O_l , which is

$$f_{R_l|O_l=o_l}(r_l) = \frac{f_{R_l,O_l}(r_l, o_l)}{f_{O_l}(o_l)} = \frac{f_{R_l}(r_l)f_{n_l}(o_l - r_l)}{\int_{-\infty}^{\infty} f_{R_l}(r_l)f_{n_l}(o_l - r_l) dr_l}$$

Conditional Mean Approach: II

- mean value of $f_{R_l|O_l=o_l}(\cdot)$ yields estimator $\hat{R}_l = E\{R_l|O_l\}$:

$$\begin{aligned} E\{R_l|O_l = o_l\} &= \int_{-\infty}^{\infty} r_l f_{R_l|O_l=o_l}(r_l) dr_l \\ &= \frac{\int_{-\infty}^{\infty} r_l f_{R_l}(r_l) f_{n_l}(o_l - r_l) dr_l}{\int_{-\infty}^{\infty} f_{R_l}(r_l) f_{n_l}(o_l - r_l) dr_l} \end{aligned}$$

- to make further progress, we need a model for the wavelet-domain representation R_l of the signal
- heuristic that signal in the wavelet domain has a few large values and lots of small values suggests a Gaussian mixture model

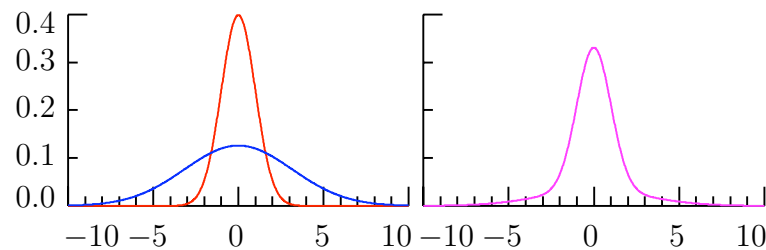
Conditional Mean Approach: III

- let \mathcal{I}_l be an RV such that $\mathbf{P} [\mathcal{I}_l = 1] = p_l$ & $\mathbf{P} [\mathcal{I}_l = 0] = 1 - p_l$
- under Gaussian mixture model, R_l has same distribution as

$$\mathcal{I}_l \mathcal{N}(0, \gamma_l^2 \sigma_{G_l}^2) + (1 - \mathcal{I}_l) \mathcal{N}(0, \sigma_{G_l}^2)$$

where $\mathcal{N}(0, \sigma^2)$ is a Gaussian RV with mean 0 and variance σ^2

- 2nd component models small # of large signal coefficients
 - 1st component models large # of small coefficients ($\gamma_l^2 \ll 1$)
- example: PDFs for case $\sigma_{G_l}^2 = 10$, $\gamma_l^2 \sigma_{G_l}^2 = 1$ and $p_l = 0.75$

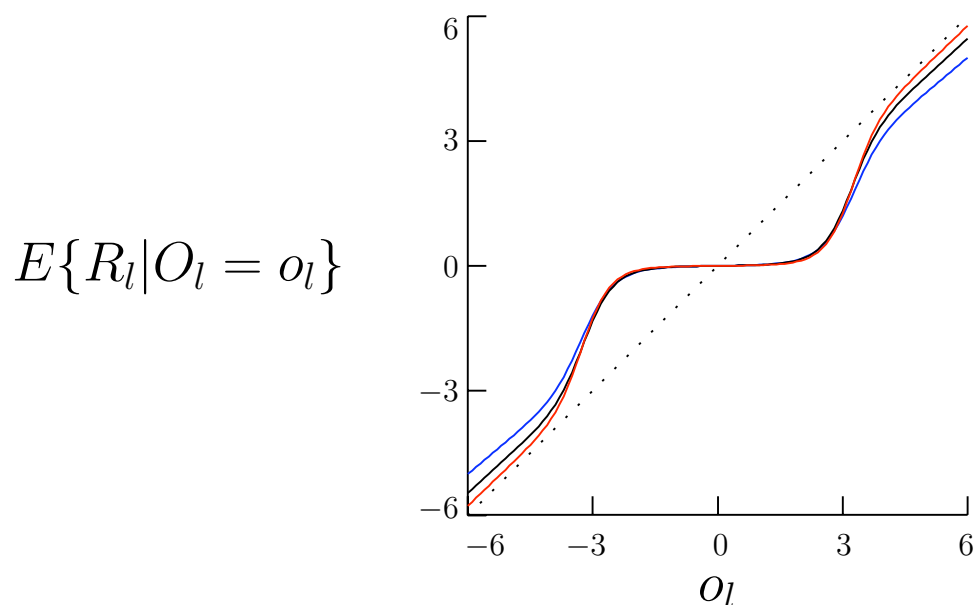


Conditional Mean Approach: VI

- let's simplify to a 'sparse' signal model by setting $\gamma_l = 0$; i.e., large # of small coefficients are all zero
- distribution for R_l same as $(1 - \mathcal{I}_l)\mathcal{N}(0, \sigma_{G_l}^2)$
- to complete model, let n_l obey a Gaussian distribution with mean 0 and variance $\sigma_{n_l}^2$
- conditional mean estimator becomes $E\{R_l|O_l = o_l\} = \frac{b_l}{1+c_l}o_l$, where

$$c_l = \frac{p_l \sqrt{(\sigma_{G_l}^2 + \sigma_{n_l}^2)} e^{-o_l^2 b_l / (2\sigma_{n_l}^2)}}{(1 - p_l)\sigma_{n_l}}$$

Conditional Mean Approach: VII



- conditional mean shrinkage rule for $p_l = 0.95$ (i.e., $\approx 95\%$ of signal coefficients are 0); $\sigma_{n_l}^2 = 1$; and $\sigma_{G_l}^2 = 5$ (curve furthest from dotted diagonal), 10 and 25 (curve nearest to diagonal)
- as $\sigma_{G_l}^2$ gets large (i.e., large signal coefficients increase in size), shrinkage rule starts to resemble mid thresholding rule

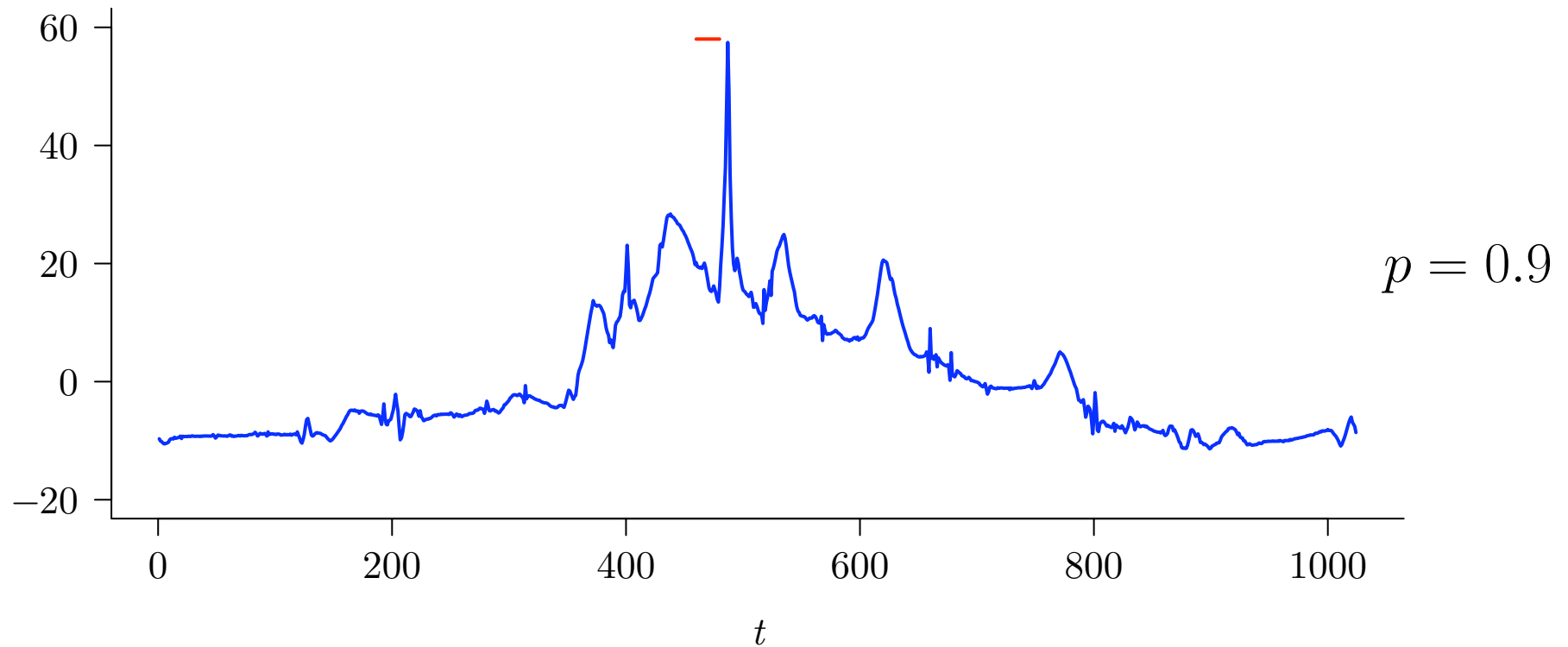
Wavelet-Based Shrinkage: I

- assume model of stochastic signal plus Gaussian IID noise:
 $\mathbf{X} = \mathbf{C} + \boldsymbol{\epsilon}$ so that $\mathbf{W} = \mathcal{W}\mathbf{X} = \mathcal{W}\mathbf{C} + \mathcal{W}\boldsymbol{\epsilon} \equiv \mathbf{R} + \mathbf{e}$
- component-wise, have $W_{j,t} = R_{j,t} + e_{j,t}$
- form partial DWT of level J_0 , shrink \mathbf{W}_j 's, but leave \mathbf{V}_{J_0} alone
- assume $E\{R_{j,t}\} = 0$ (reasonable for \mathbf{W}_j , but not for \mathbf{V}_{J_0})
- use a conditional mean approach with the sparse signal model
 - $R_{j,t}$ has distribution dictated by $(1 - \mathcal{I}_{j,t})\mathcal{N}(0, \sigma_G^2)$, where
$$\mathbf{P} [\mathcal{I}_{j,t} = 1] = p \text{ and } \mathbf{P} [\mathcal{I}_{j,t} = 0] = 1 - p$$
 - $R_{j,t}$'s are assumed to be IID
 - model for $e_{j,t}$ is Gaussian with mean 0 and variance σ_ϵ^2
 - note: parameters do not vary with j or t

Wavelet-Based Shrinkage: II

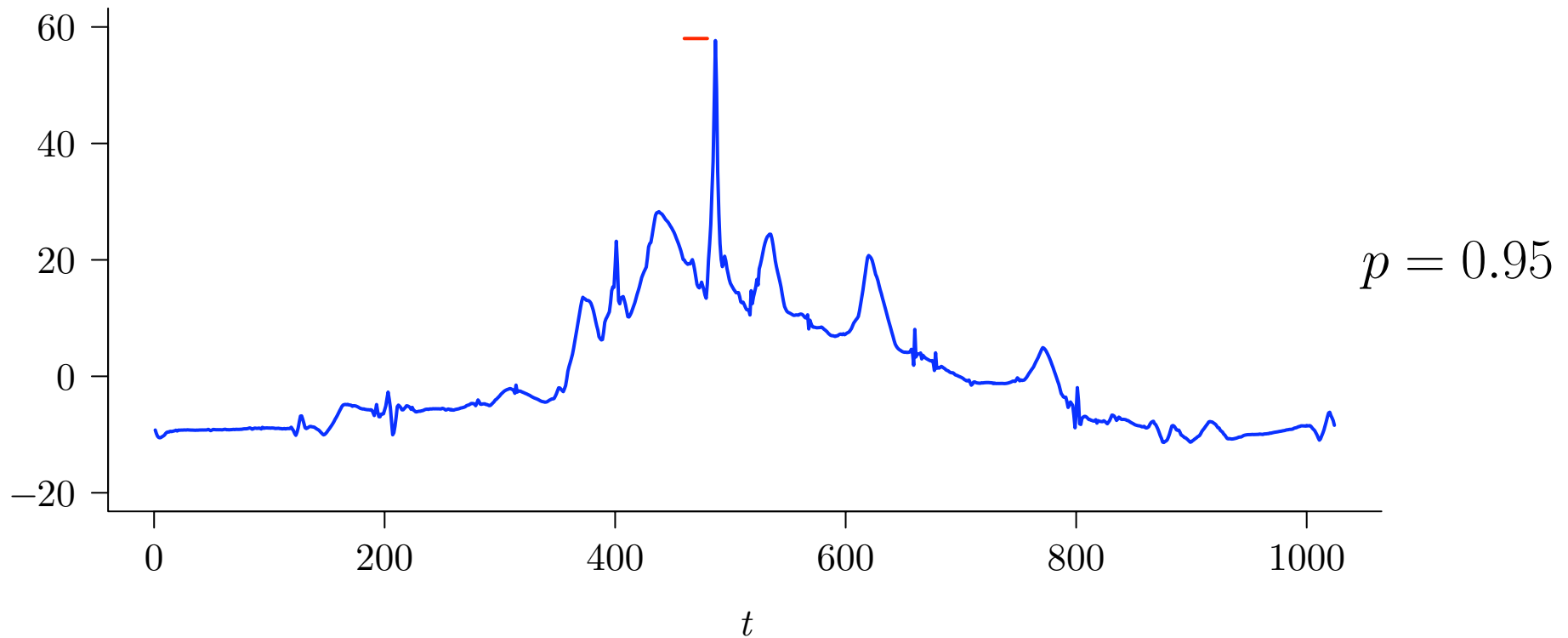
- model has three parameters σ_G^2 , p and σ_ϵ^2 , which need to be set
- let σ_R^2 and σ_W^2 be variances of RVs $R_{j,t}$ and $W_{j,t}$
- have relationships $\sigma_R^2 = (1 - p)\sigma_G^2$ and $\sigma_W^2 = \sigma_R^2 + \sigma_\epsilon^2$
 - set $\hat{\sigma}_\epsilon^2 = \hat{\sigma}_{(\text{mad})}^2$ using \mathbf{W}_1
 - let $\hat{\sigma}_W^2$ be sample mean of all $W_{j,t}^2$
 - given p , let $\hat{\sigma}_G^2 = (\hat{\sigma}_W^2 - \hat{\sigma}_\epsilon^2)/(1 - p)$
 - p usually chosen subjectively, keeping in mind that p is proportion of noise-dominated coefficients (can set based on rough estimate of proportion of ‘small’ coefficients)

Examples of Wavelet-Based Shrinkage: I



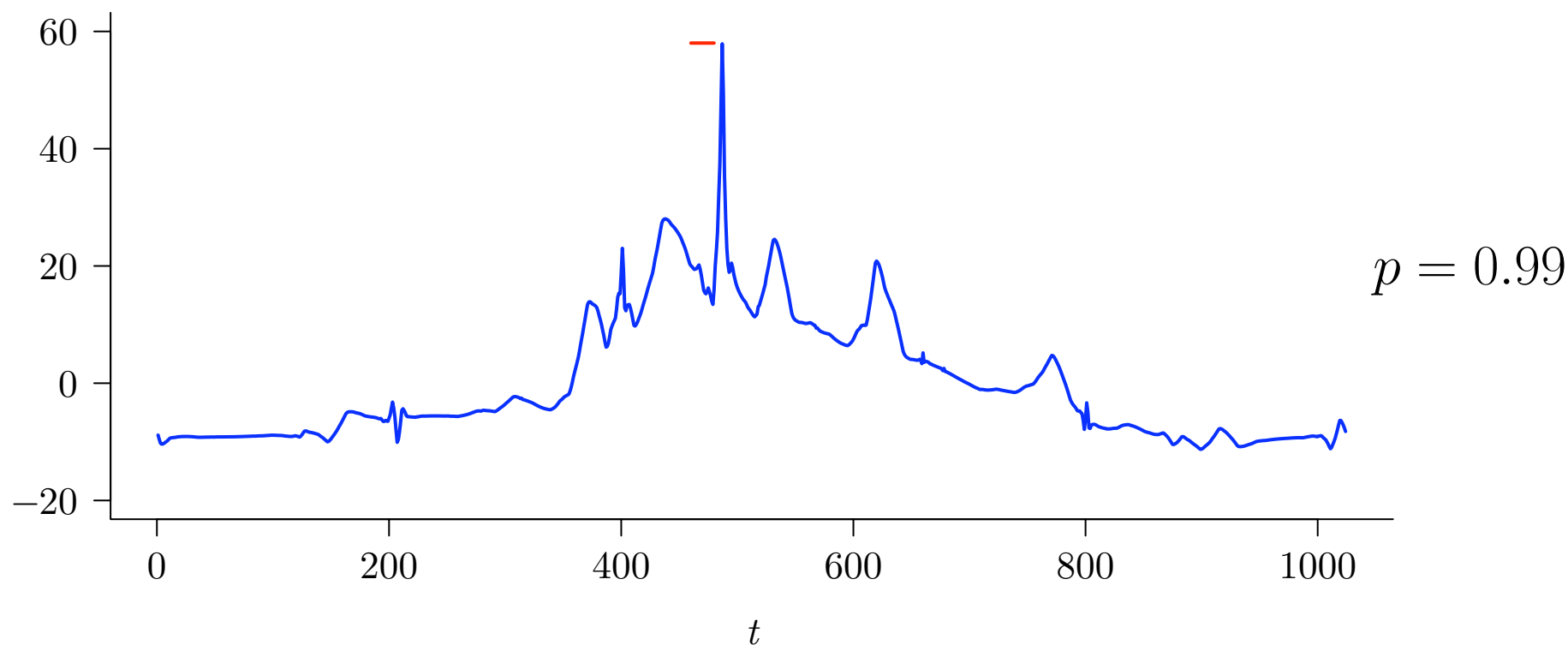
- shrinkage signal estimates of NMR spectrum based upon level $J_0 = 6$ partial LA(8) DWT and conditional mean with $p = 0.9$

Examples of Wavelet-Based Shrinkage: II



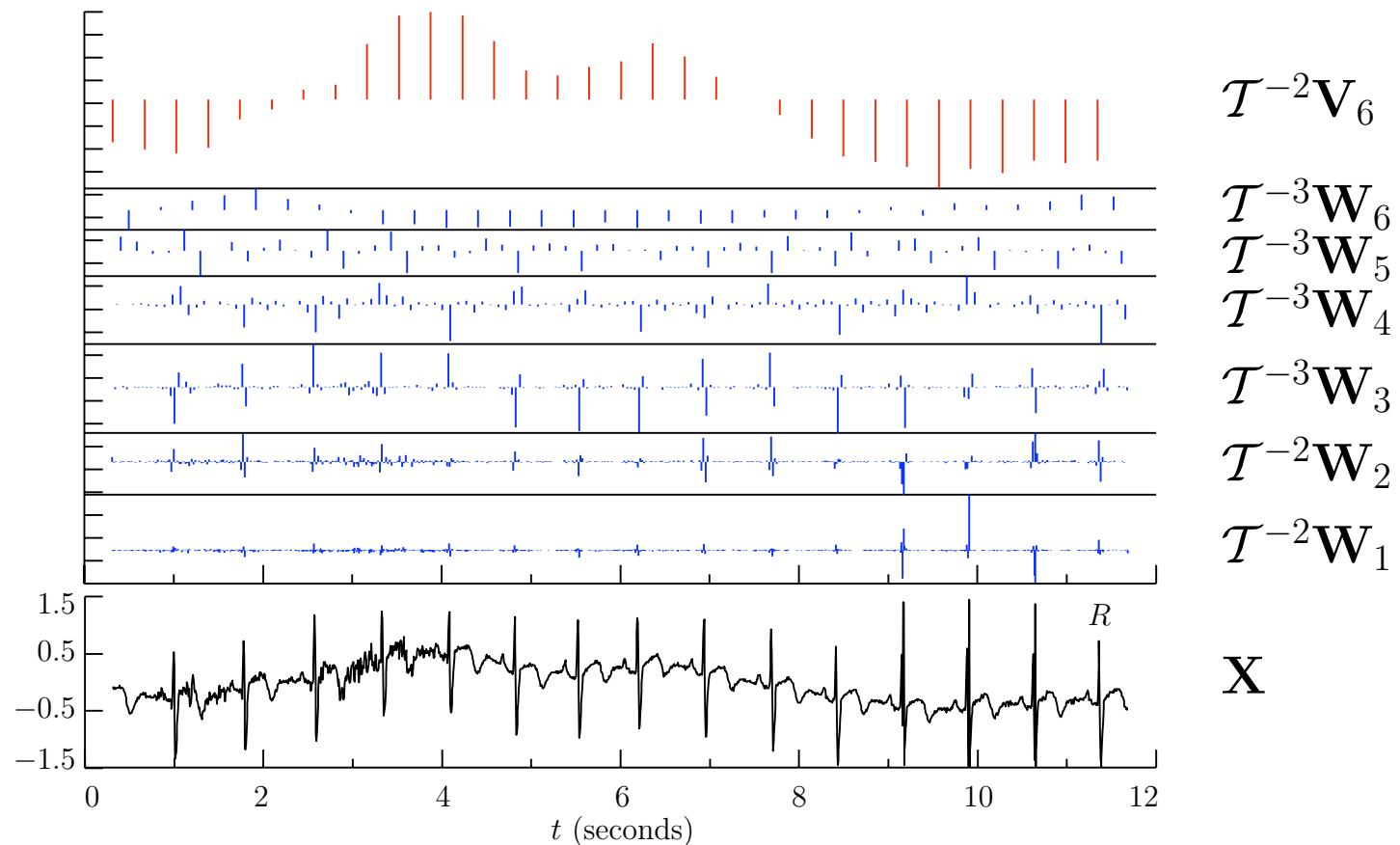
- same as before, but now with $p = 0.95$

Examples of Wavelet-Based Shrinkage: III



- same as before, but now with $p = 0.99$ (as $p \rightarrow 1$, we declare there are proportionately fewer significant signal coefficients, implying need for heavier shrinkage)

Comments on ‘Next Generation’ Denoising: I



- ‘classical’ denoising looks at each $W_{j,t}$ alone; for ‘real world’ signals, coefficients often cluster within a given level and persist across adjacent levels (ECG series offers an example)

Comments on ‘Next Generation’ Denoising: II

- here are some ‘next generation’ approaches that exploit these ‘real world’ properties:
 - Crouse *et al.* (1998) use hidden Markov models for stochastic signal DWT coefficients to handle clustering, persistence and non-Gaussianity
 - Huang and Cressie (2000) consider scale-dependent multi-scale graphical models to handle clustering and persistence
 - Cai and Silverman (2001) consider ‘block’ thresholding in which coefficients are thresholded in blocks rather than individually (handles clustering)
 - Dragotti and Vetterli (2003) introduce the notion of ‘wavelet footprints’ to track discontinuities in a signal across different scales (handles persistence)

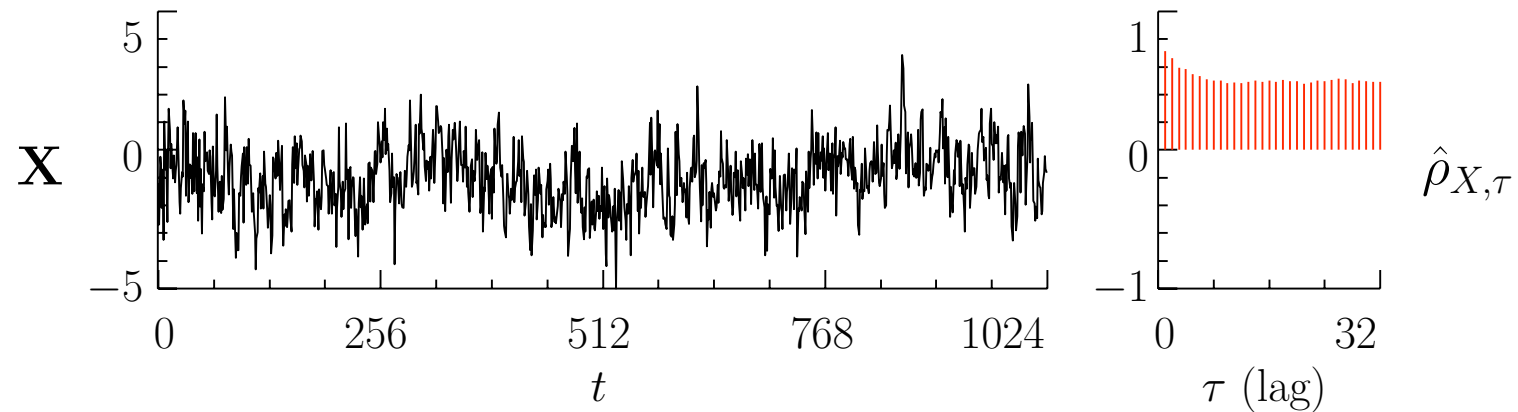
Comments on ‘Next Generation’ Denoising: III

- ‘classical’ denoising also suffers from problem of overall significance of multiple hypothesis tests
- ‘next generation’ work integrates idea of ‘false discovery rate’ (Benjamini and Hochberg, 1995) into denoising (see Wink and Roerdink, 2004, for an applications-oriented discussion)
- for more recent developments (there are a lot!!!), see
 - review article by Antoniadis (2007)
 - Chapters 3 and 4 of book by Nason (2008)
 - October 2009 issue of *Statistica Sinica*, which has a special section entitled ‘Multiscale Methods and Statistics: A Productive Marriage’

Wavelet-Based Decorrelation of Time Series: Overview

- DWT well-suited for decorrelating certain time series, including ones generated from a fractionally differenced (FD) process
- on synthesis side, leads to
 - DWT-based simulation of FD processes
 - wavelet-based bootstrapping
- on analysis side, leads to
 - wavelet-based estimators for FD parameters
 - test for homogeneity of variance (will cover briefly)
 - test for trends (won't discuss – see Craigmile *et al.*, 2004, for details)

DWT of an FD Process: I

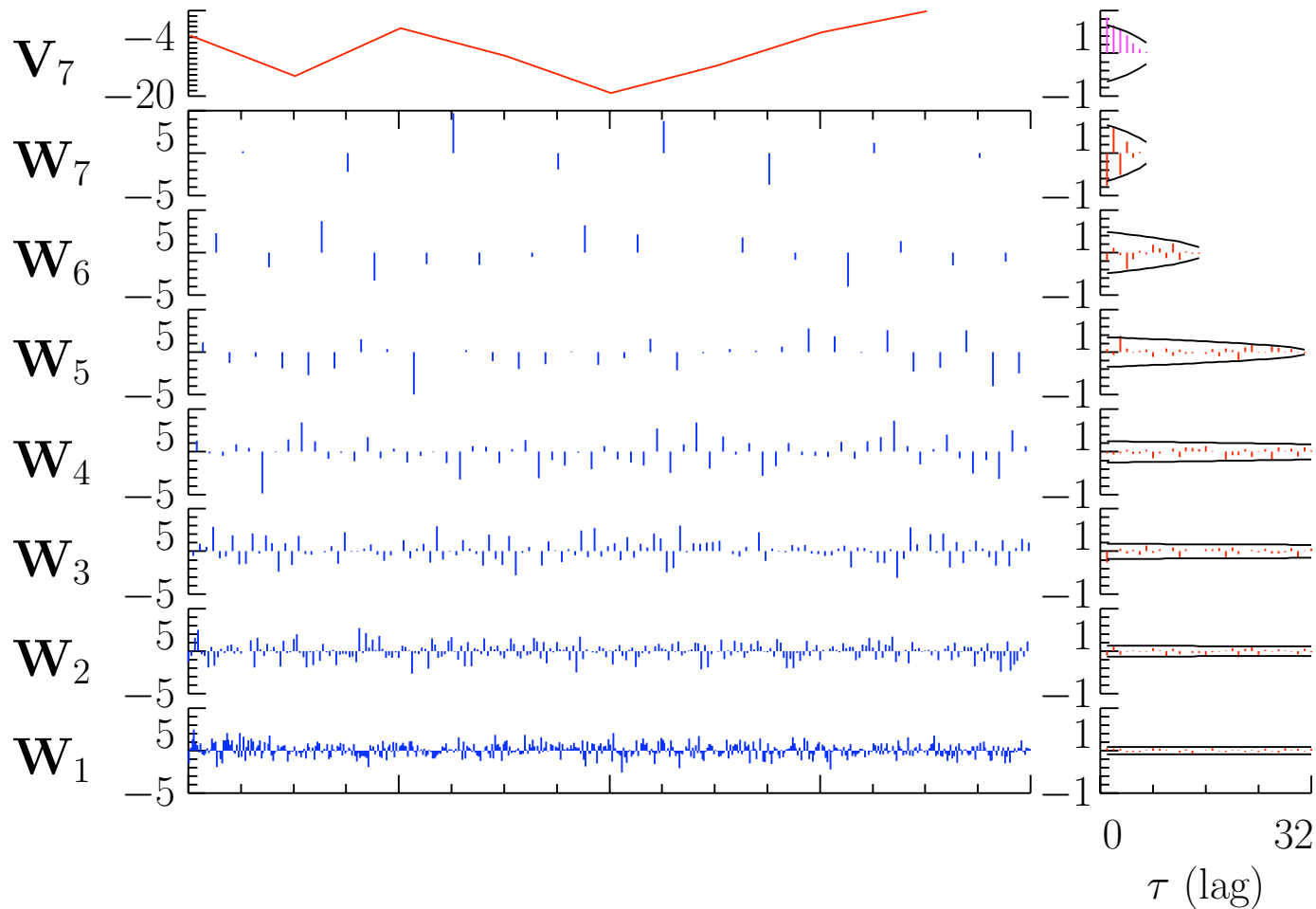


- realization of an FD(0.4) time series \mathbf{X} along with its sample autocorrelation sequence (ACS): for $\tau \geq 0$,

$$\hat{\rho}_{X,\tau} \equiv \frac{\sum_{t=0}^{N-1-\tau} X_t X_{t+\tau}}{\sum_{t=0}^{N-1} X_t^2}$$

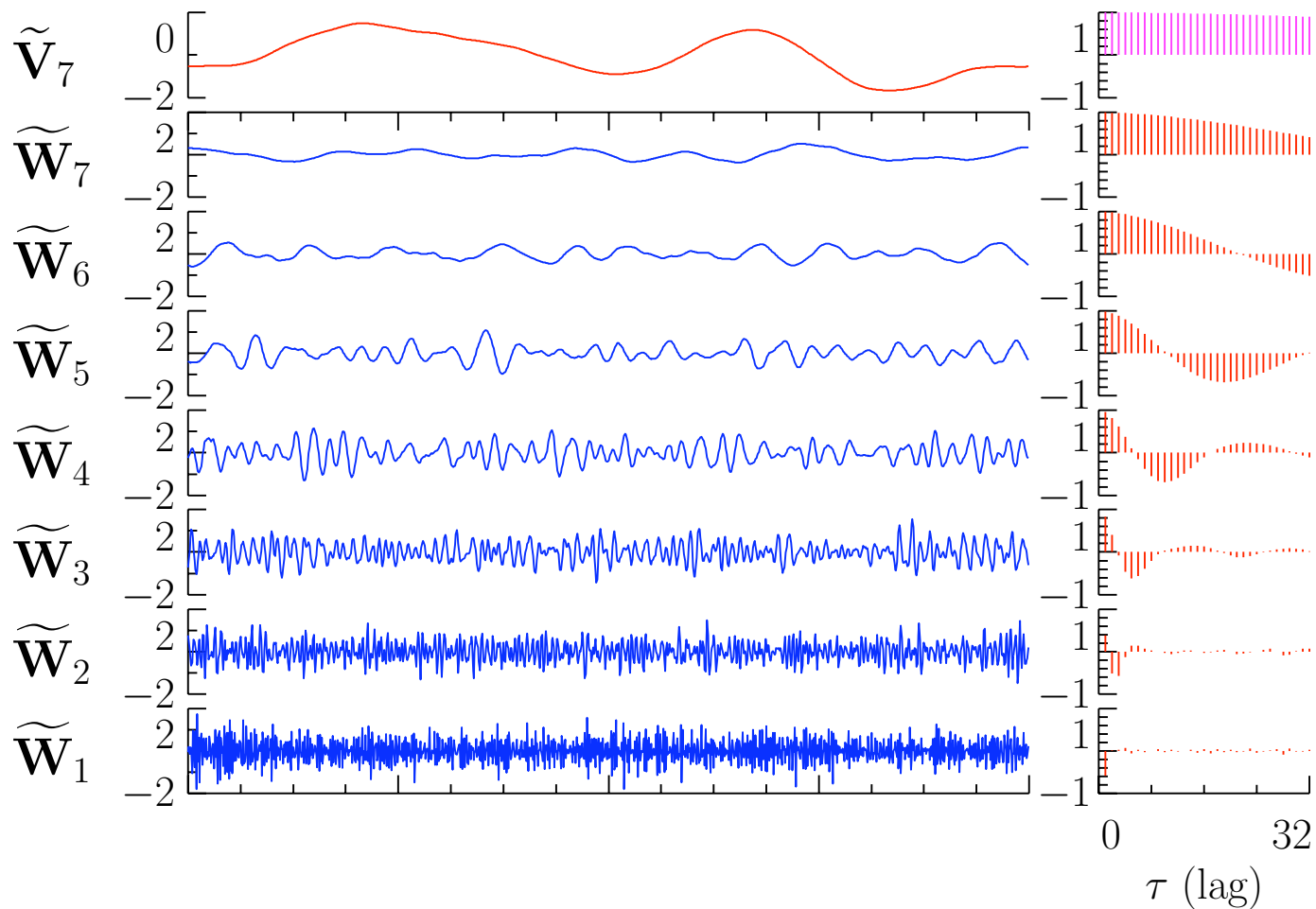
- note that ACS dies down slowly

DWT of an FD Process: II



- LA(8) DWT of FD(0.4) series and sample ACSs for each \mathbf{W}_j & \mathbf{V}_7 , along with 95% confidence intervals for white noise

MODWT of an FD Process



- LA(8) MODWT of FD(0.4) series & sample ACSs for MODWT coefficients, none of which are approximately uncorrelated

DWT of an FD Process: III

- in contrast to \mathbf{X} , ACSs for \mathbf{W}_j consistent with white noise
- variance of RVs in \mathbf{W}_j increases with j : for FD process,

$$\text{var} \{W_{j,t}\} \approx c\tau_j^{2\delta} \equiv C_j,$$

where c is a constant depending on δ but not j , and $\tau_j = 2^{j-1}$ is scale associated with \mathbf{W}_j

- for white noise ($\delta = 0$), $\text{var} \{W_{j,t}\}$ is the same for all j
- dependence in \mathbf{X} thus manifests itself in wavelet domain by different variances for wavelet coefficients at different scales

Correlations Within a Scale and Between Two Scales

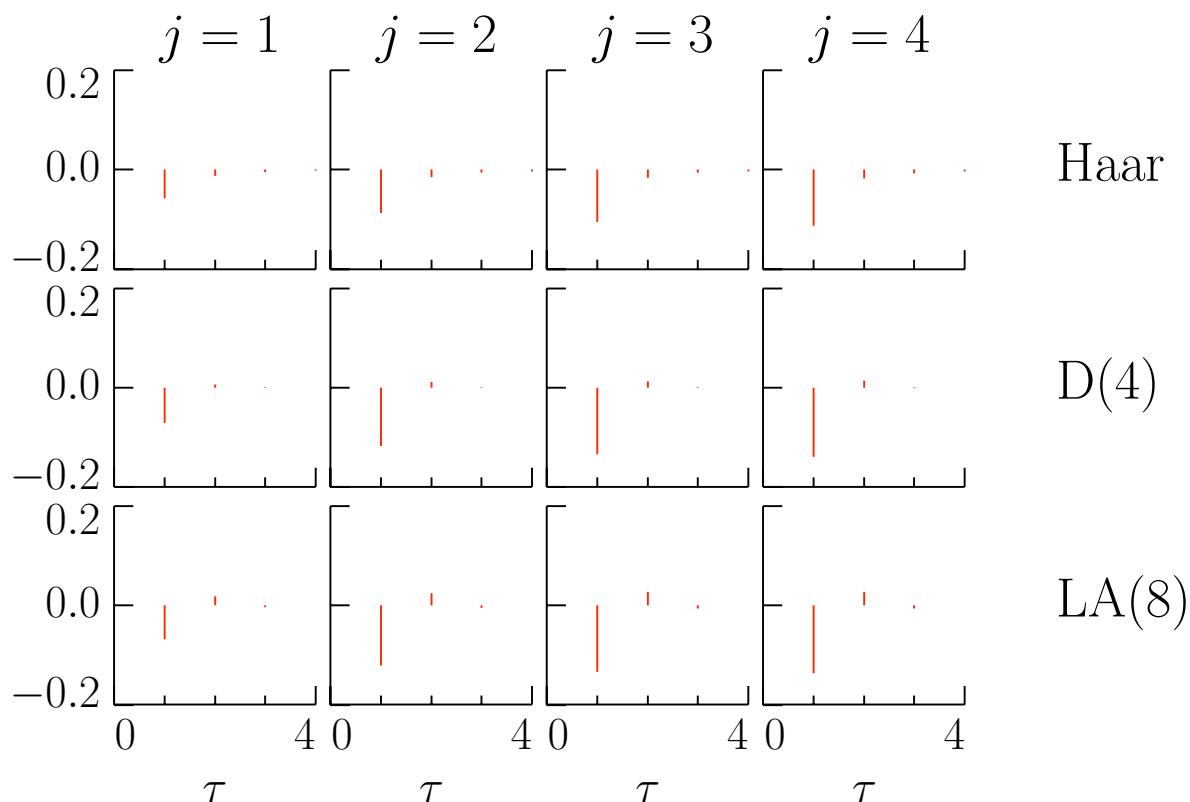
- let $\{s_{X,\tau}\}$ denote autocovariance sequence (ACVS) for $\{X_t\}$; i.e., $s_{X,\tau} = \text{cov}\{X_t, X_{t+\tau}\}$
- let $\{h_{j,l}\}$ denote equivalent wavelet filter for j th level
- to quantify decorrelation, can write

$$\text{cov}\{W_{j,t}, W_{j',t'}\} = \sum_{l=0}^{L_j-1} \sum_{l'=0}^{L_{j'}-1} h_{j,l} h_{j',l'} s_{X,2^j(t+1)-l-2^{j'}(t'+1)+l'}$$

from which we can get ACVS (and hence within-scale correlations) for $\{W_{j,t}\}$:

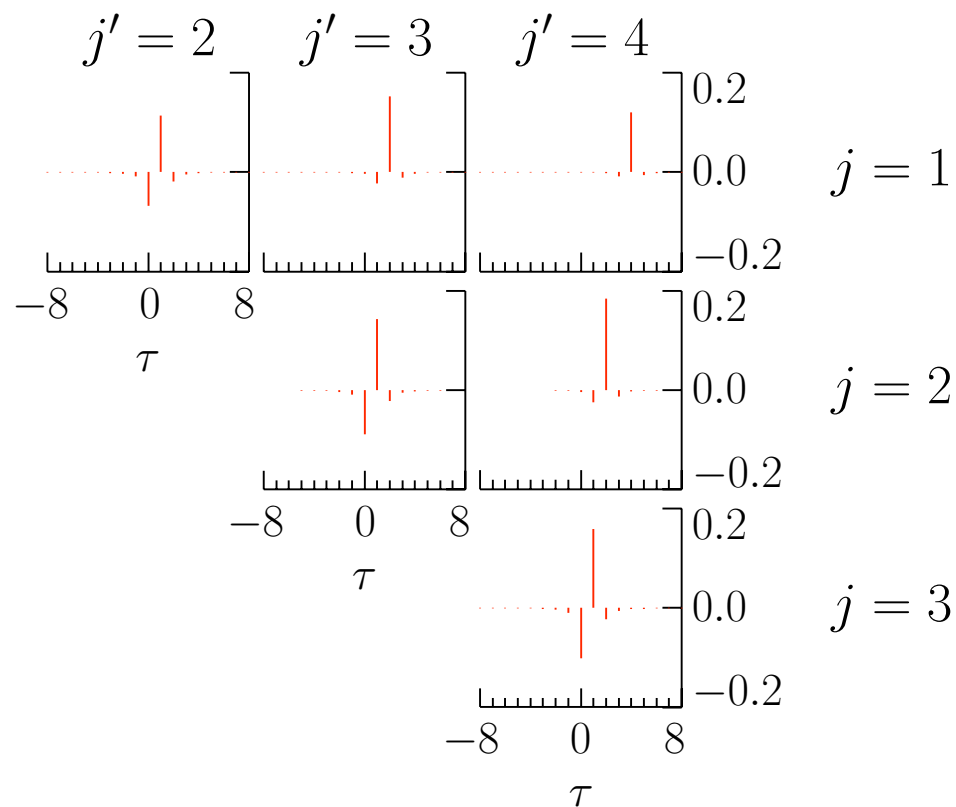
$$\text{cov}\{W_{j,t}, W_{j,t+\tau}\} = \sum_{m=-(L_j-1)}^{L_j-1} s_{X,2^j\tau+m} \sum_{l=0}^{L_j-|m|-1} h_{j,l} h_{j,l+|m|}$$

Correlations Within a Scale



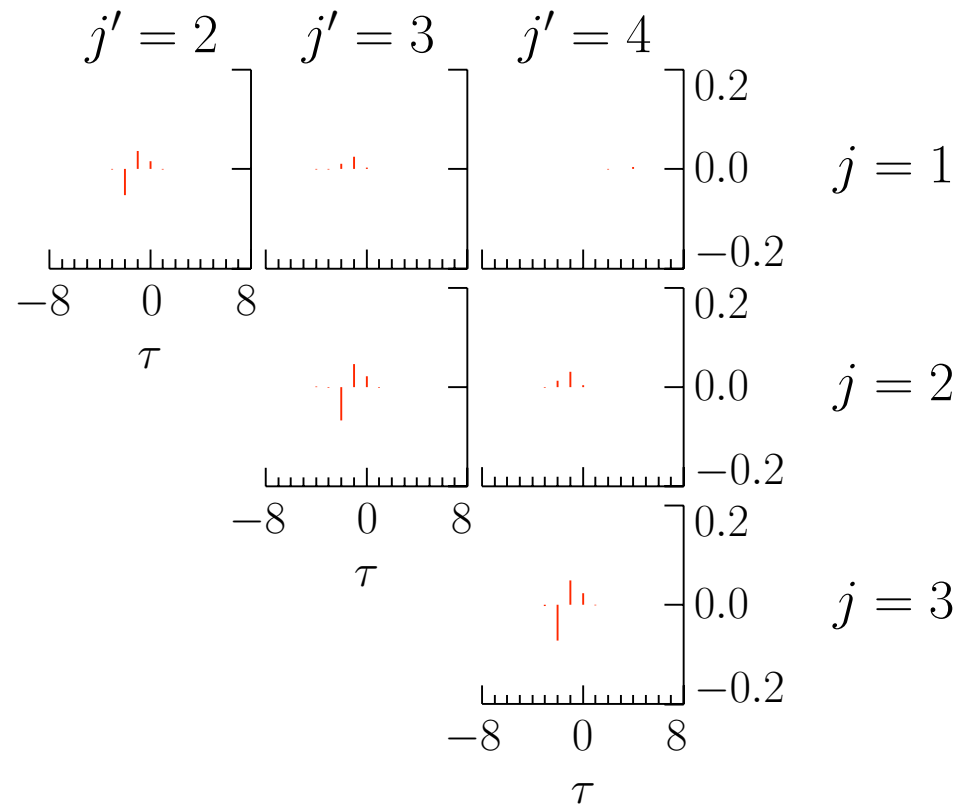
- correlations between $W_{j,t}$ and $W_{j,t+\tau}$ for an FD(0.4) process
- correlations within scale are slightly smaller for Haar
- maximum magnitude of correlation is less than 0.2

Correlations Between Two Scales: I



- correlation between Haar wavelet coefficients $W_{j,t}$ and $W_{j',t'}$ from FD(0.4) process and for levels satisfying $1 \leq j < j' \leq 4$

Correlations Between Two Scales: II



- same as before, but now for LA(8) wavelet coefficients
- correlations between scales decrease as L increases

Wavelet Domain Description of FD Process

- DWT acts as a decorrelating transform for FD processes and other (but not all!) intrinsically stationary processes
- wavelet domain description is simple
 - wavelet coefficients within a given scale approximately uncorrelated (refinement: assume 1st order autoregressive model)
 - wavelet coefficients have scale-dependent variance controlled by the two FD parameters (δ and σ_ε^2)
 - wavelet coefficients between scales also approximately uncorrelated (approximation improves as filter width L increases)

DWT-Based Simulation

- properties of DWT of FD processes lead to schemes for simulating time series $\mathbf{X} \equiv [X_0, \dots, X_{N-1}]^T$ with zero mean and with a multivariate Gaussian distribution
- with $N = 2^J$, recall that $\mathbf{X} = \mathcal{W}^T \mathbf{W}$, where

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_j \\ \vdots \\ \mathbf{W}_J \\ \mathbf{V}_J \end{bmatrix}$$

Basic DWT-Based Simulation Scheme

- assume \mathbf{W} to contain N uncorrelated Gaussian (normal) random variables (RVs) with zero mean
- assume \mathbf{W}_j to have variance $C_j = c\tau_j^{2\delta}$
- assume single RV in \mathbf{V}_J to have variance C_{J+1} (see Percival and Walden, 2000, for details on how to set C_{J+1})
- approximate FD time series \mathbf{X} via $\mathbf{Y} \equiv \mathcal{W}^T \Lambda^{1/2} \mathbf{Z}$, where
 - $\Lambda^{1/2}$ is $N \times N$ diagonal matrix with diagonal elements

$$\underbrace{C_1^{1/2}, \dots, C_1^{1/2}}_{\frac{N}{2} \text{ of these}}, \underbrace{C_2^{1/2}, \dots, C_2^{1/2}}_{\frac{N}{4} \text{ of these}}, \dots, \underbrace{C_{J-1}^{1/2}, C_{J-1}^{1/2}}_{2 \text{ of these}}, C_J^{1/2}, C_{J+1}^{1/2}$$
 - \mathbf{Z} is vector of deviations drawn from a Gaussian distribution with zero mean and unit variance

Refinements to Basic Scheme: I

- covariance matrix for approximation \mathbf{Y} does not correspond to that of a stationary process
- recall \mathcal{W} treats \mathbf{X} as if it were circular
- let \mathcal{T} be $N \times N$ ‘circular shift’ matrix:

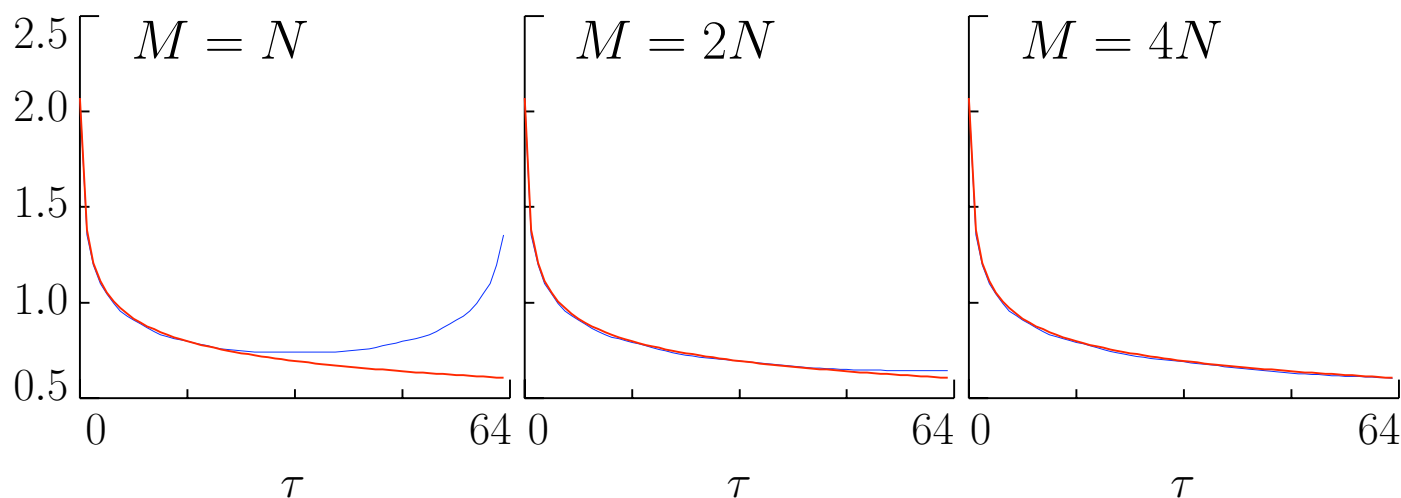
$$\mathcal{T} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_0 \end{bmatrix}; \quad \mathcal{T}^2 \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_2 \\ Y_3 \\ Y_0 \\ Y_1 \end{bmatrix}; \quad \text{etc.}$$

- let κ be uniformly distributed over $0, \dots, N - 1$
- define $\tilde{\mathbf{Y}} \equiv \mathcal{T}^\kappa \mathbf{Y}$
- $\tilde{\mathbf{Y}}$ is stationary with ACVS given by, say, $s_{\tilde{\mathbf{Y}}, \tau}$

Refinements to Basic Scheme: II

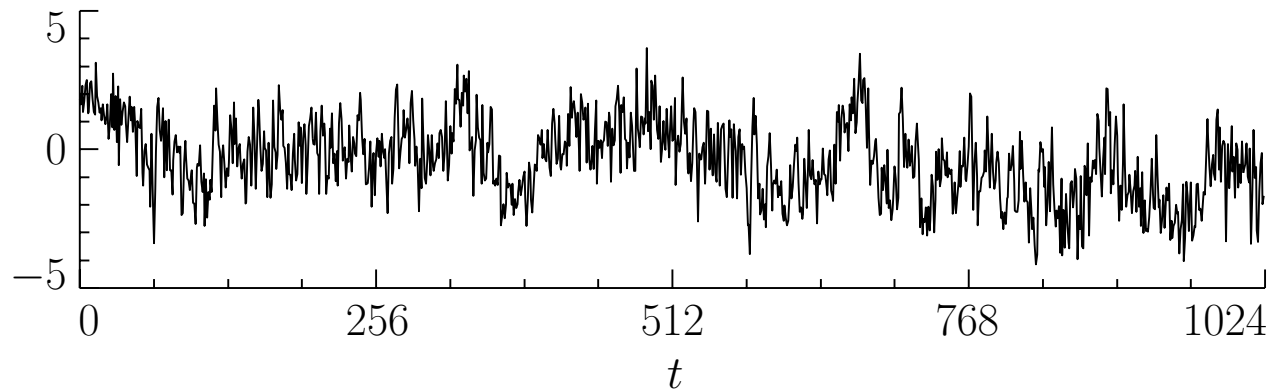
- Q: how well does $\{s_{\tilde{Y},\tau}\}$ match $\{s_{X,\tau}\}$?
- due to circularity, find that $s_{\tilde{Y},N-\tau} = s_{\tilde{Y},\tau}$ for $\tau = 1, \dots, N/2$
- implies $s_{\tilde{Y},\tau}$ cannot approximate $s_{X,\tau}$ well for τ close to N
- can patch up by simulating $\tilde{\mathbf{Y}}$ with $M > N$ elements and then extracting first N deviates ($M = 4N$ works well)

Refinements to Basic Scheme: III



- plot shows **true** ACVS $\{s_{X,\tau}\}$ (**thick** curves) for FD(0.4) process and wavelet-based **approximate** ACVSs $\{s_{\tilde{Y},\tau}\}$ (**thin** curves) based on an LA(8) DWT in which an $N = 64$ series is extracted from $M = N$, $M = 2N$ and $M = 4N$ series

Example and Some Notes



- simulated FD(0.4) series (LA(8), $N = 1024$ and $M = 4N$)
- notes:
 - can form realizations faster than best exact method
 - can efficiently simulate extremely long time series in ‘real-time’ (e.g, $N = 2^{30} = 1,073,741,824$ or even longer!)
 - effect of random circular shifting is to render time series slightly non-Gaussian (a Gaussian mixture model)

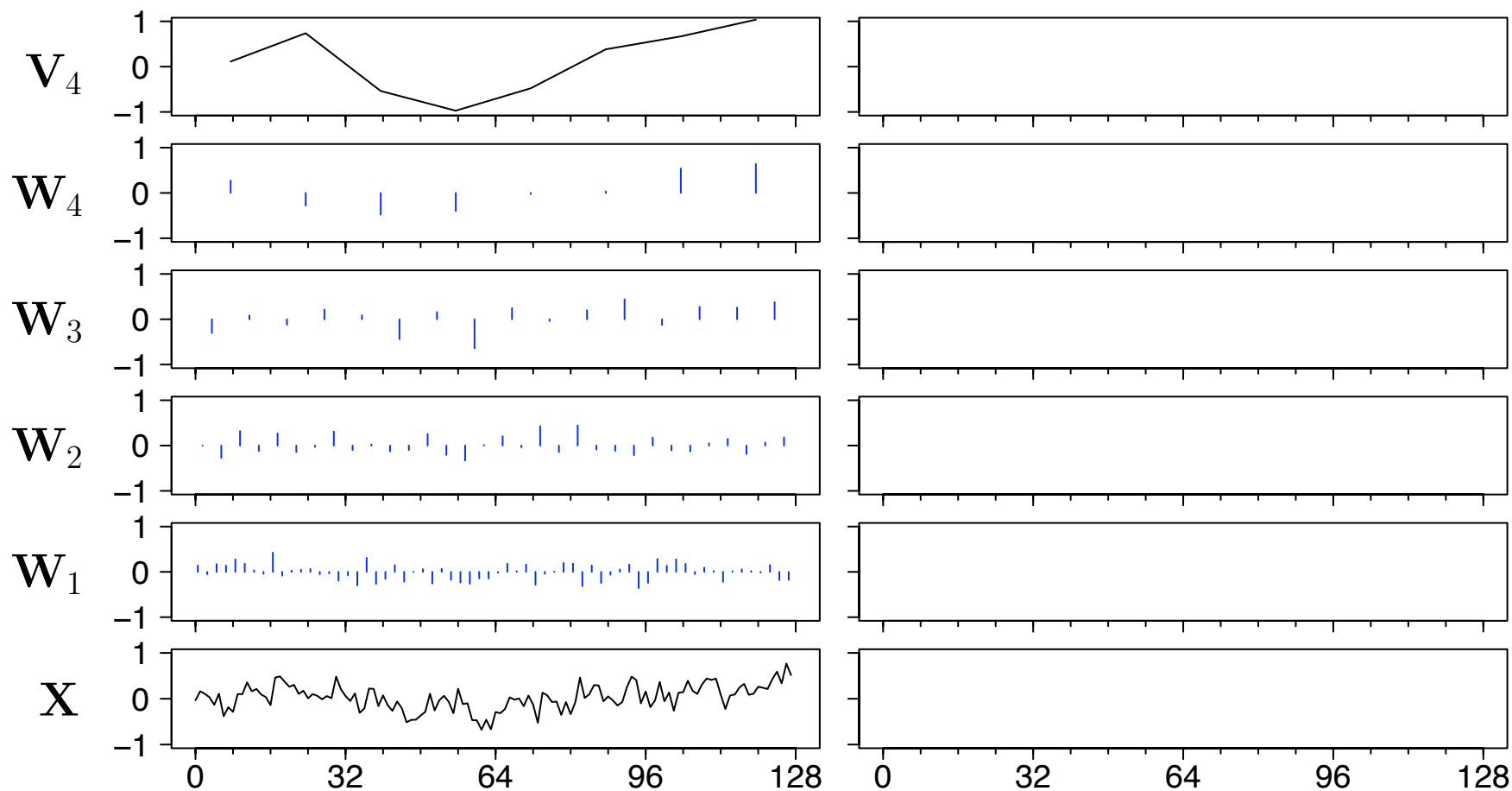
Wavelet-Domain Bootstrapping

- for many (but not all!) time series, DWT acts as a decorrelating transform: to a good approximation, each \mathbf{W}_j is a sample of a white noise process, and coefficients from different sub-vectors \mathbf{W}_j and $\mathbf{W}_{j'}$ are also pairwise uncorrelated
- variance of coefficients in \mathbf{W}_j depends on j
- scaling coefficients \mathbf{V}_{J_0} are still autocorrelated, but there will be just a few of them if J_0 is selected to be large
- decorrelating property holds particularly well for FD and other processes with long-range dependence
- above suggests the following recipe for wavelet-domain bootstrapping of a statistic of interest, e.g., sample autocorrelation sequence $\hat{\rho}_{X,\tau}$ at unit lag $\tau = 1$

Recipe for Wavelet-Domain Bootstrapping

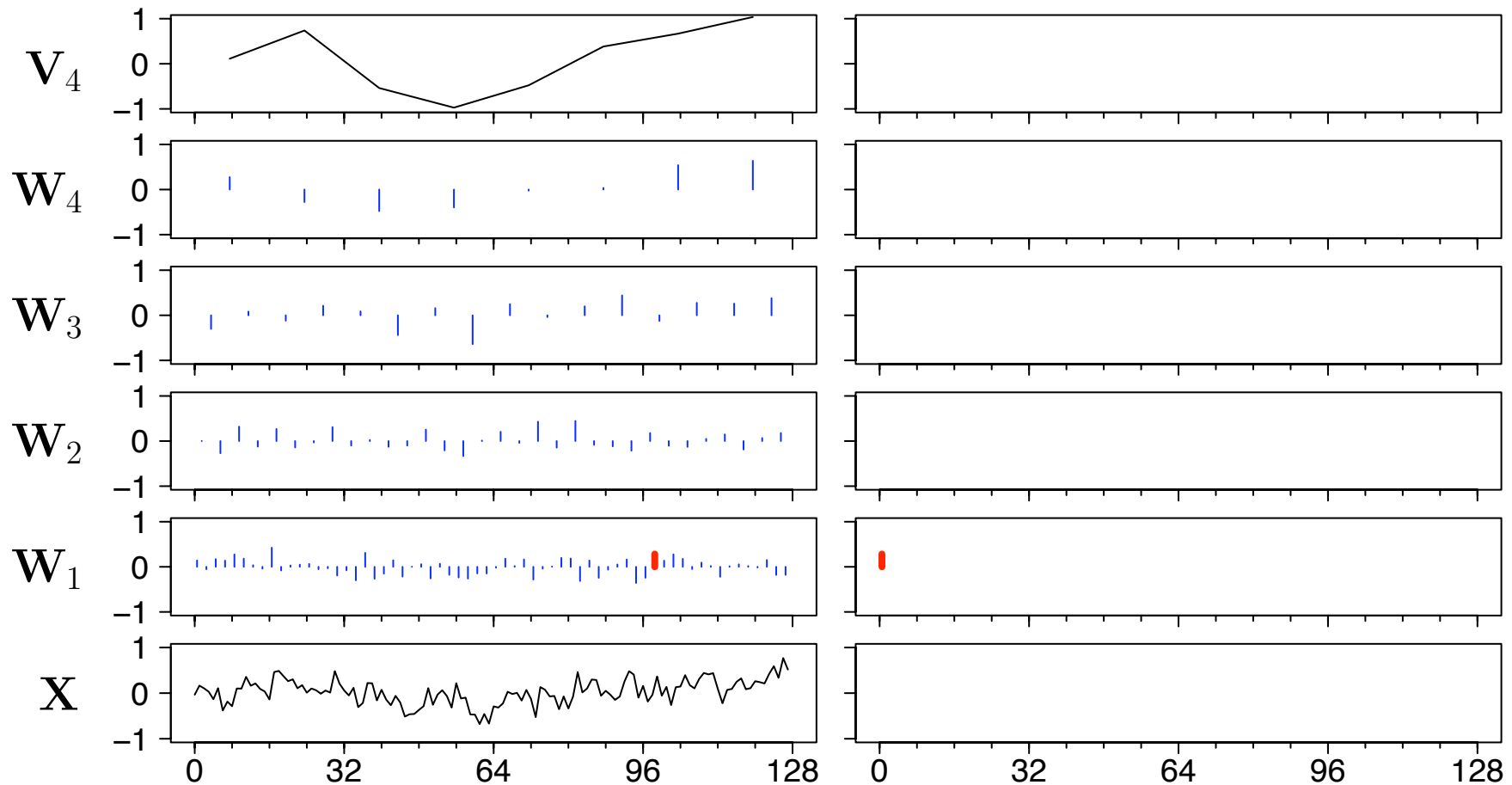
1. given \mathbf{X} of length $N = 2^J$, compute level J_0 DWT (the choice $J_0 = J - 3$ yields 8 coefficients in \mathbf{W}_{J_0} and \mathbf{V}_{J_0})
 2. randomly sample with replacement from \mathbf{W}_j to create bootstrapped vector $\mathbf{W}_j^{(b)}$, $j = 1, \dots, J_0$
 3. create $\mathbf{V}_{J_0}^{(b)}$ using 1st-order autoregressive parametric bootstrap
 4. apply \mathcal{W}^T to $\mathbf{W}_1^{(b)}, \dots, \mathbf{W}_{J_0}^{(b)}$ and $\mathbf{V}_{J_0}^{(b)}$ to obtain bootstrapped time series $\mathbf{X}^{(b)}$ and then form $\hat{\rho}_{X,1}^{(b)}$
- repeat above many times to build up sample distribution of bootstrapped autocorrelations

Illustration of Wavelet-Domain Bootstrapping



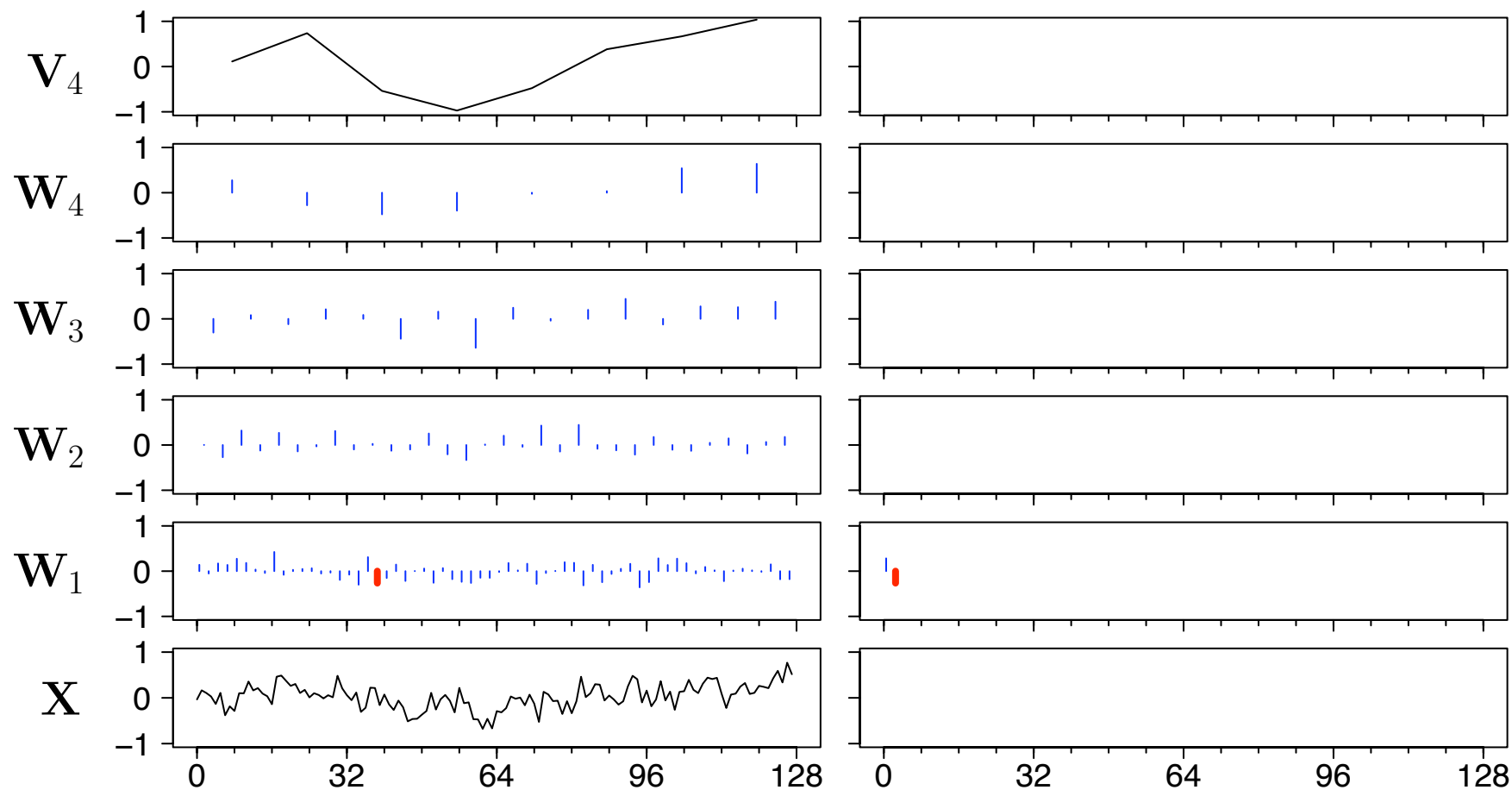
- Haar DWT of FD(0.45) series \mathbf{X} (left-hand column) and wavelet-domain bootstrap thereof (right-hand)

Illustration of Wavelet-Domain Bootstrapping



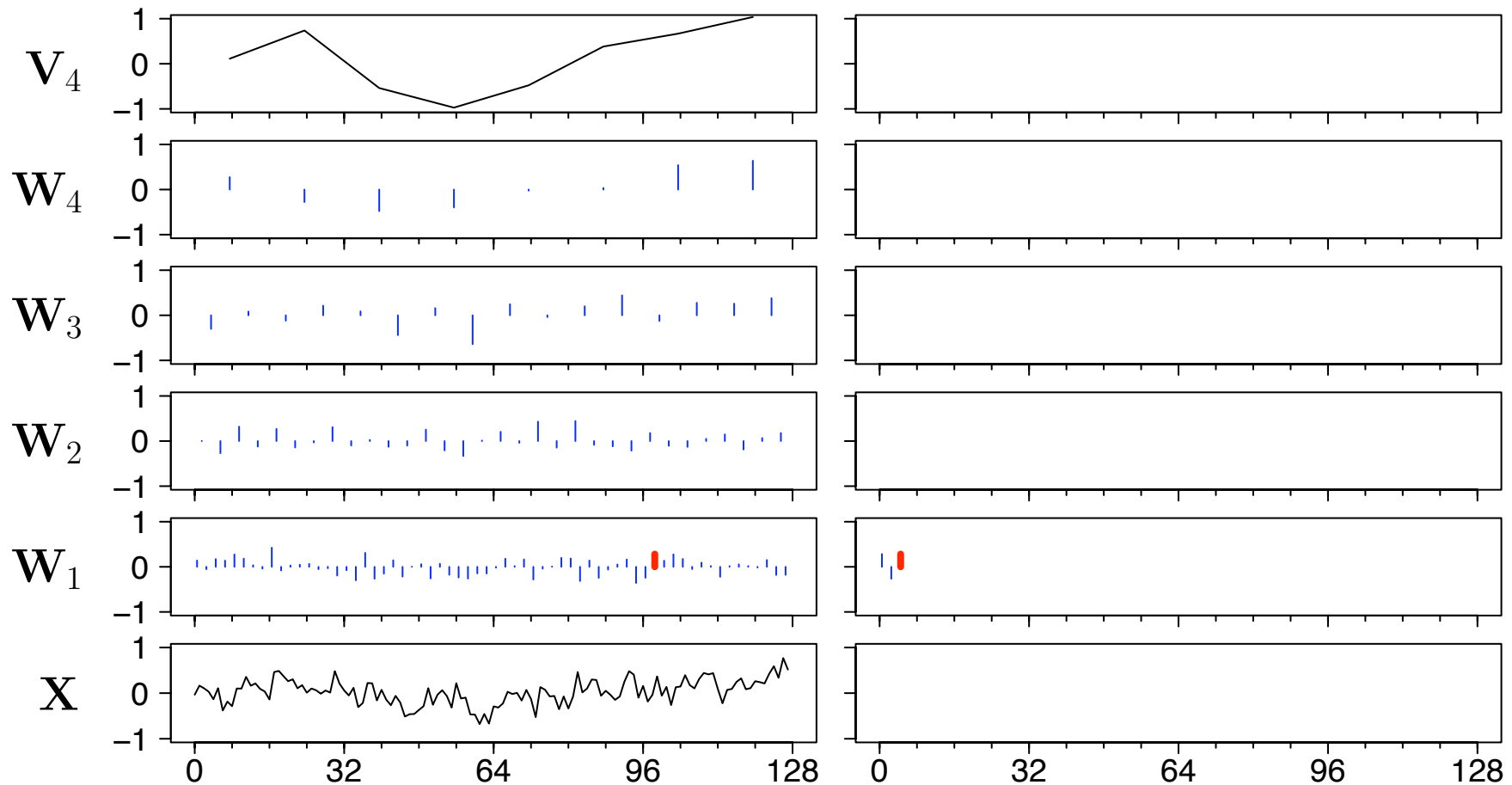
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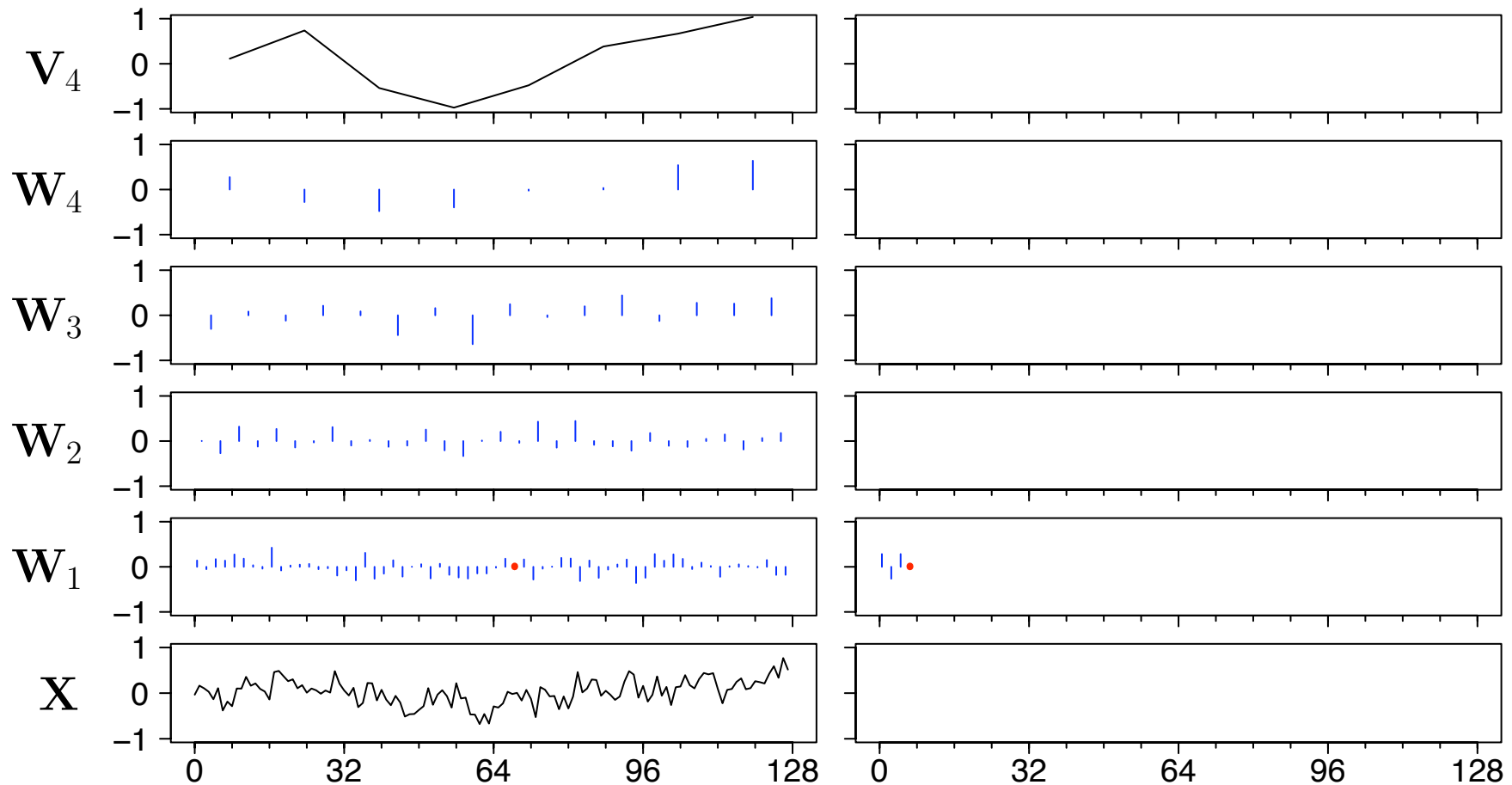
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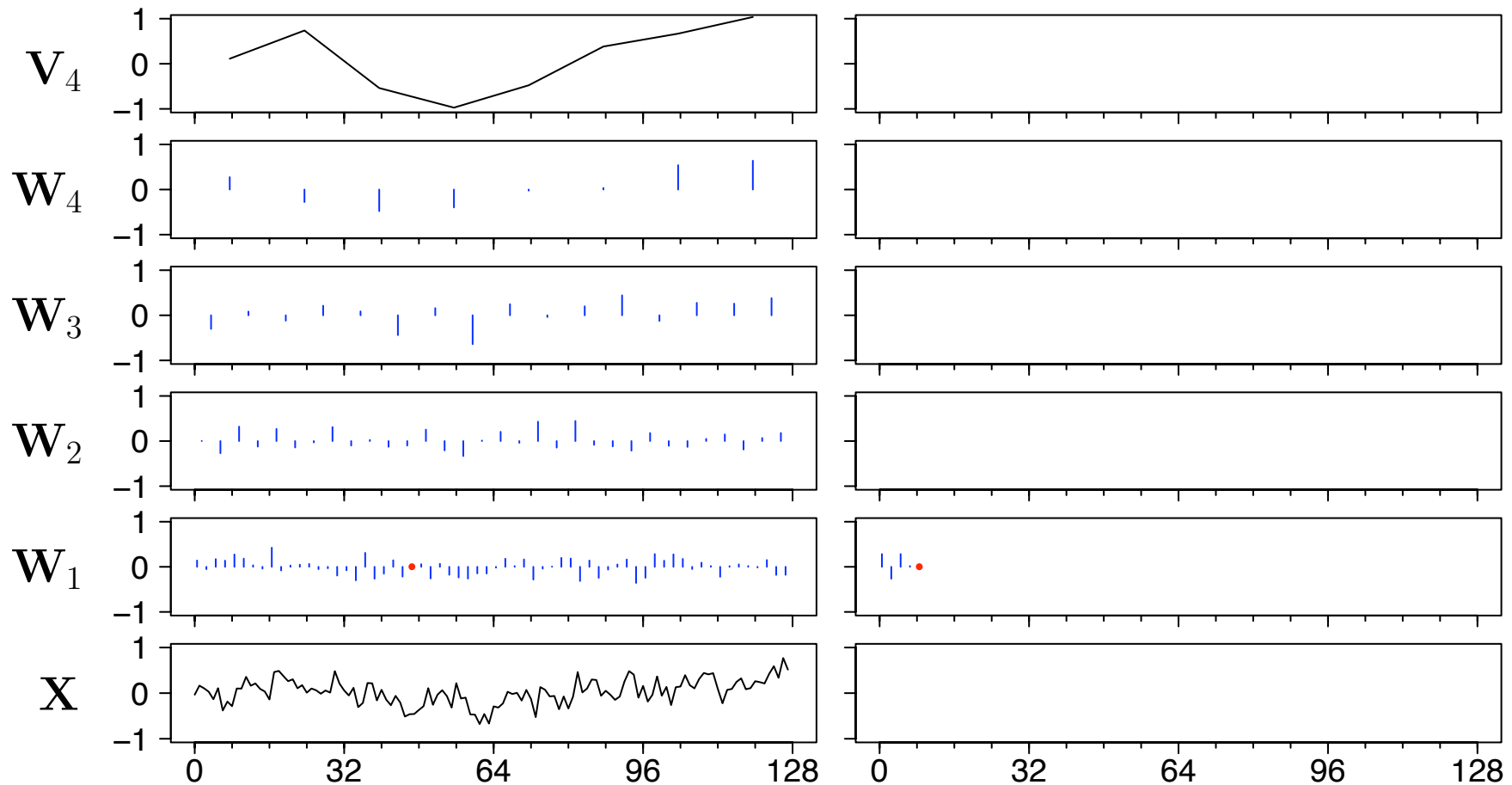
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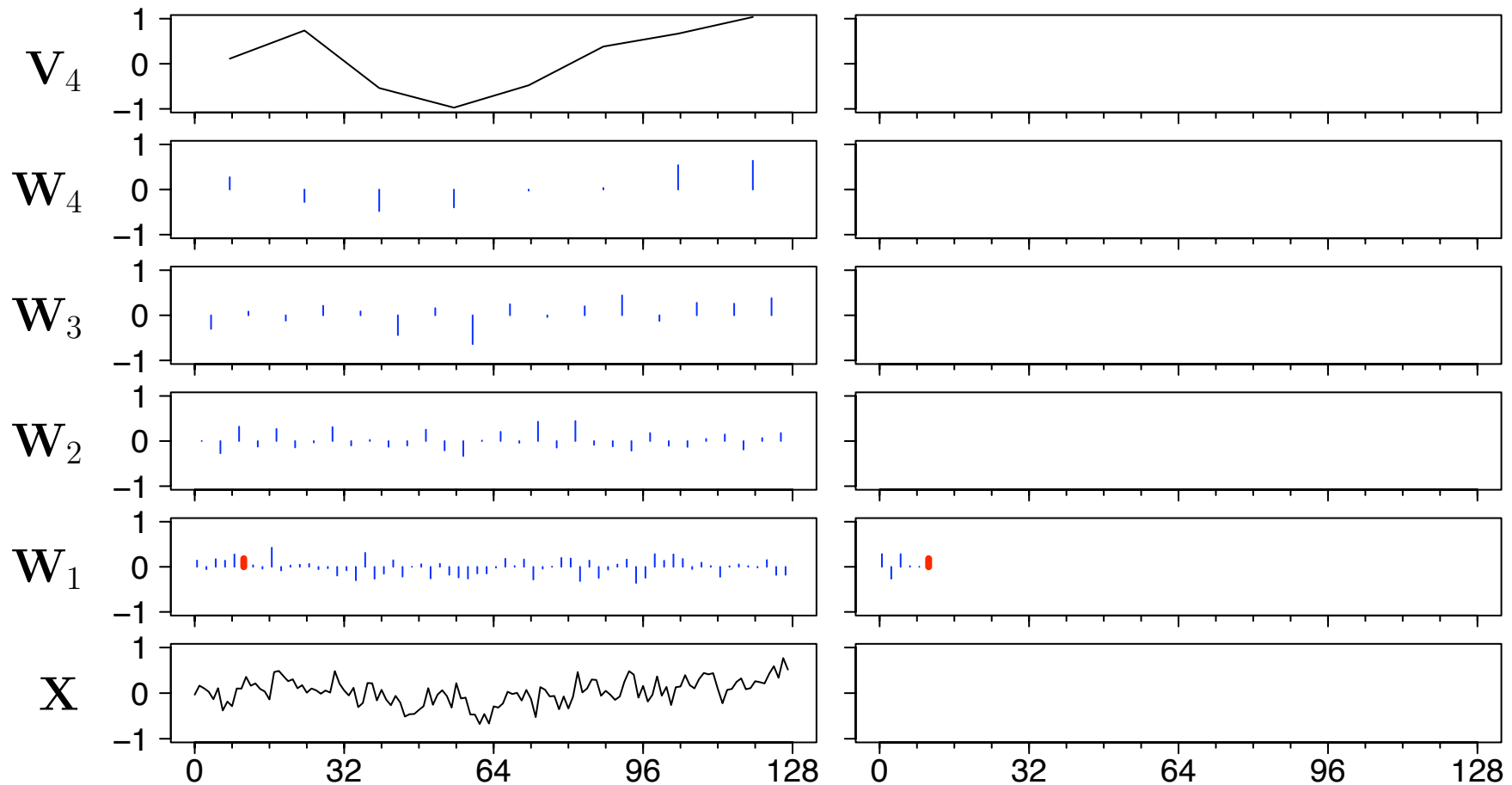
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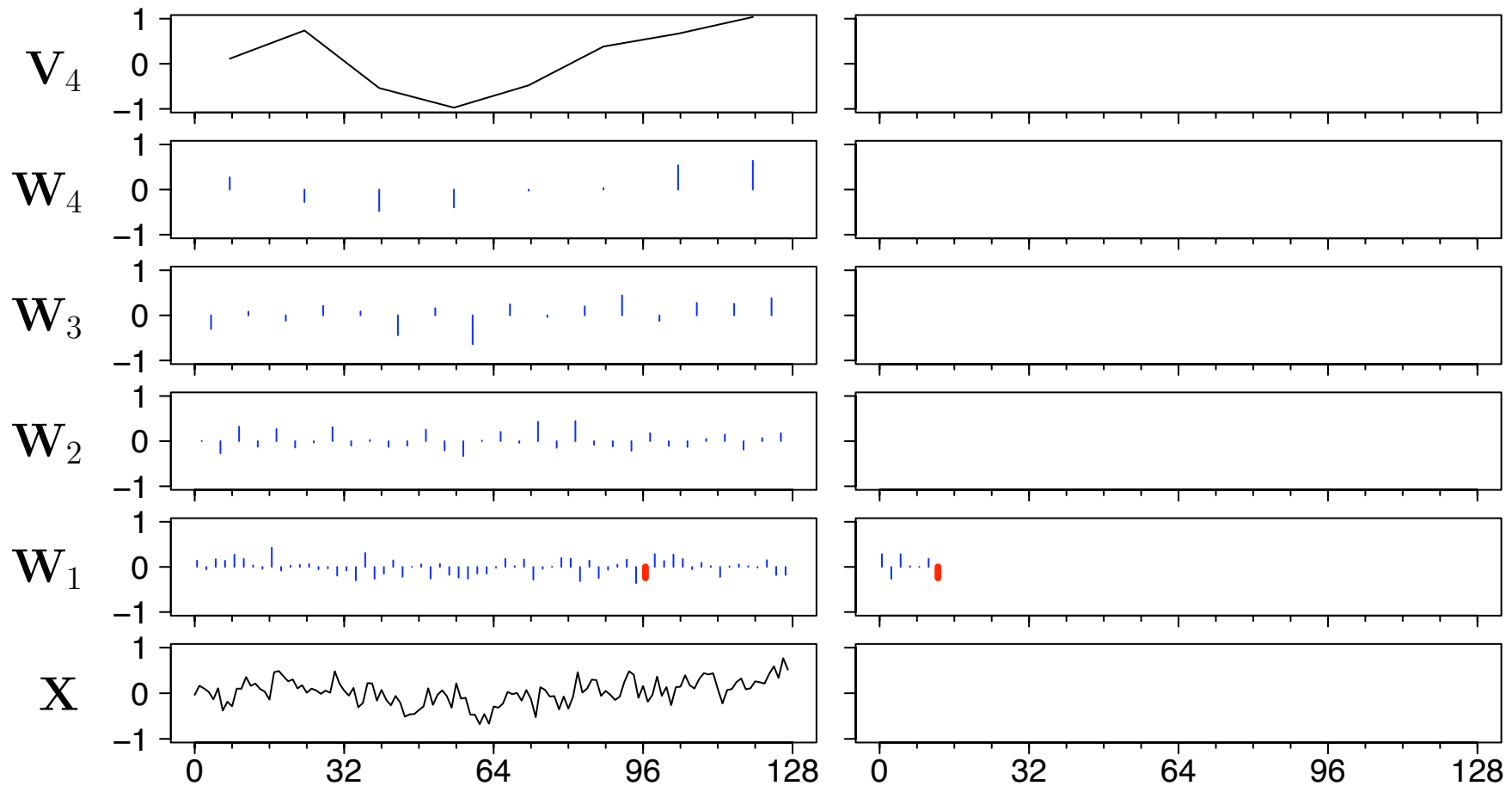
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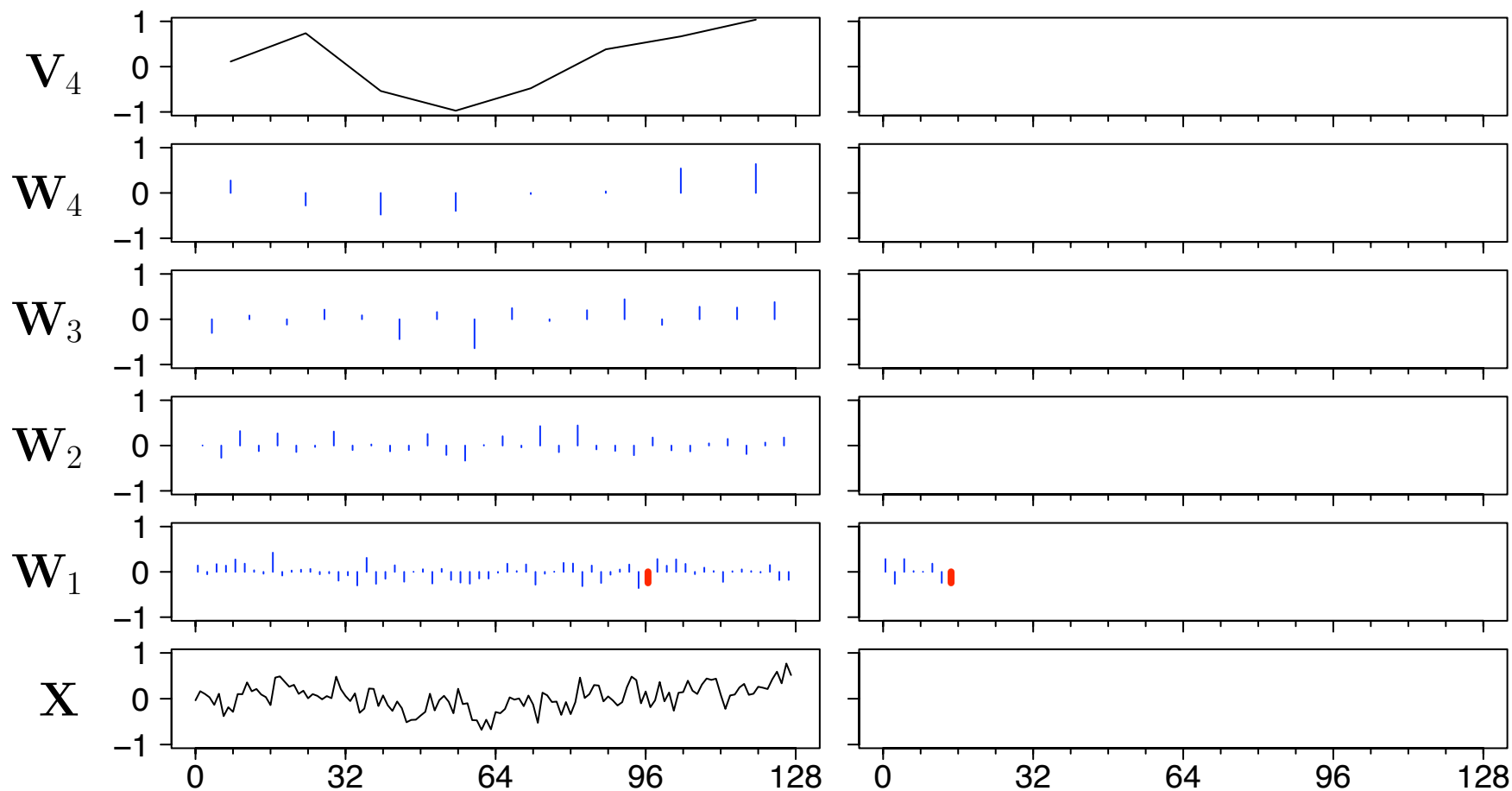
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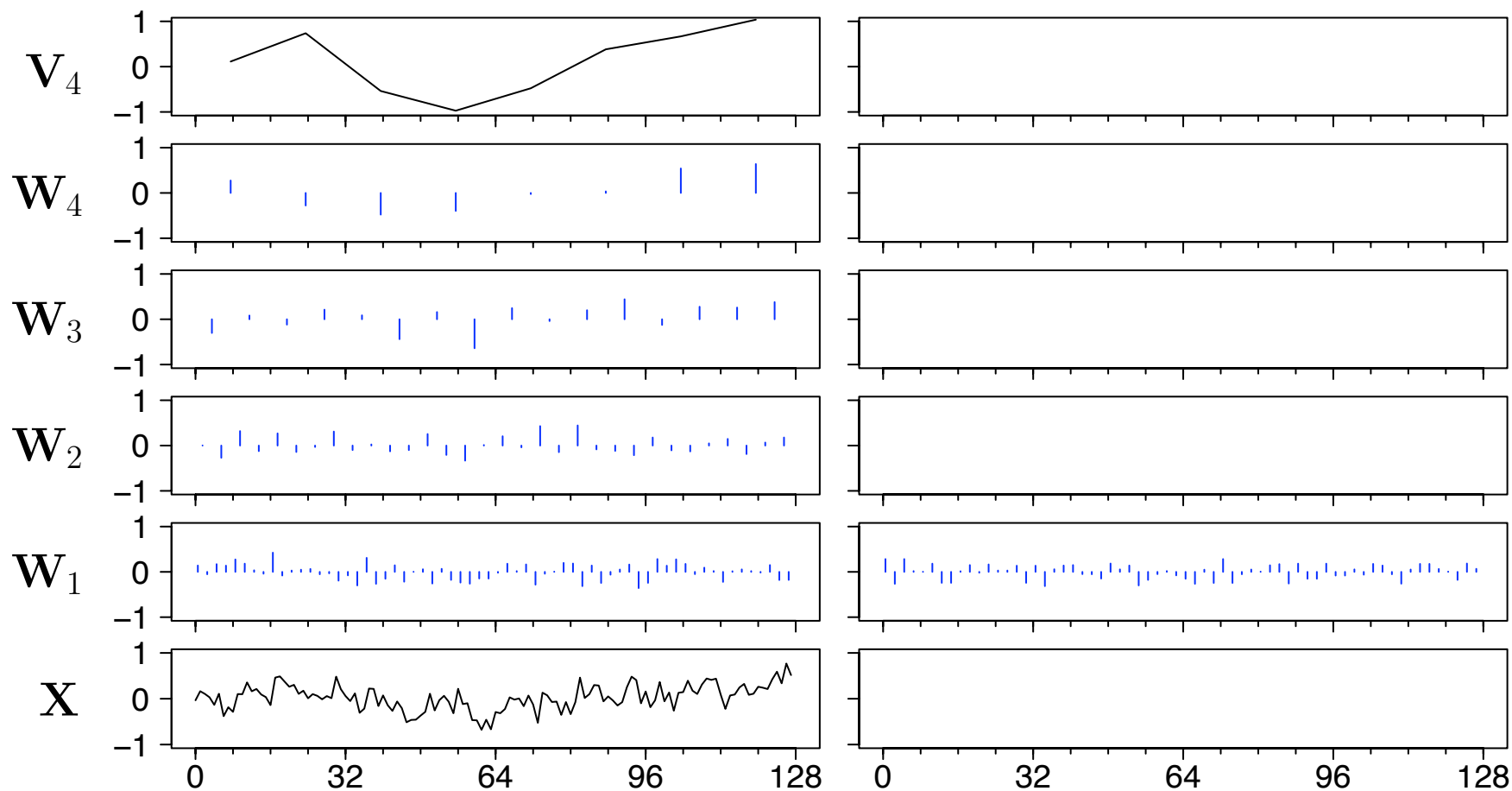
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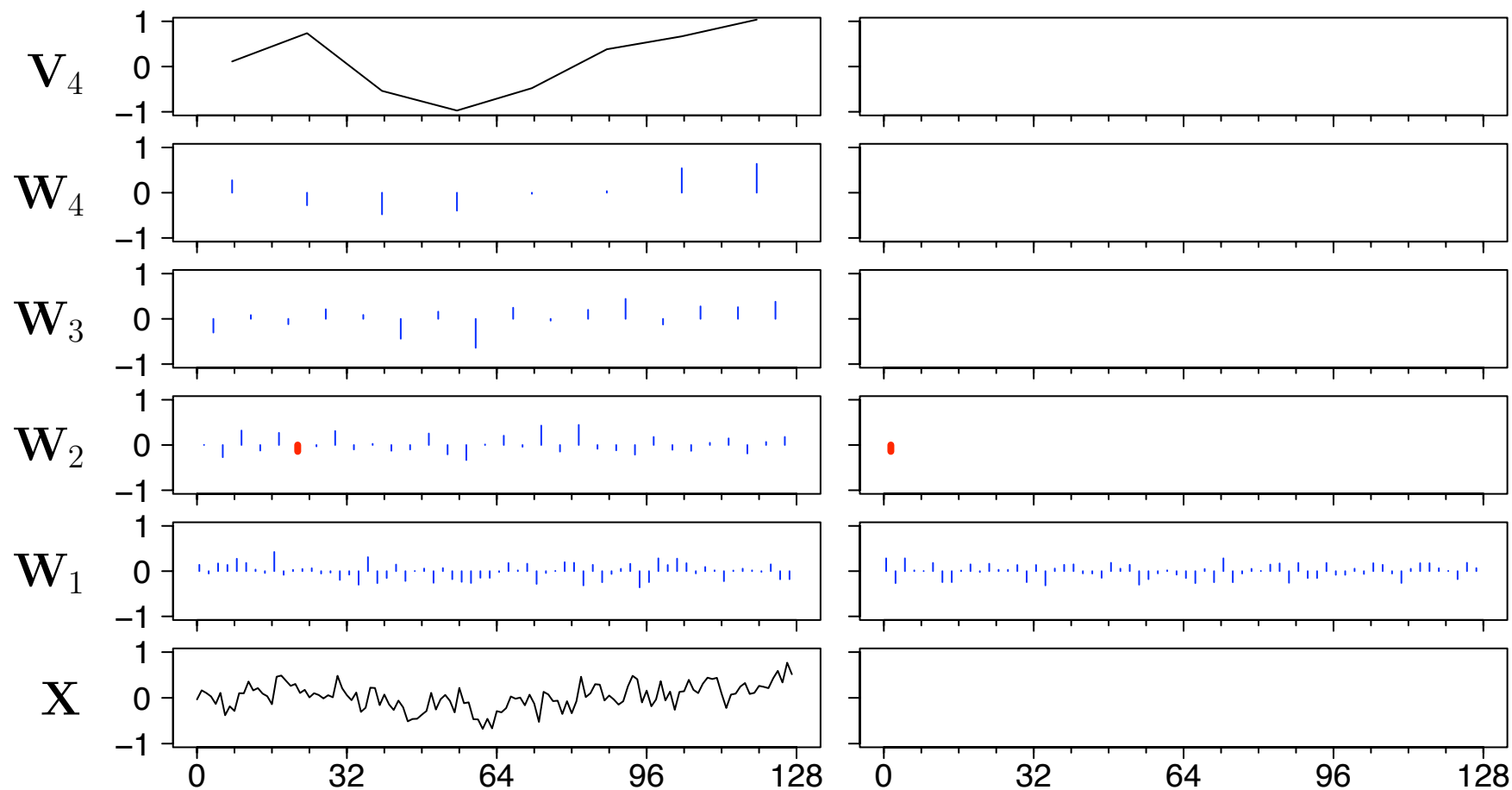
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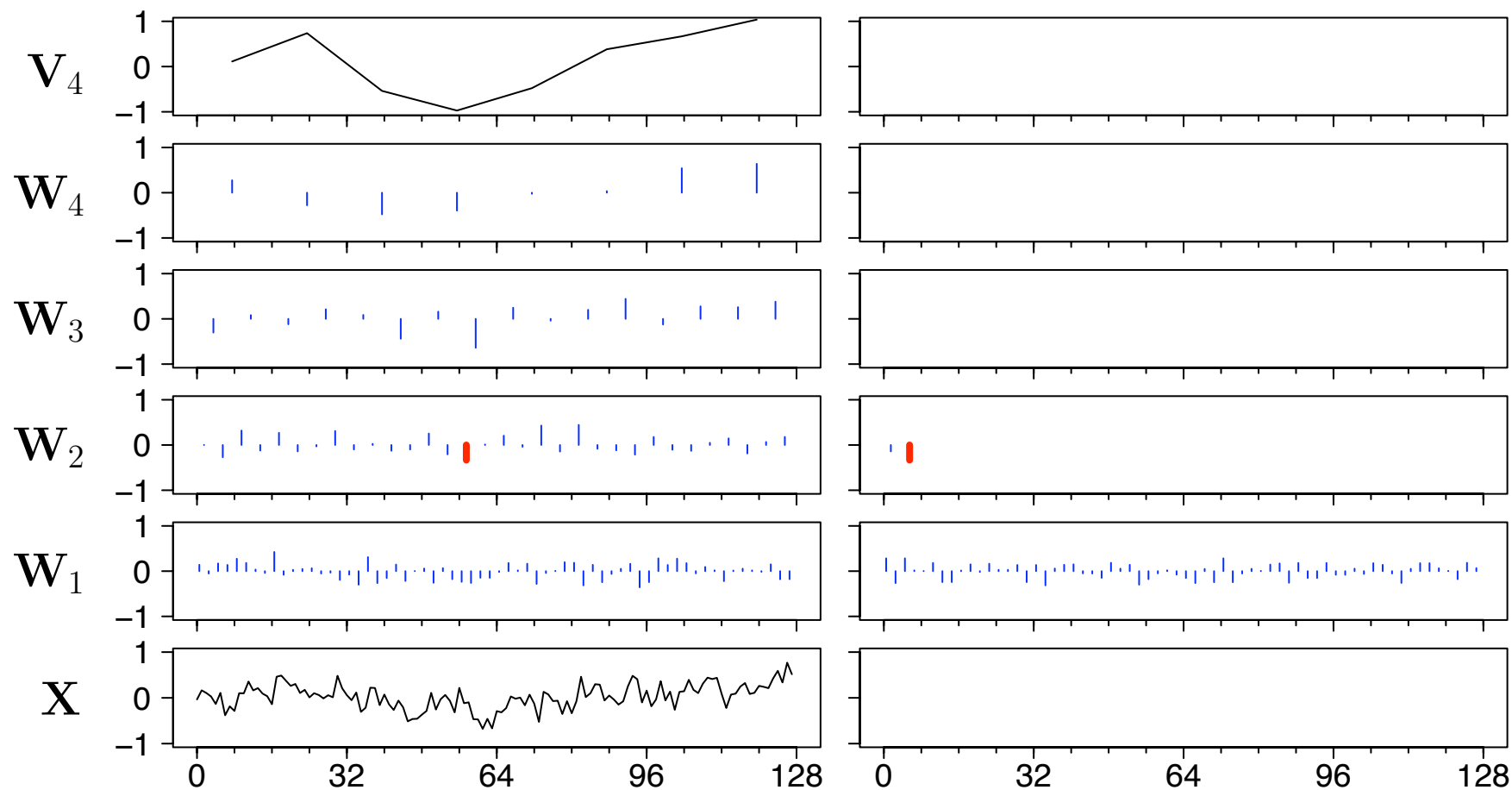
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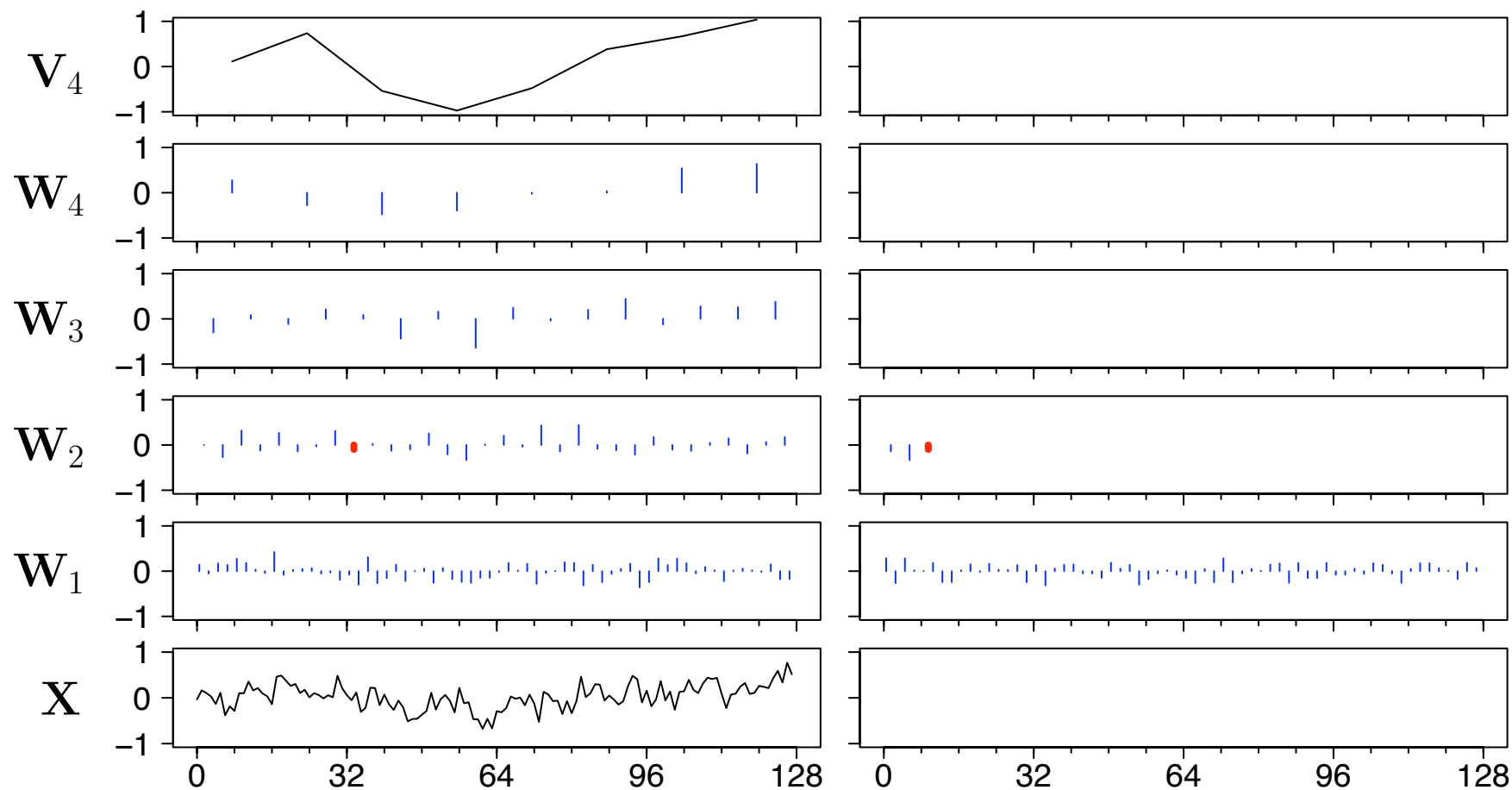
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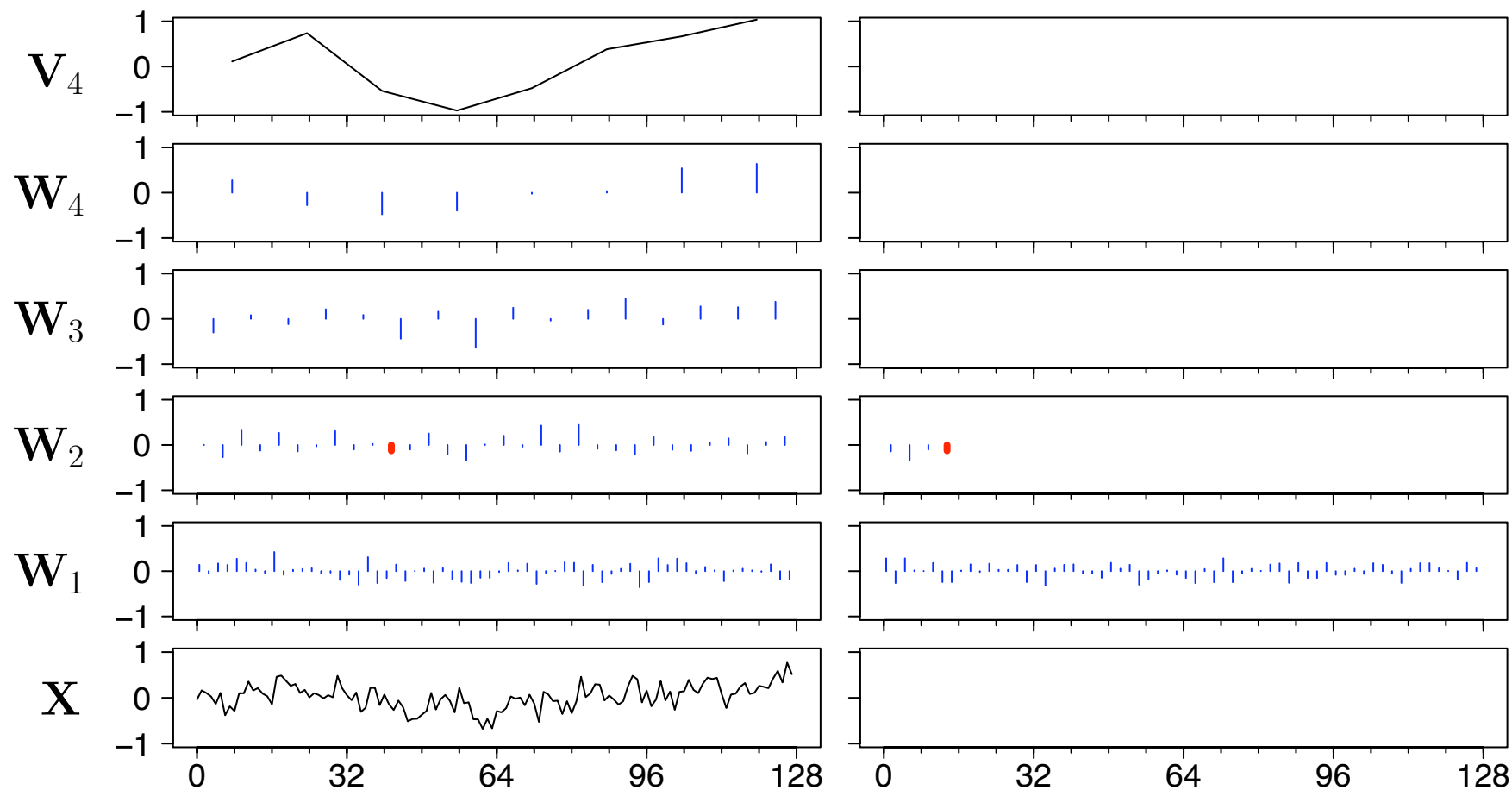
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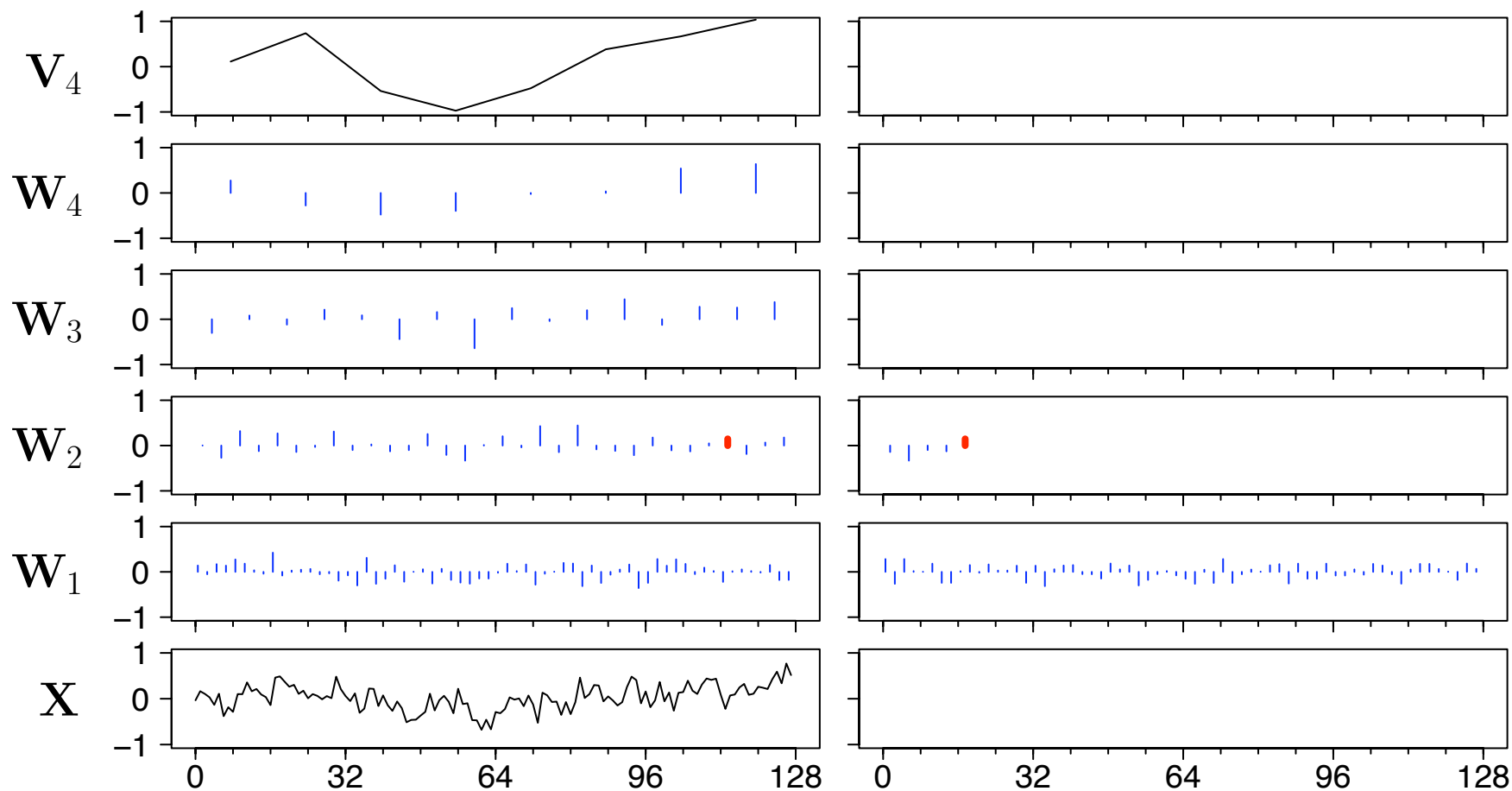
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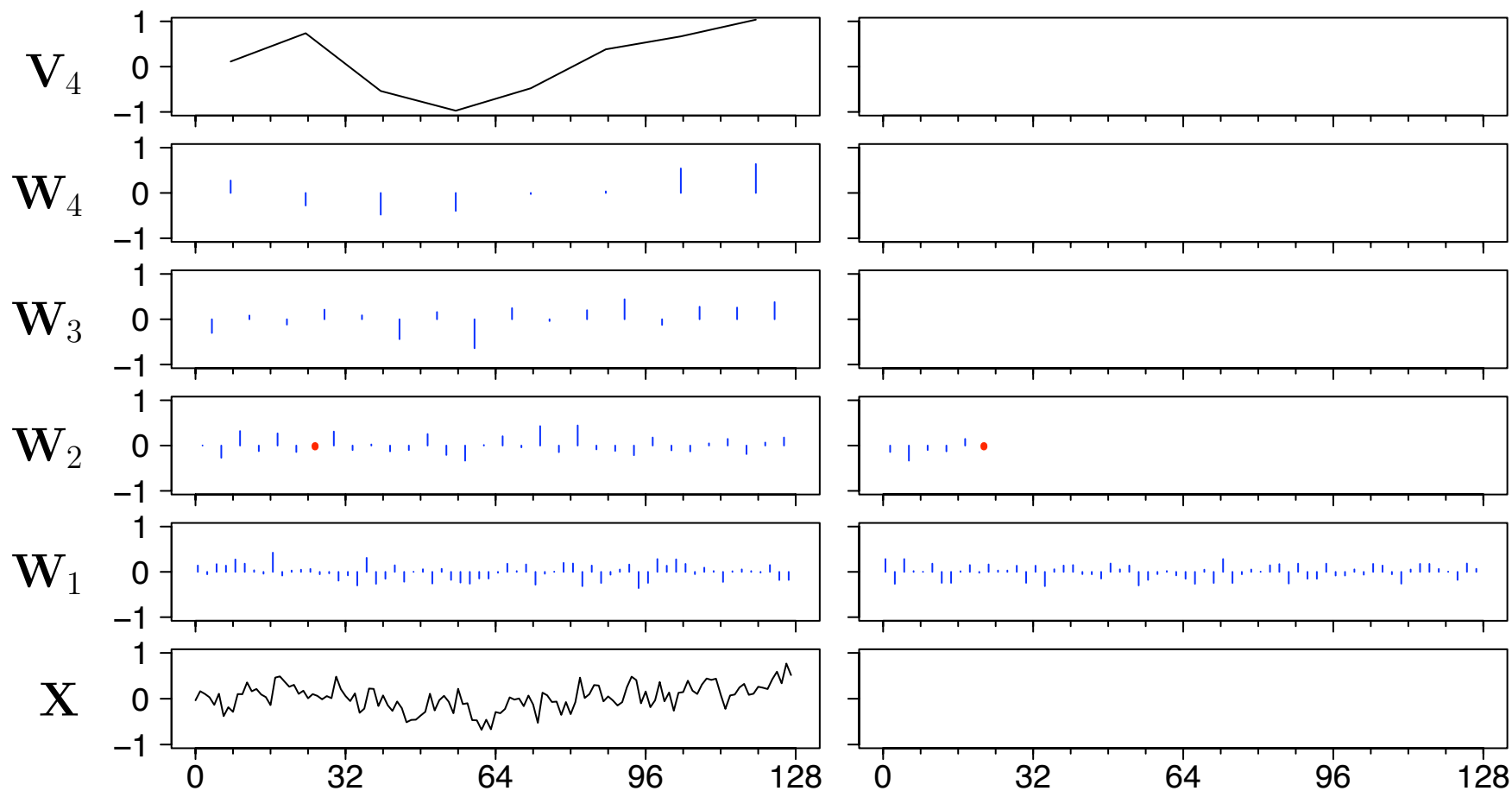
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Illustration of Wavelet-Domain Bootstrapping



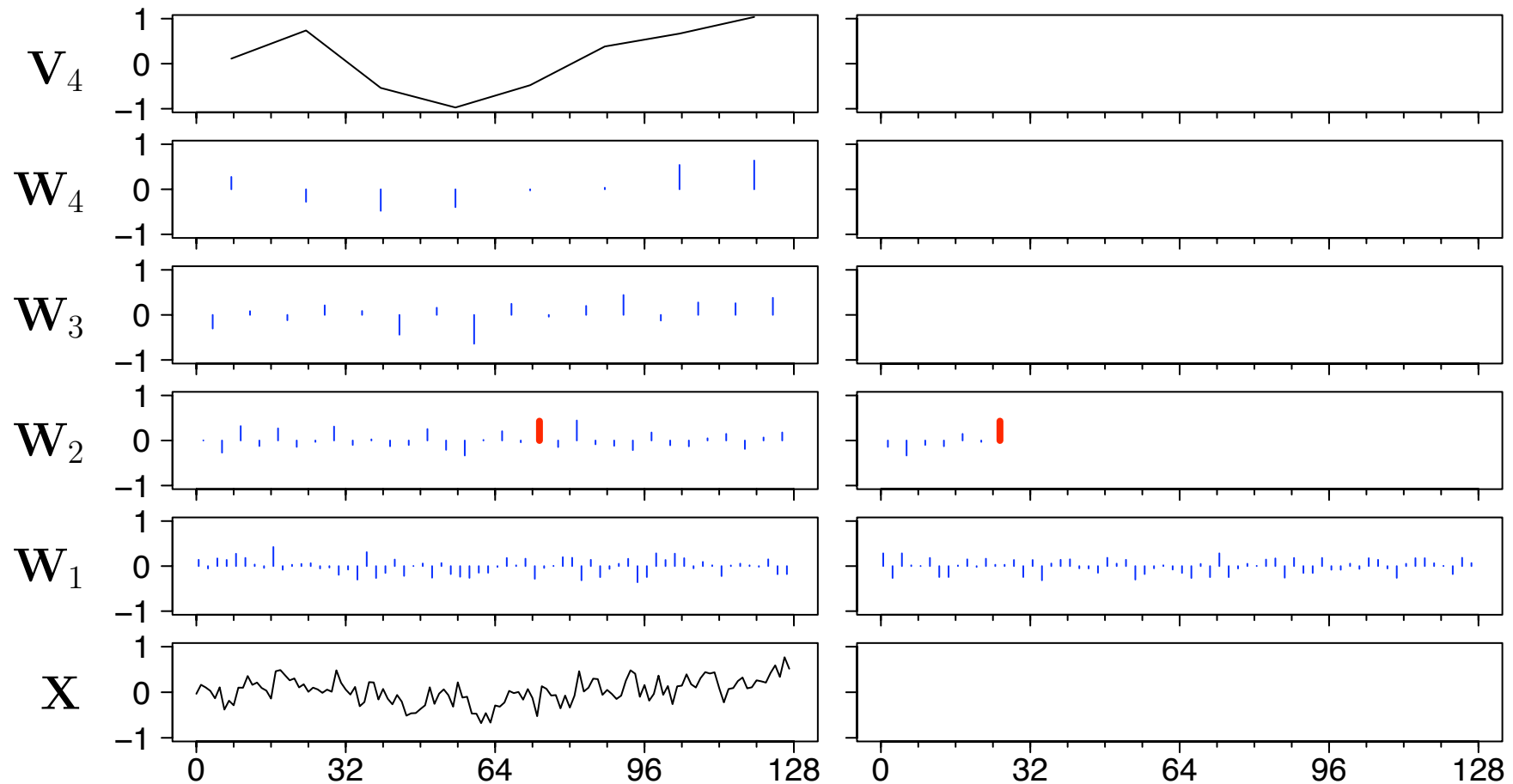
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Illustration of Wavelet-Domain Bootstrapping



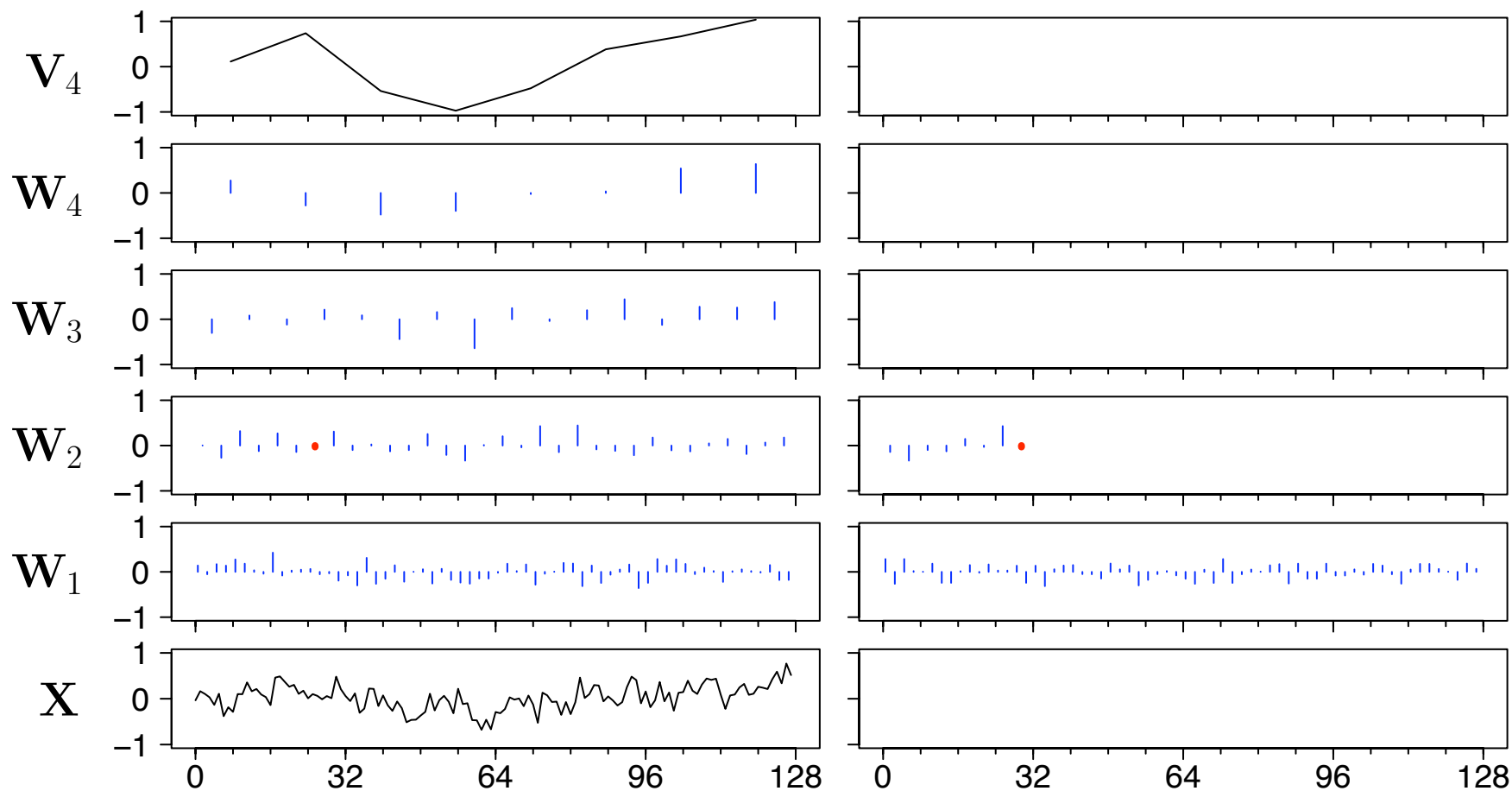
- Haar DWT of FD(0.45) series X (left-hand column) and wavelet-domain bootstrap thereof (right-hand)

Illustration of Wavelet-Domain Bootstrapping



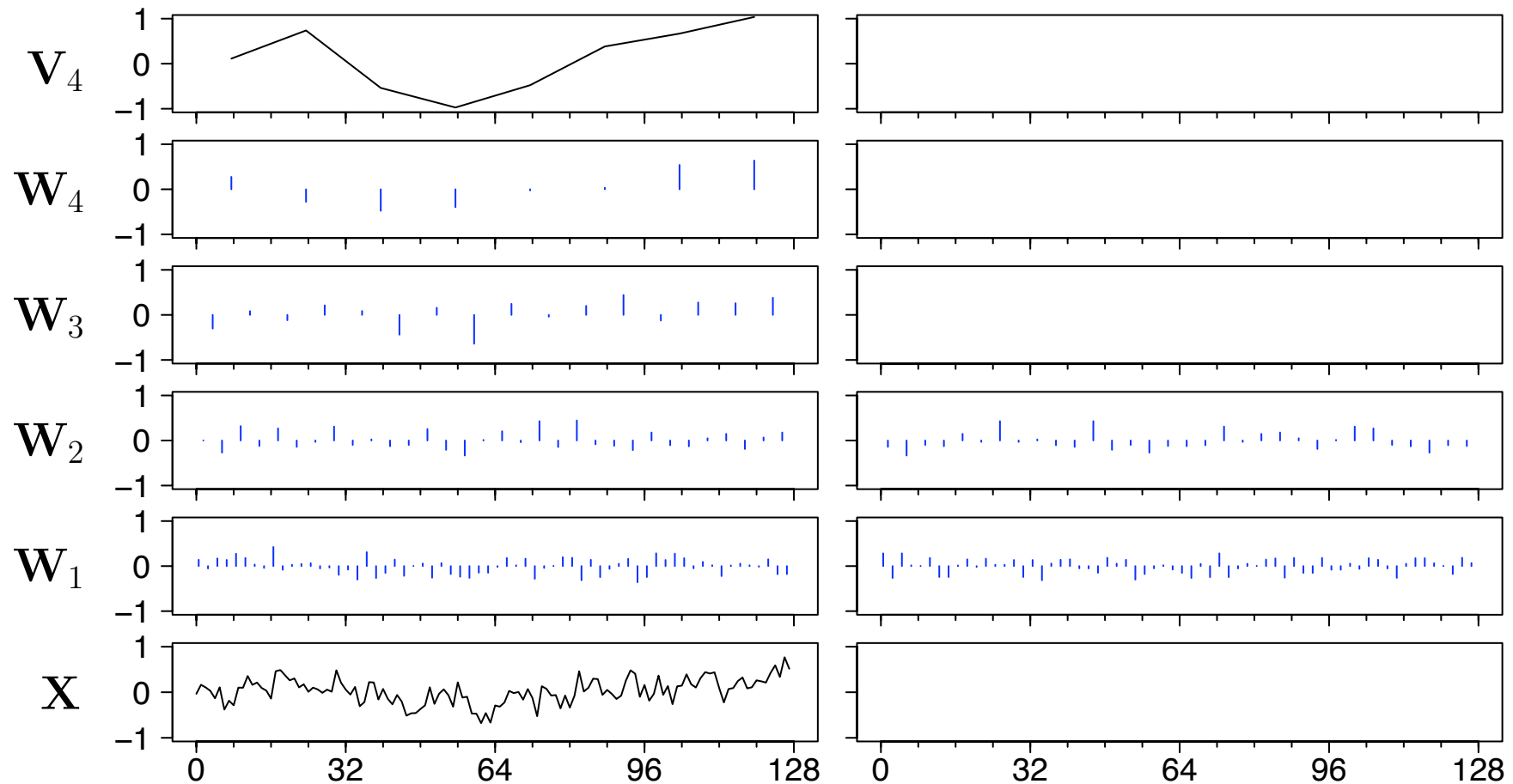
- Haar DWT of FD(0.45) series \mathbf{X} (left-hand column) and wavelet-domain bootstrap thereof (right-hand)

Illustration of Wavelet-Domain Bootstrapping



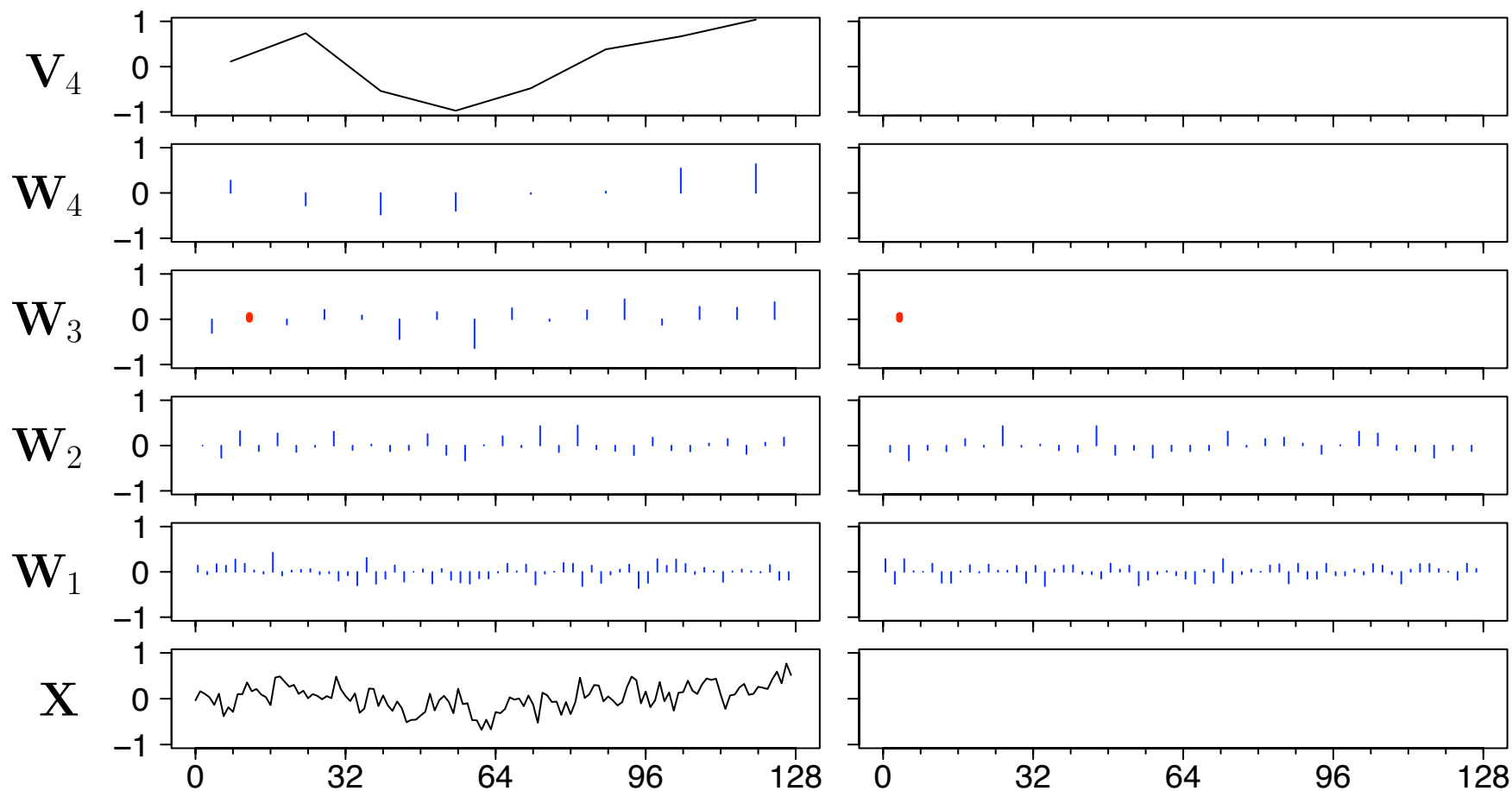
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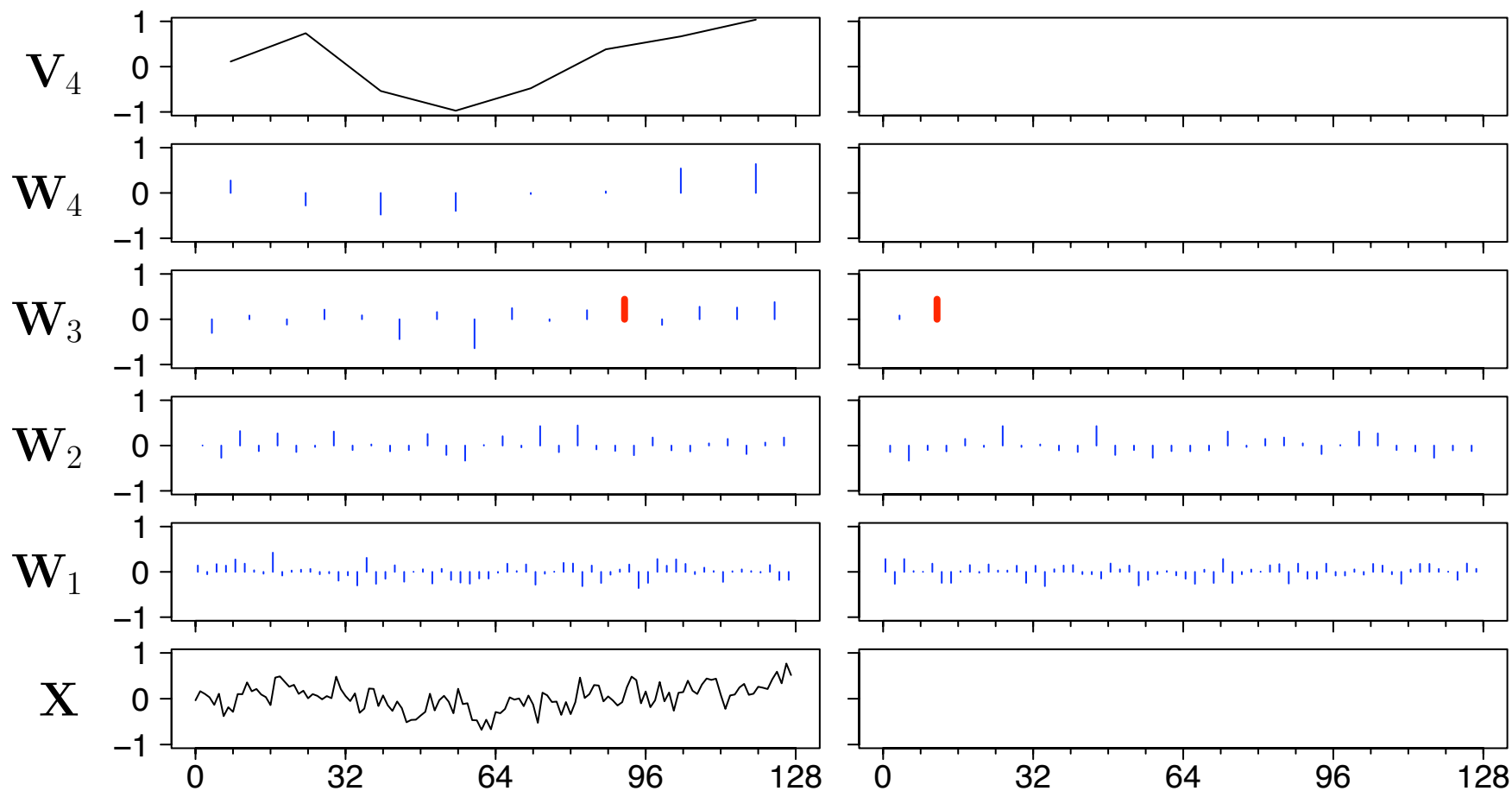
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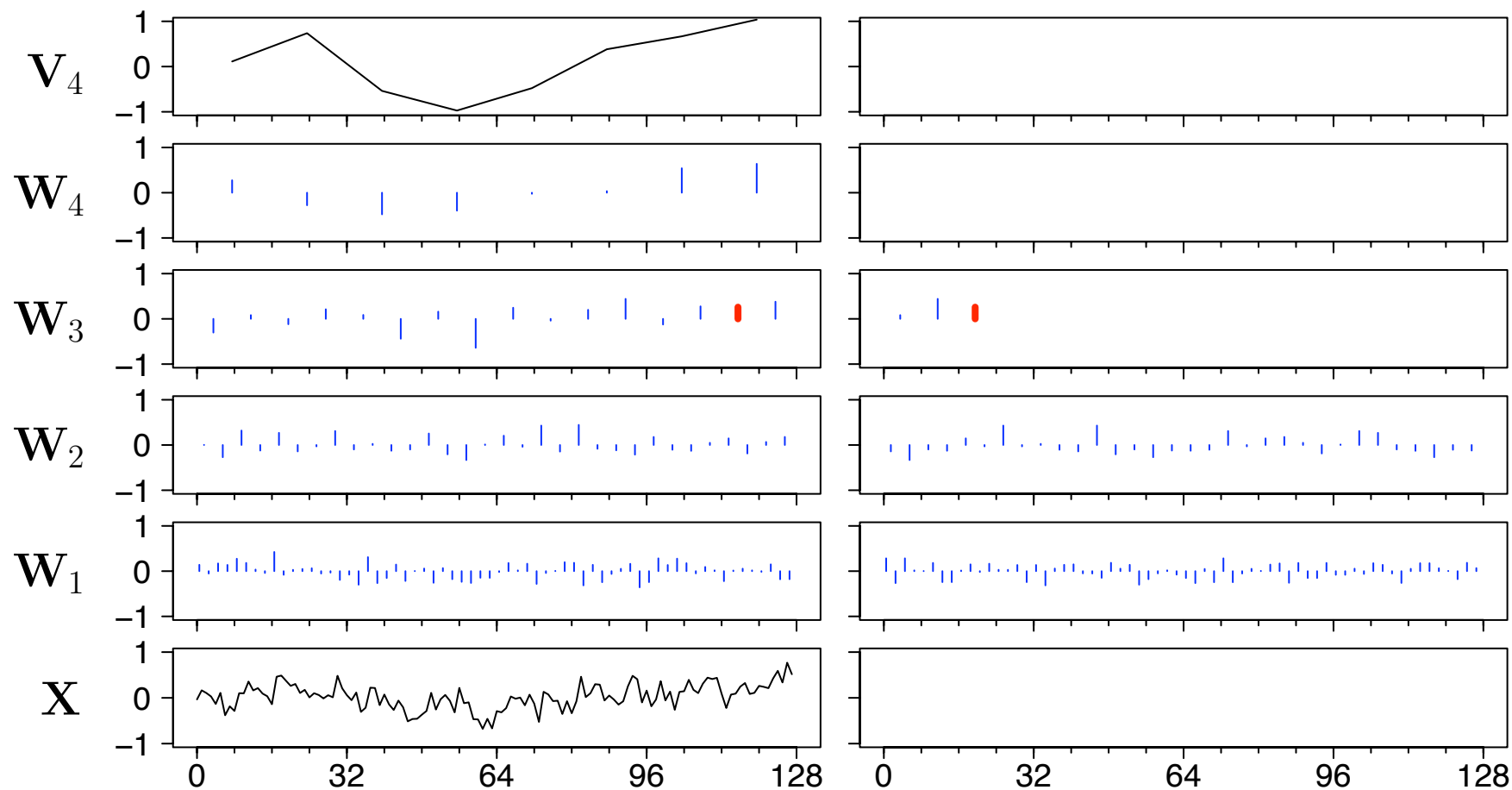
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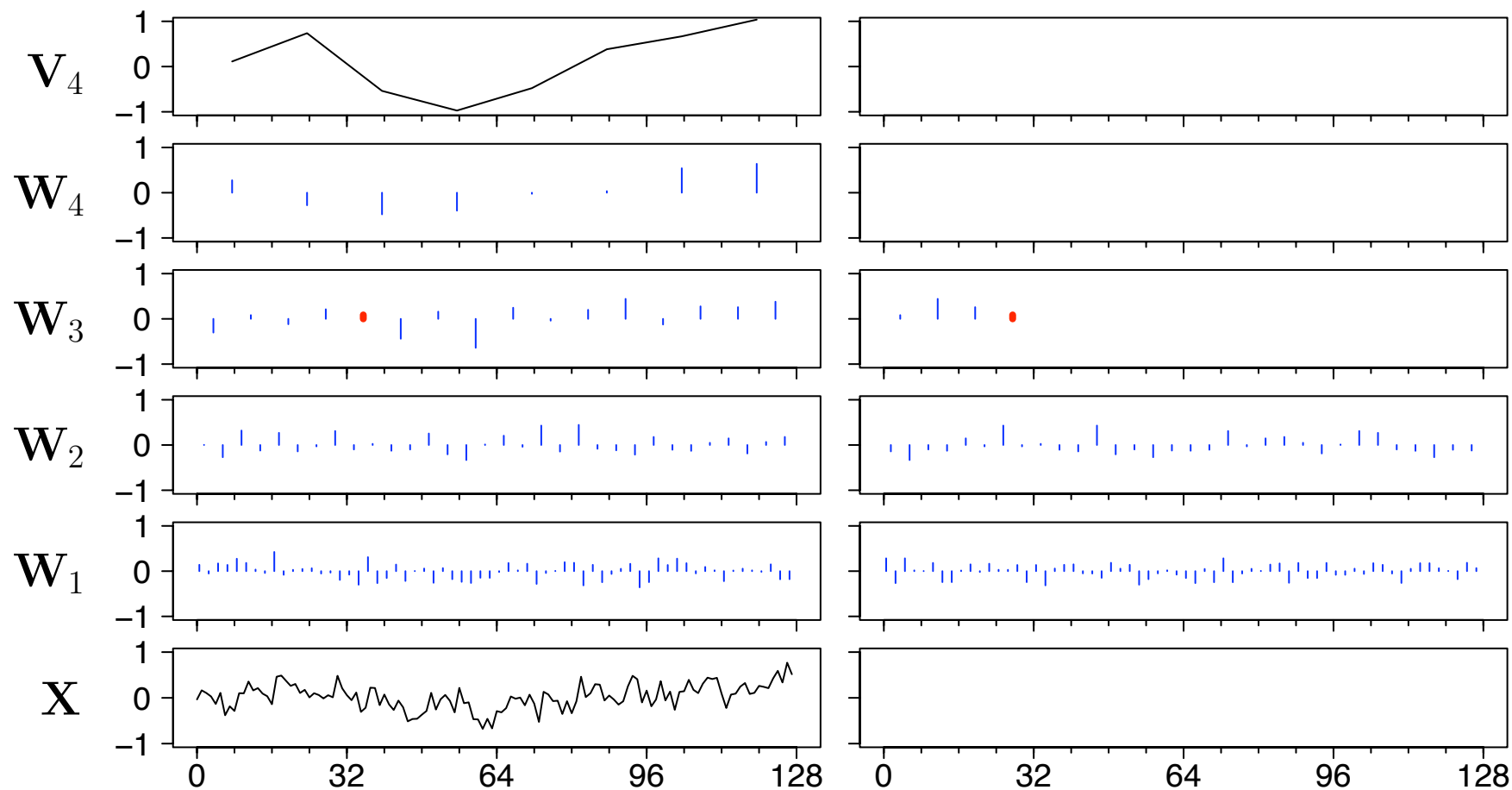
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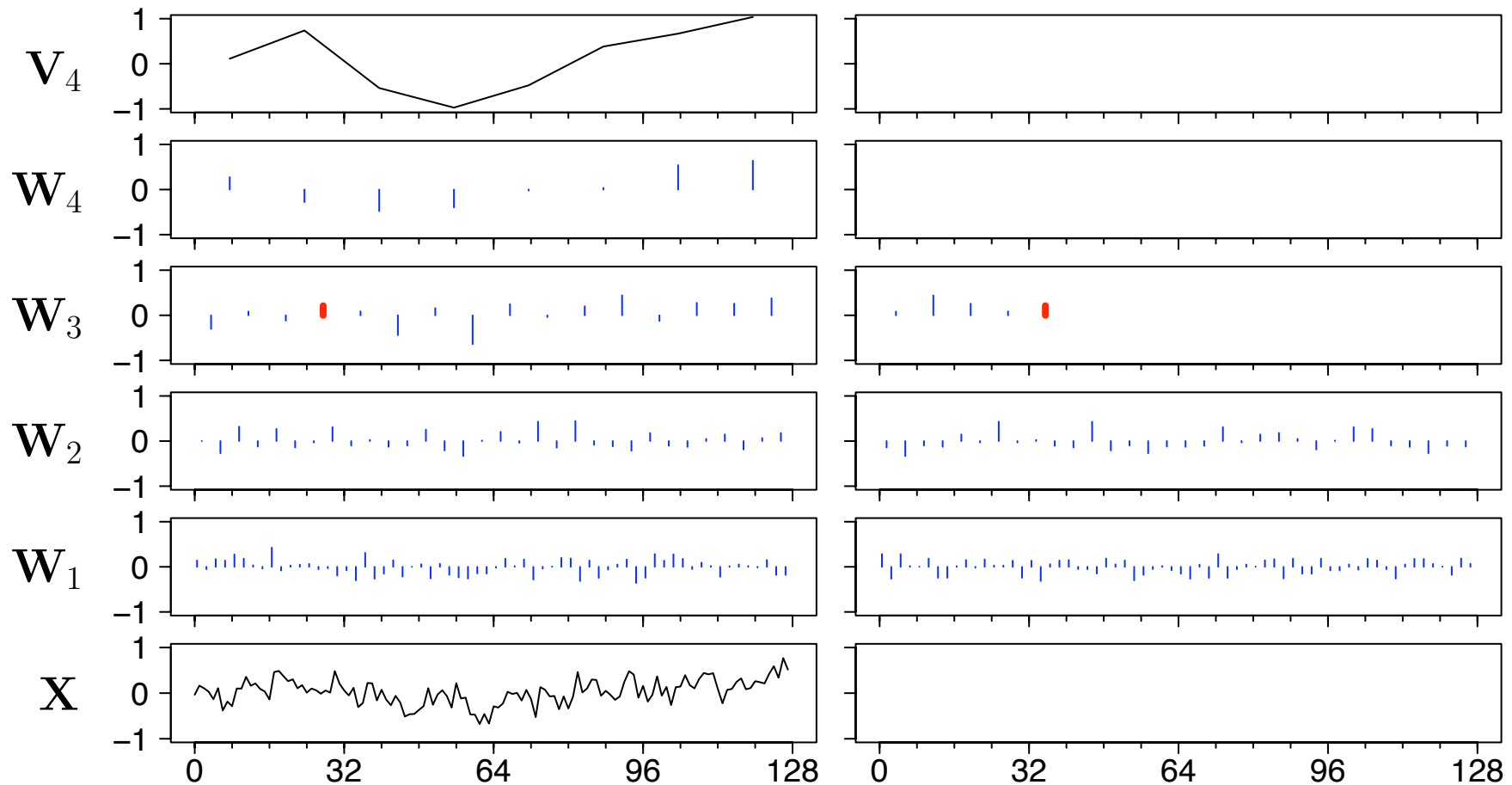
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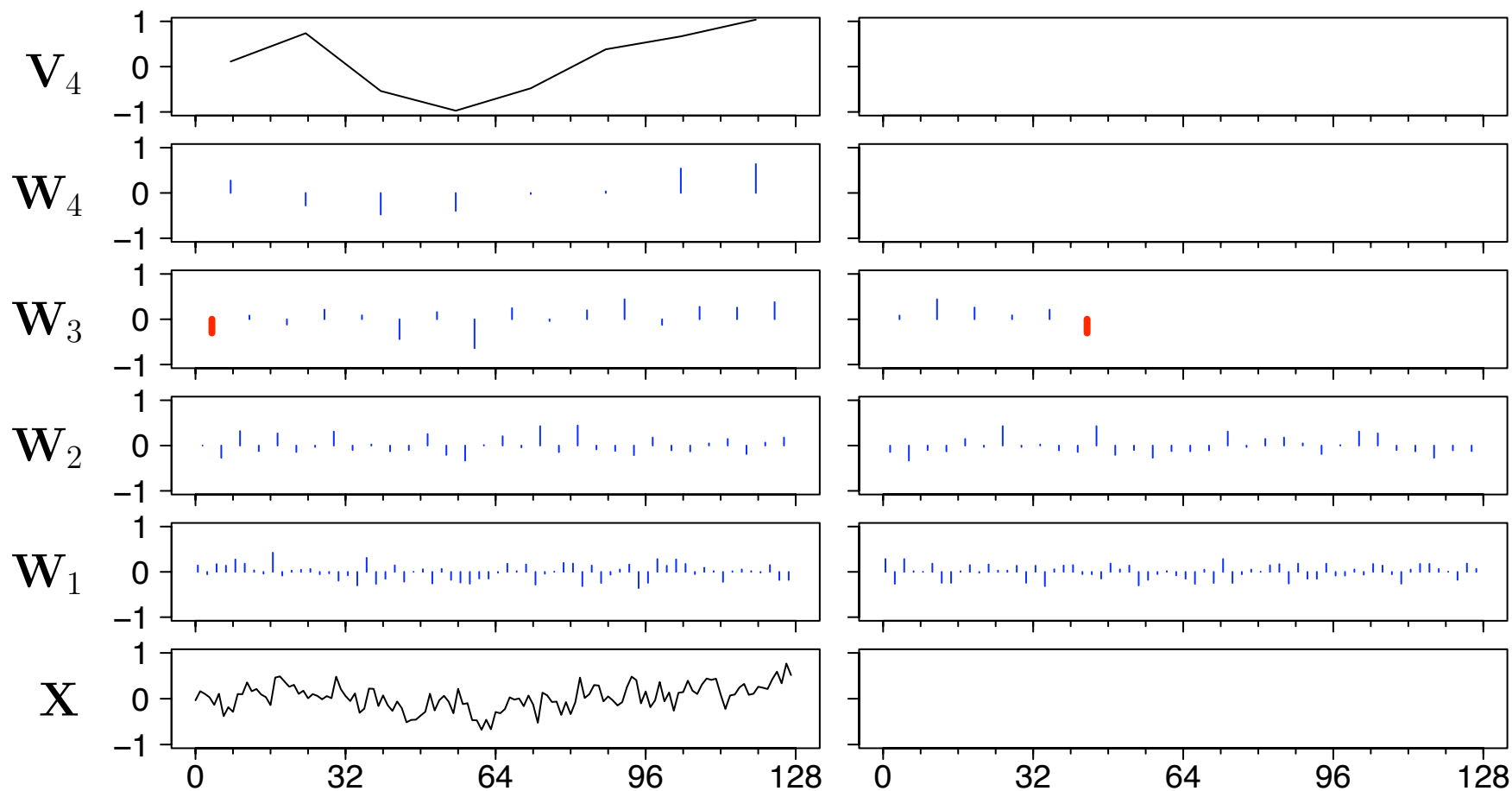
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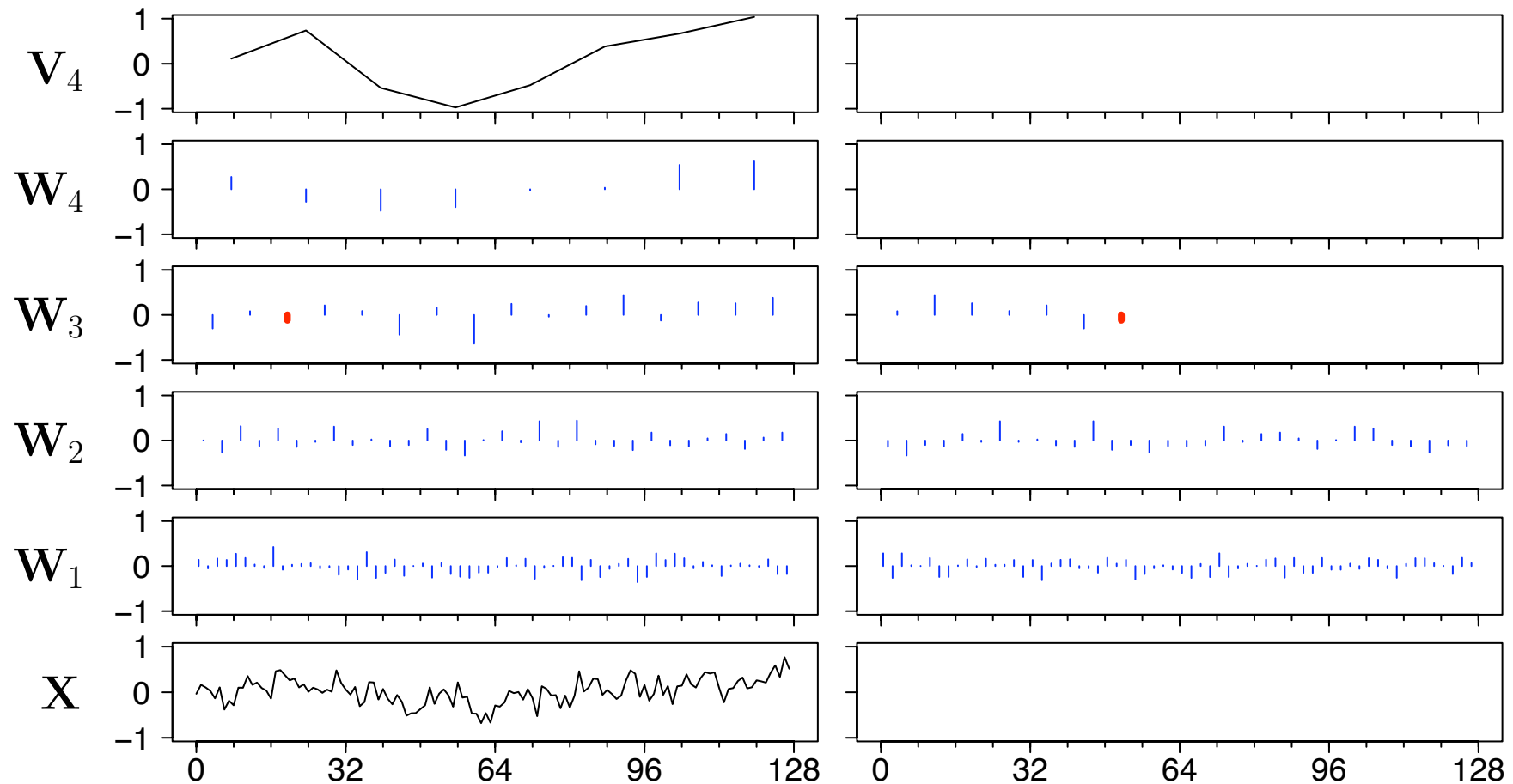
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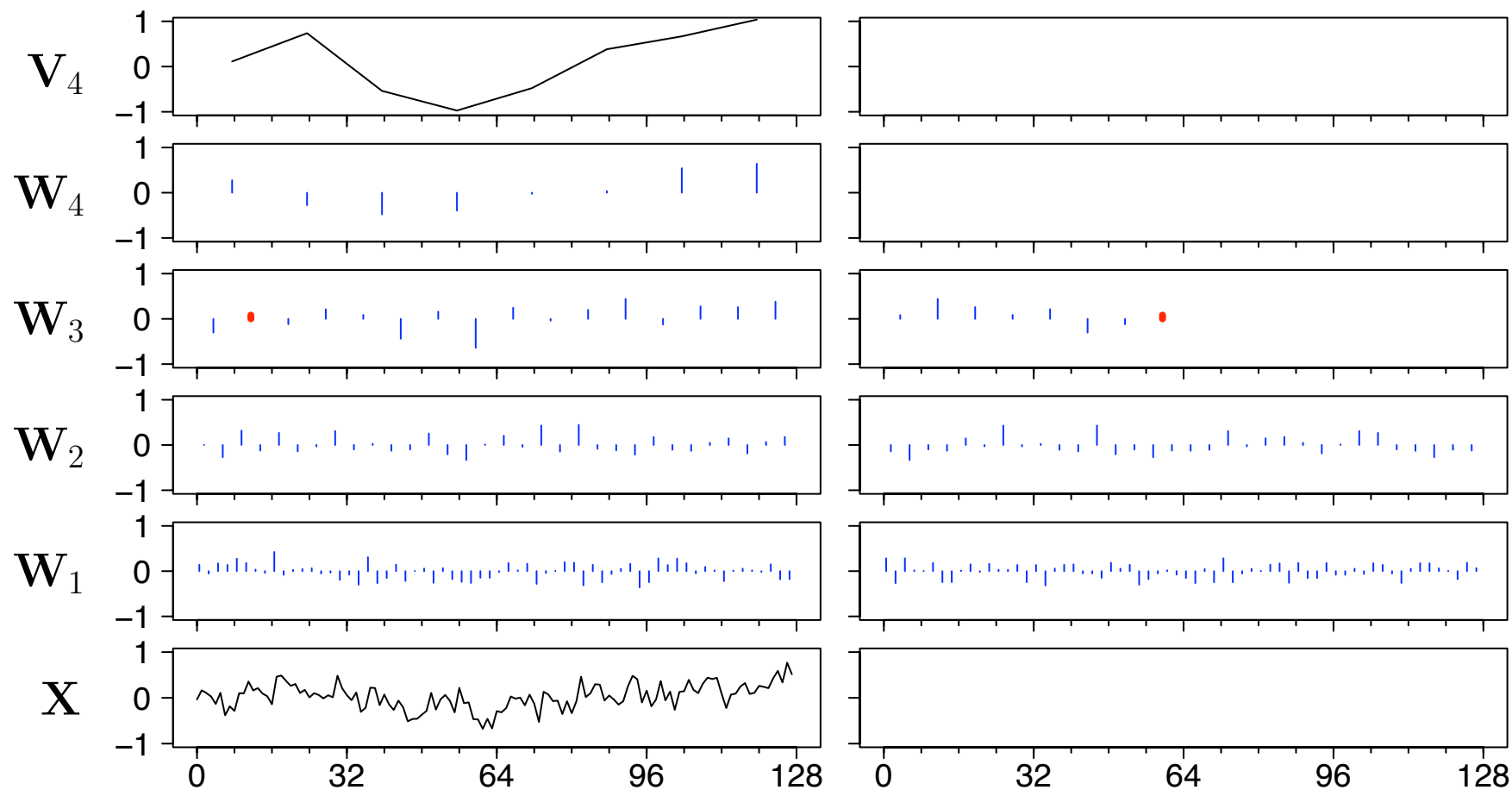
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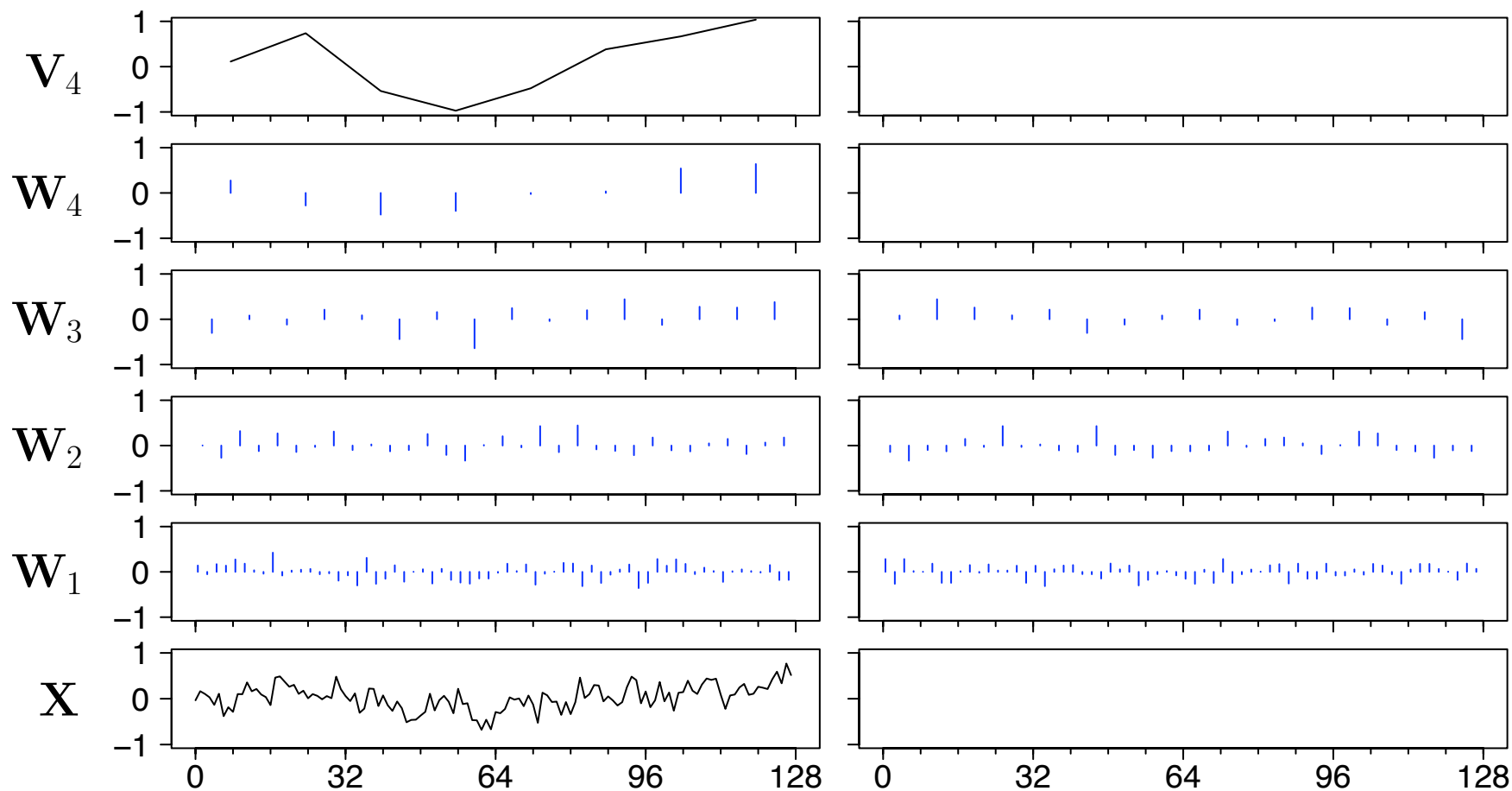
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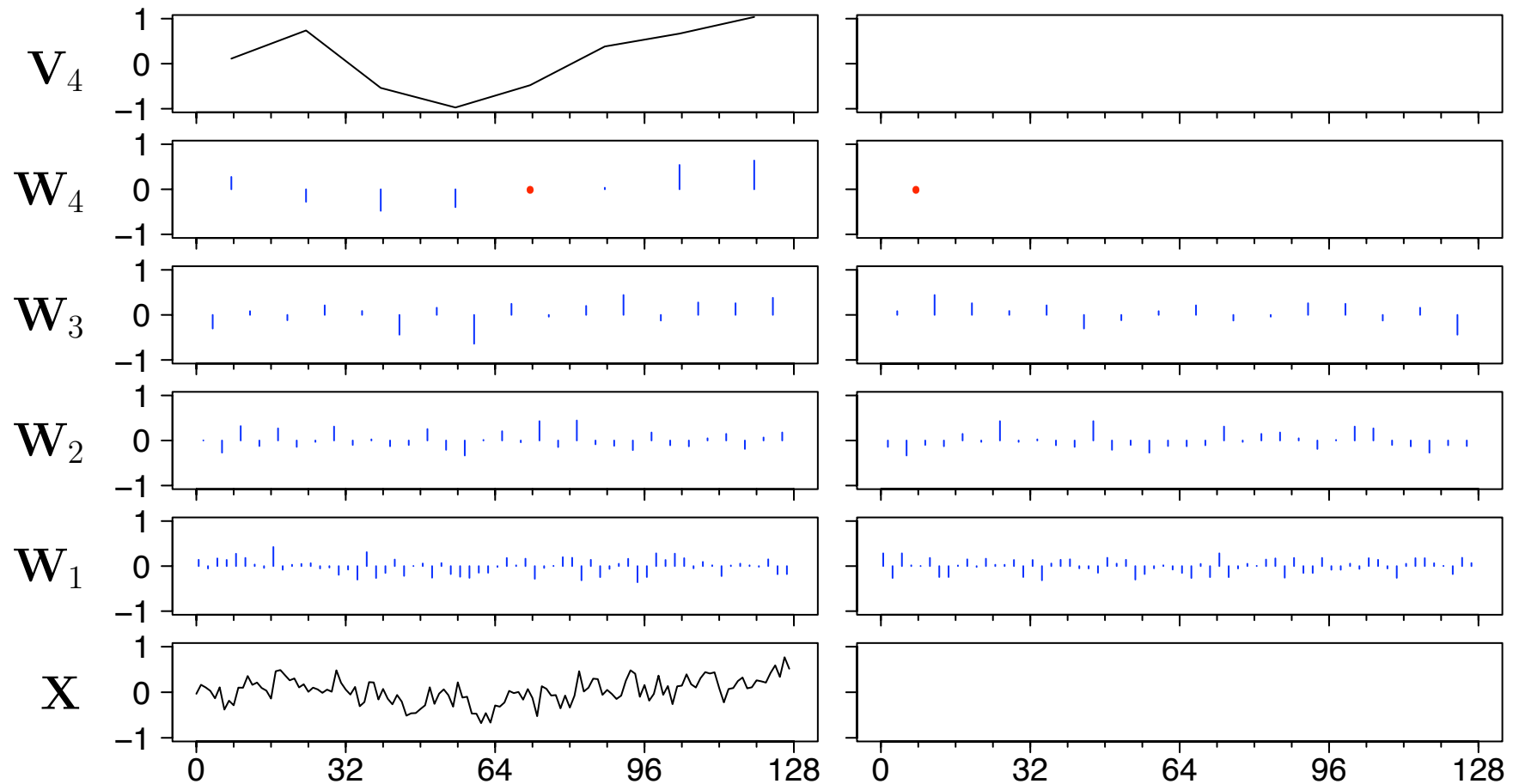
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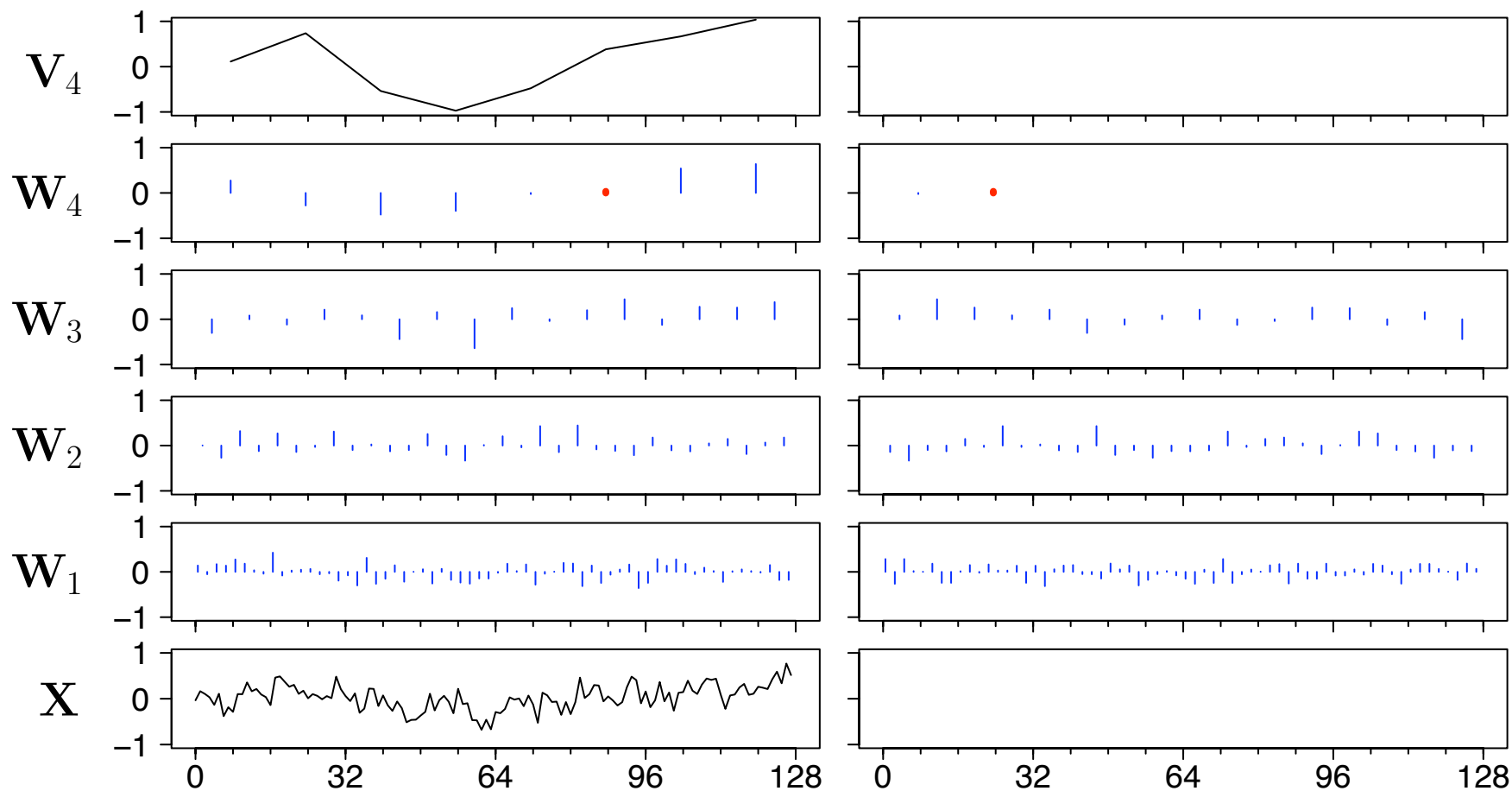
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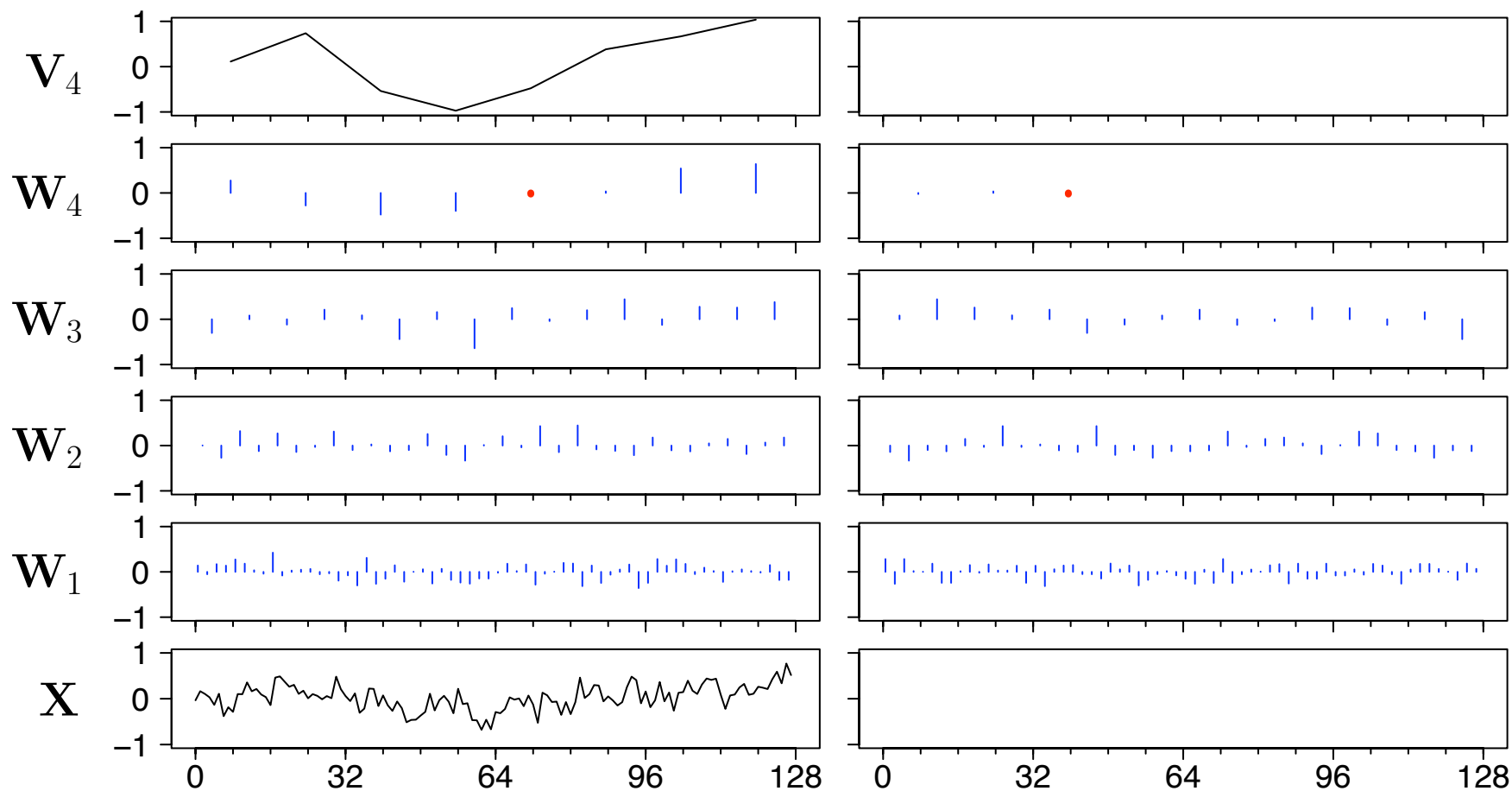
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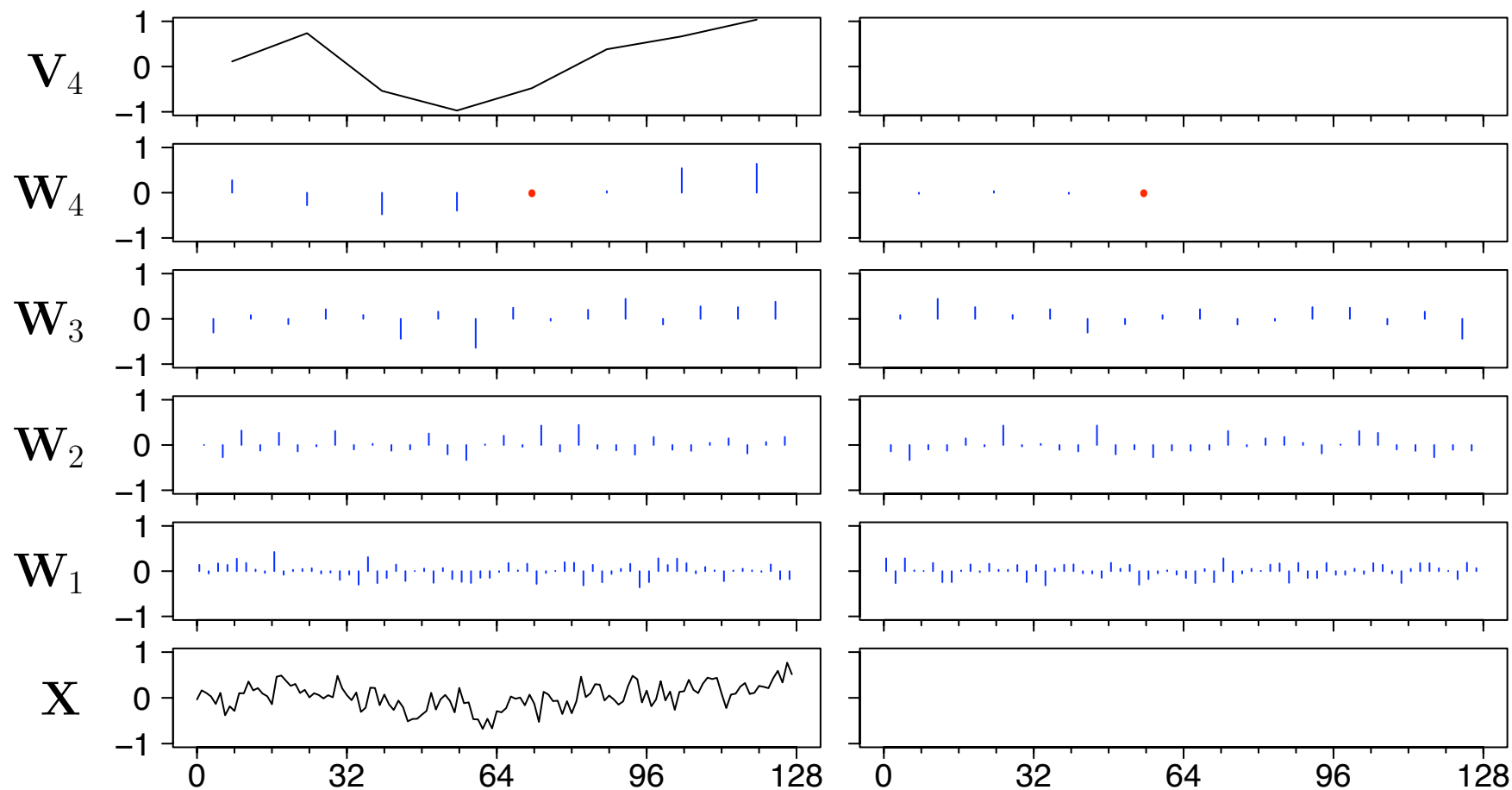
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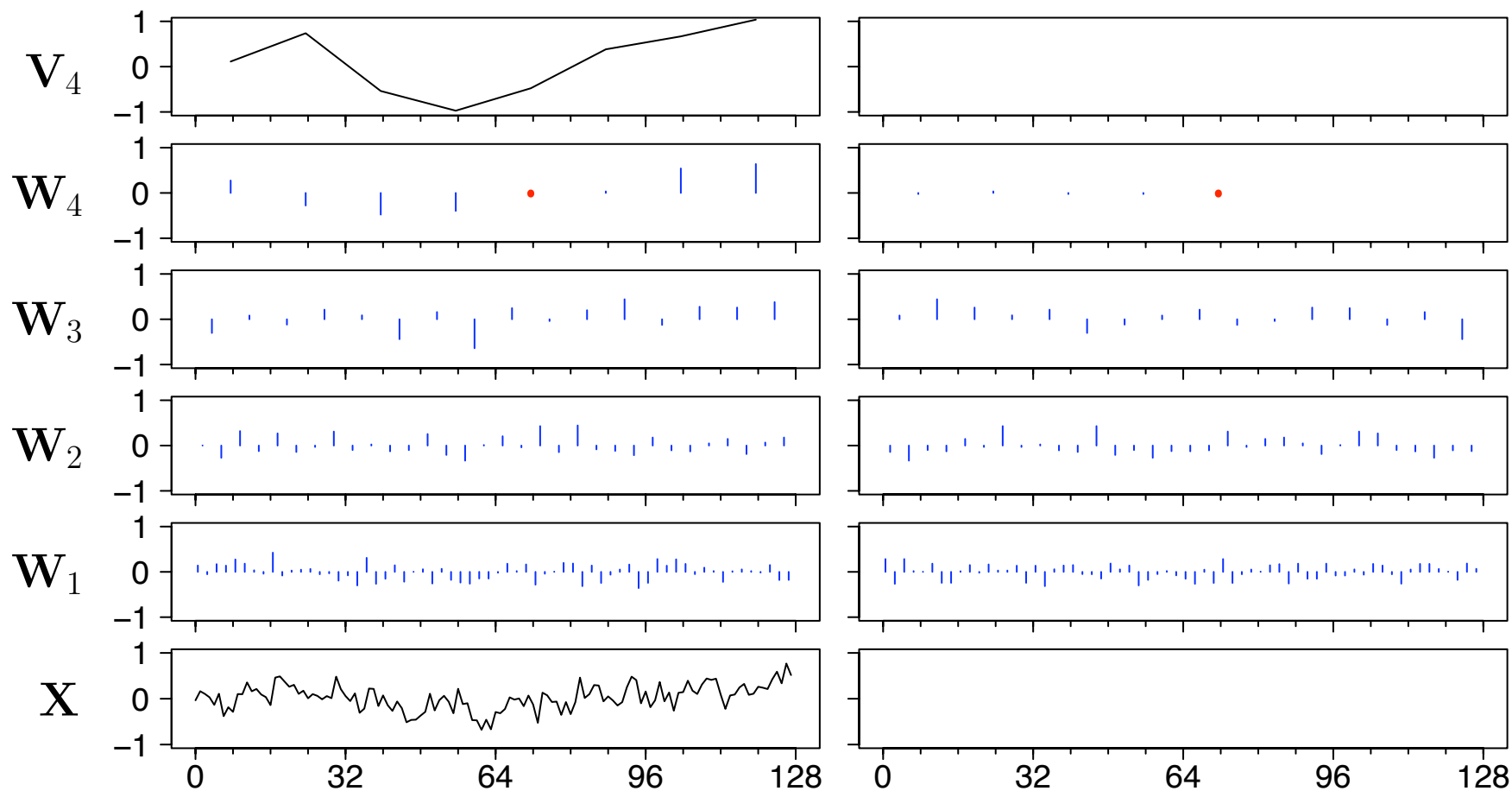
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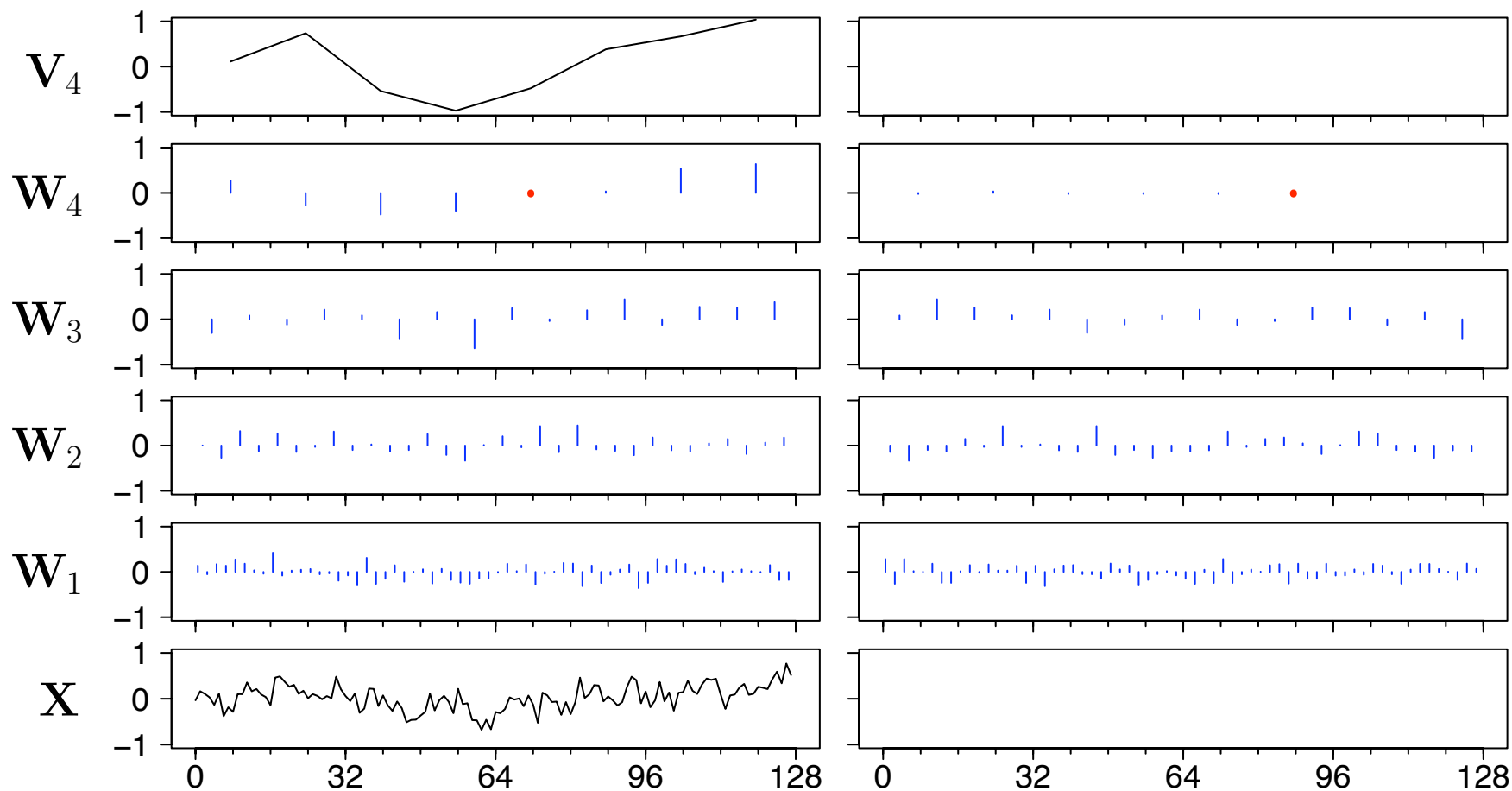
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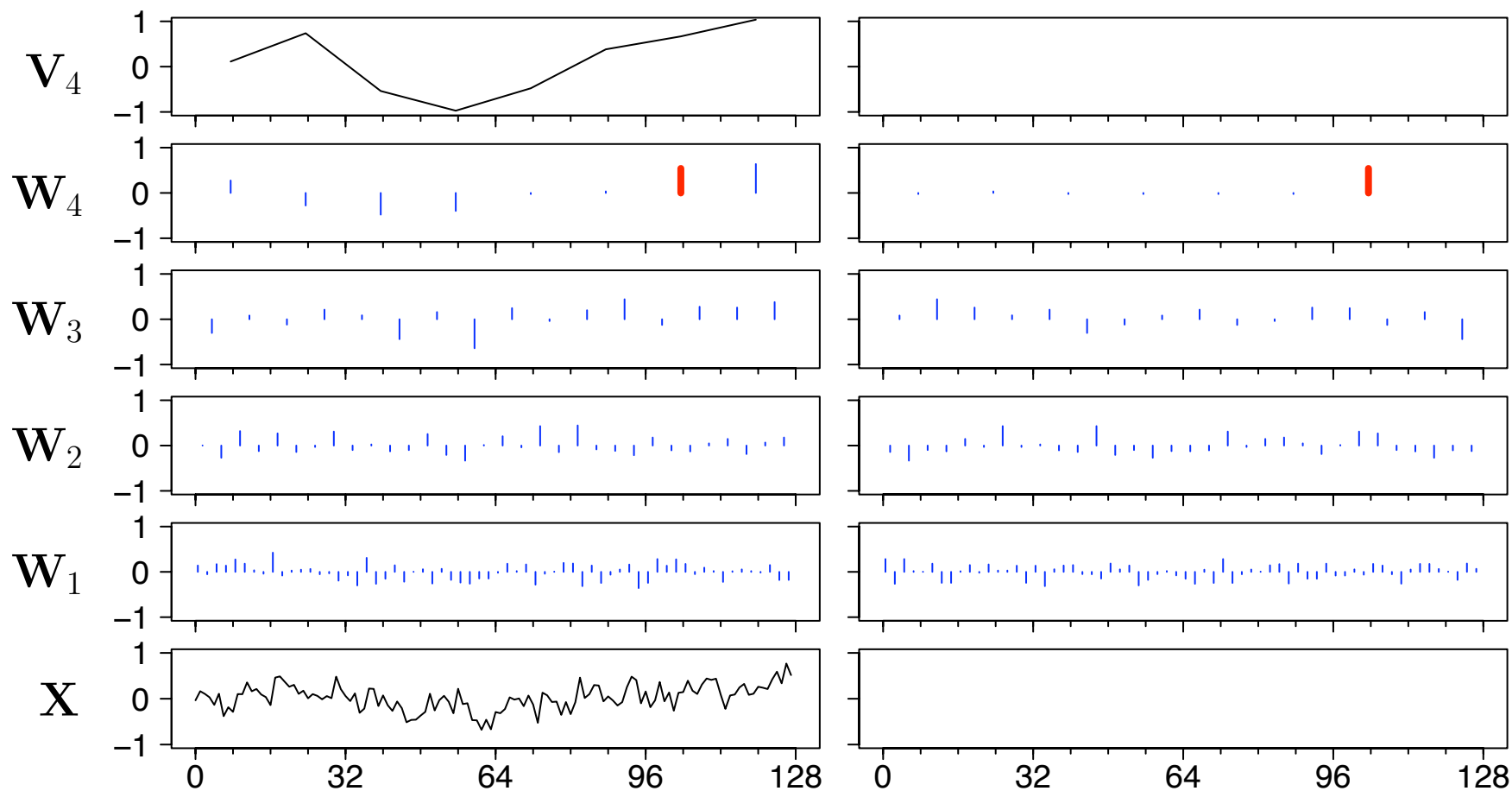
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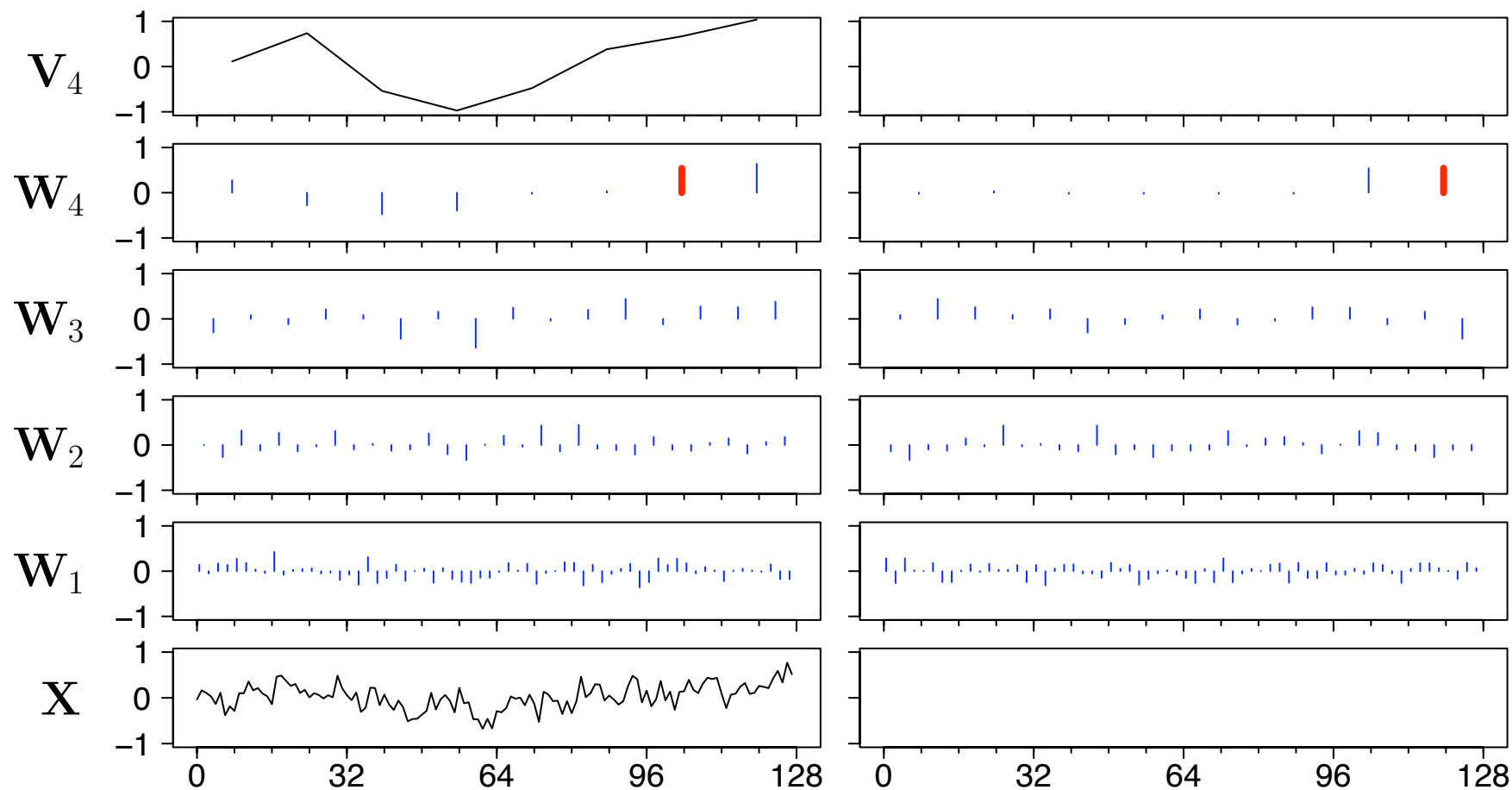
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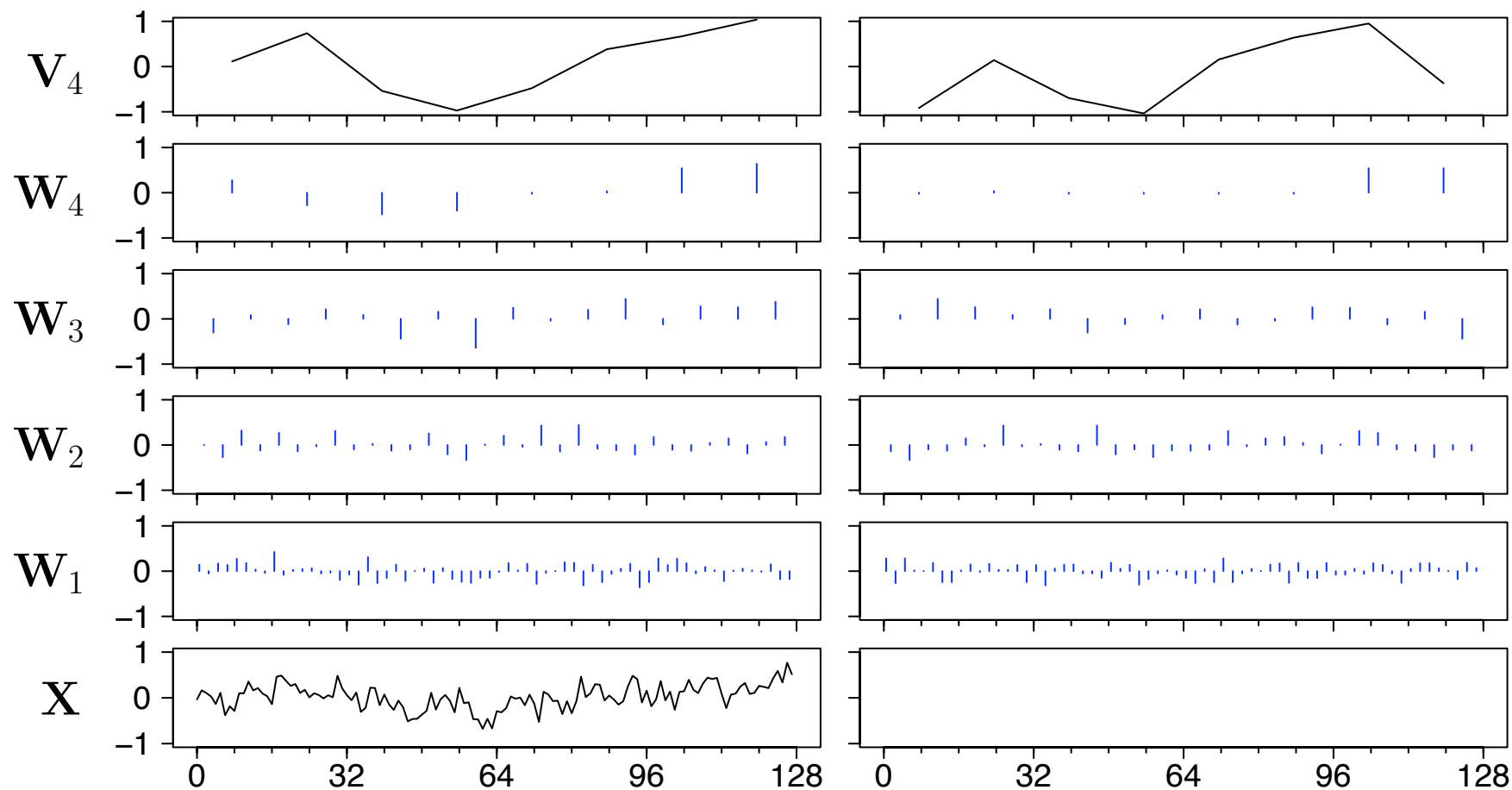
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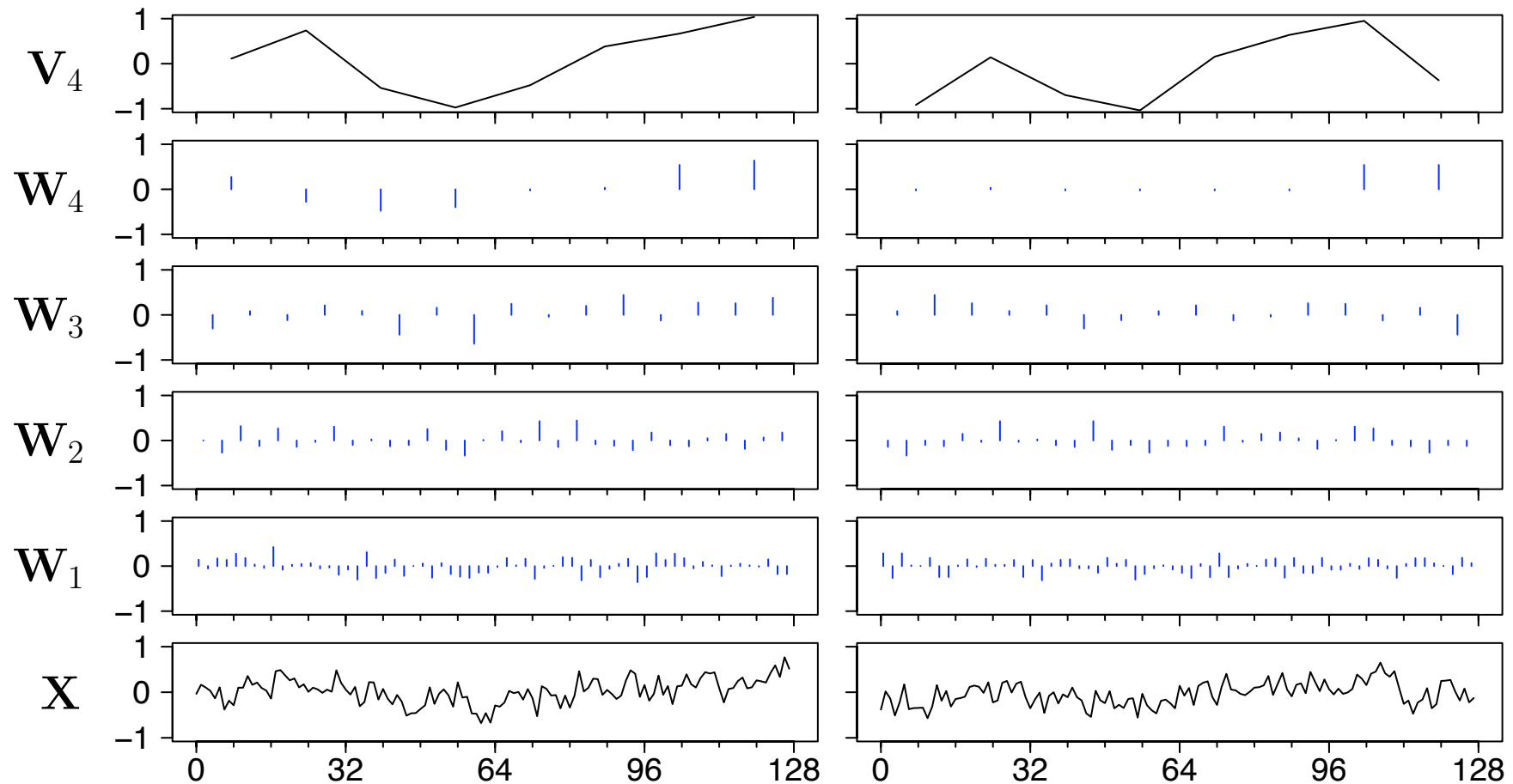
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Illustration of Wavelet-Domain Bootstrapping



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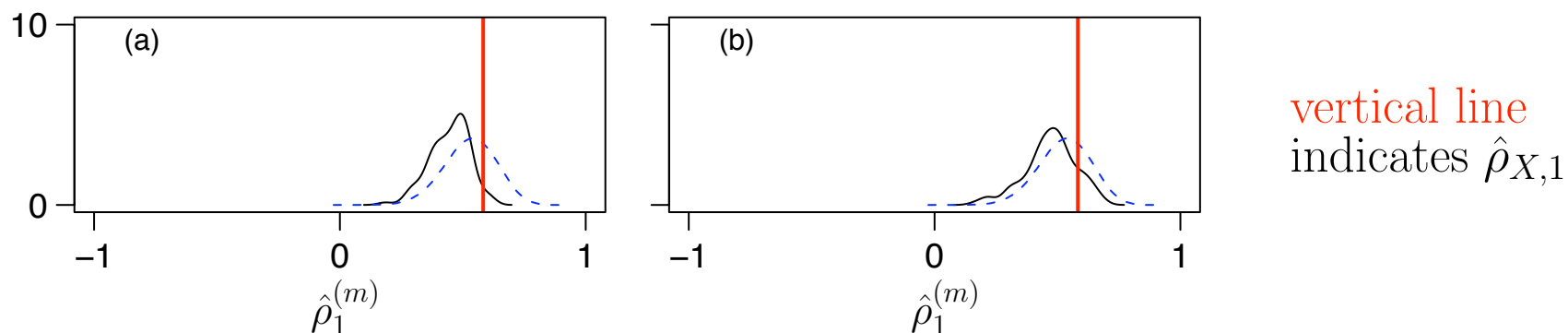
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- Haar DWT of FD(0.45) series X (left-hand column) and wavelet-domain bootstrap thereof (right-hand)

Wavelet-Domain Bootstrapping of FD Series

- approximations to true PDF using (a) Haar & (b) LA(8) wavelets



- using 50 FD time series and the Haar DWT yields:
 - average of 50 sample means $\doteq 0.35$ (truth $\doteq 0.53$)
 - average of 50 sample SDs $\doteq 0.096$ (truth $\doteq 0.107$)
- using 50 FD time series and the LA(8) DWT yields:
 - average of 50 sample means $\doteq 0.43$ (truth $\doteq 0.53$)
 - average of 50 sample SDs $\doteq 0.098$ (truth $\doteq 0.107$)

MLEs of FD Parameters: I

- FD process depends on 2 parameters, namely, δ and σ_ε^2
- given $\mathbf{X} = [X_0, X_1, \dots, X_{N-1}]^T$ with $N = 2^J$, suppose we want to estimate δ and σ_ε^2
- if \mathbf{X} is stationary (i.e. $\delta < 1/2$) and multivariate Gaussian, can use the maximum likelihood (ML) method

MLEs of FD Parameters: II

- definition of Gaussian likelihood function:

$$L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X}) \equiv \frac{1}{(2\pi)^{N/2} |\Sigma_{\mathbf{X}}|^{1/2}} e^{-\mathbf{X}^T \Sigma_{\mathbf{X}}^{-1} \mathbf{X} / 2}$$

where $\Sigma_{\mathbf{X}}$ is covariance matrix for \mathbf{X} , with (s, t) th element given by $s_{X, s-t}$, and $|\Sigma_{\mathbf{X}}|$ & $\Sigma_{\mathbf{X}}^{-1}$ denote determinant & inverse

- ML estimators of δ and σ_ε^2 maximize $L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})$ or, equivalently, minimize

$$-2 \log (L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})) = N \log (2\pi) + \log (|\Sigma_{\mathbf{X}}|) + \mathbf{X}^T \Sigma_{\mathbf{X}}^{-1} \mathbf{X}$$

- exact MLEs computationally intensive, mainly because of the need to deal with $|\Sigma_{\mathbf{X}}|$ and $\Sigma_{\mathbf{X}}^{-1}$
- good approximate MLEs of considerable interest

MLEs of FD Parameters: III

- key ideas behind first wavelet-based approximate MLEs
 - have seen that we can approximate FD time series \mathbf{X} by $\mathbf{Y} = \mathcal{W}^T \Lambda^{1/2} \mathbf{Z}$, where $\Lambda^{1/2}$ is a diagonal matrix, all of whose diagonal elements are positive
 - since covariance matrix for \mathbf{Z} is I_N , the one for \mathbf{Y} is
$$\mathcal{W}^T \Lambda^{1/2} I_N (\mathcal{W}^T \Lambda^{1/2})^T = \mathcal{W}^T \Lambda^{1/2} \Lambda^{1/2} \mathcal{W} = \mathcal{W}^T \Lambda \mathcal{W} \equiv \tilde{\Sigma}_{\mathbf{X}},$$
where $\Lambda \equiv \Lambda^{1/2} \Lambda^{1/2}$ is also diagonal
 - can consider $\tilde{\Sigma}_{\mathbf{X}}$ to be an approximation to $\Sigma_{\mathbf{X}}$
- leads to approximation of log likelihood:
$$-2 \log (L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})) \approx N \log (2\pi) + \log (|\tilde{\Sigma}_{\mathbf{X}}|) + \mathbf{X}^T \tilde{\Sigma}_{\mathbf{X}}^{-1} \mathbf{X}$$

MLEs of FD Parameters: IV

- Q: so how does this help us?

- easy to invert $\tilde{\Sigma}_{\mathbf{X}}$:

$$\tilde{\Sigma}_{\mathbf{X}}^{-1} = (\mathcal{W}^T \Lambda \mathcal{W})^{-1} = (\mathcal{W})^{-1} \Lambda^{-1} (\mathcal{W}^T)^{-1} = \mathcal{W}^T \Lambda^{-1} \mathcal{W},$$

where Λ^{-1} is another diagonal matrix, leading to

$$\mathbf{X}^T \tilde{\Sigma}_{\mathbf{X}}^{-1} \mathbf{X} = \mathbf{X}^T \mathcal{W}^T \Lambda^{-1} \mathcal{W} \mathbf{X} = \mathbf{W}^T \Lambda^{-1} \mathbf{W}$$

- easy to compute the determinant of $\tilde{\Sigma}_{\mathbf{X}}$:

$$|\tilde{\Sigma}_{\mathbf{X}}| = |\mathcal{W}^T \Lambda \mathcal{W}| = |\Lambda \mathcal{W} \mathcal{W}^T| = |\Lambda I_N| = |\Lambda|,$$

and the determinant of a diagonal matrix is just the product of its diagonal elements

MLEs of FD Parameters: V

- define the following three functions of δ :

$$C'_j(\delta) \equiv \int_{1/2^{j+1}}^{1/2^j} \frac{2^{j+1}}{[4 \sin^2(\pi f)]^\delta} df \approx \int_{1/2^{j+1}}^{1/2^j} \frac{2^{j+1}}{[2\pi f]^{2\delta}} df$$

$$C'_{J+1}(\delta) \equiv \frac{N\Gamma(1-2\delta)}{\Gamma^2(1-\delta)} - \sum_{j=1}^J \frac{N}{2^j} C'_j(\delta)$$

$$\sigma_\varepsilon^2(\delta) \equiv \frac{1}{N} \left(\frac{V_{J,0}^2}{C'_{J+1}(\delta)} + \sum_{j=1}^J \frac{1}{C'_j(\delta)} \sum_{t=0}^{\frac{N}{2^j}-1} W_{j,t}^2 \right)$$

MLEs of FD Parameters: VI

- wavelet-based approximate MLE $\tilde{\delta}$ for δ is the value that minimizes the following function of δ :

$$\tilde{l}(\delta | \mathbf{X}) \equiv N \log(\sigma_{\varepsilon}^2(\delta)) + \log(C'_{J+1}(\delta)) + \sum_{j=1}^J \frac{N}{2^j} \log(C'_j(\delta))$$

- once $\tilde{\delta}$ has been determined, MLE for σ_{ε}^2 is given by $\sigma_{\varepsilon}^2(\tilde{\delta})$
- computer experiments indicate scheme works quite well

Other Wavelet-Based Estimators of FD Parameters

- second MLE approach: formulate likelihood directly in terms of nonboundary wavelet coefficients
 - handles stationary or nonstationary FD processes (i.e., need not assume $\delta < 1/2$)
 - handles certain deterministic trends
- alternative to MLEs are least square estimators (LSEs)
 - recall that, for large τ and for $\beta = 2\delta - 1$, have
$$\log(\nu_X^2(\tau_j)) \approx \zeta + \beta \log(\tau_j)$$
 - suggests determining δ by regressing $\log(\hat{\nu}_X^2(\tau_j))$ on $\log(\tau_j)$ over range of τ_j
 - weighted LSE takes into account fact that variance of $\log(\hat{\nu}_X^2(\tau_j))$ depends upon scale τ_j (increases as τ_j increases)

Homogeneity of Variance: I

- because DWT decorrelates FD and related processes, nonboundary coefficients in \mathbf{W}_j should resemble white noise; i.e.,

$$\text{cov} \{W_{j,t}, W_{j,t'}\} \approx 0$$

when $t \neq t'$, and $\text{var} \{W_{j,t}\}$ should not depend upon t

- can test for homogeneity of variance in \mathbf{X} using \mathbf{W}_j over a range of levels j
- suppose U_0, \dots, U_{N-1} are independent normal RVs with $E\{U_t\} = 0$ and $\text{var} \{U_t\} = \sigma_t^2$
- want to test null hypothesis $H_0 : \sigma_0^2 = \sigma_1^2 = \dots = \sigma_{N-1}^2$
- can test H_0 versus a variety of alternatives, e.g.,
$$H_1 : \sigma_0^2 = \dots = \sigma_k^2 \neq \sigma_{k+1}^2 = \dots = \sigma_{N-1}^2$$
using normalized cumulative sum of squares

Homogeneity of Variance: II

- to define test statistic D , start with

$$\mathcal{P}_k \equiv \frac{\sum_{j=0}^k U_j^2}{\sum_{j=0}^{N-1} U_j^2}, \quad k = 0, \dots, N-2$$

and then compute $D \equiv \max(D^+, D^-)$, where

$$D^+ \equiv \max_{0 \leq k \leq N-2} \left(\frac{k+1}{N-1} - \mathcal{P}_k \right) \quad \& \quad D^- \equiv \max_{0 \leq k \leq N-2} \left(\mathcal{P}_k - \frac{k}{N-1} \right)$$

- can reject H_0 if observed D is ‘too large,’ where ‘too large’ is quantified by considering distribution of D under H_0
- need to find critical value x_α such that $\mathbf{P}[D \geq x_\alpha] = \alpha$ for, e.g., $\alpha = 0.01, 0.05$ or 0.1

Homogeneity of Variance: III

- once determined, can perform α level test of H_0 :
 - compute D statistic from data U_0, \dots, U_{N-1}
 - reject H_0 at level α if $D \geq x_\alpha$
 - fail to reject H_0 at level α if $D < x_\alpha$
- can determine critical values x_α in two ways
 - Monte Carlo simulations
 - large sample approximation to distribution of D :

$$\mathbf{P}[(N/2)^{1/2}D \geq x] \approx 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-2l^2 x^2}$$

(reasonable approximation for $N \geq 128$)

Homogeneity of Variance: IV

- idea: given time series $\{X_t\}$, compute D using nonboundary wavelet coefficients $W_{j,t}$ (there are $M'_j \equiv N_j - L'_j$ of these):

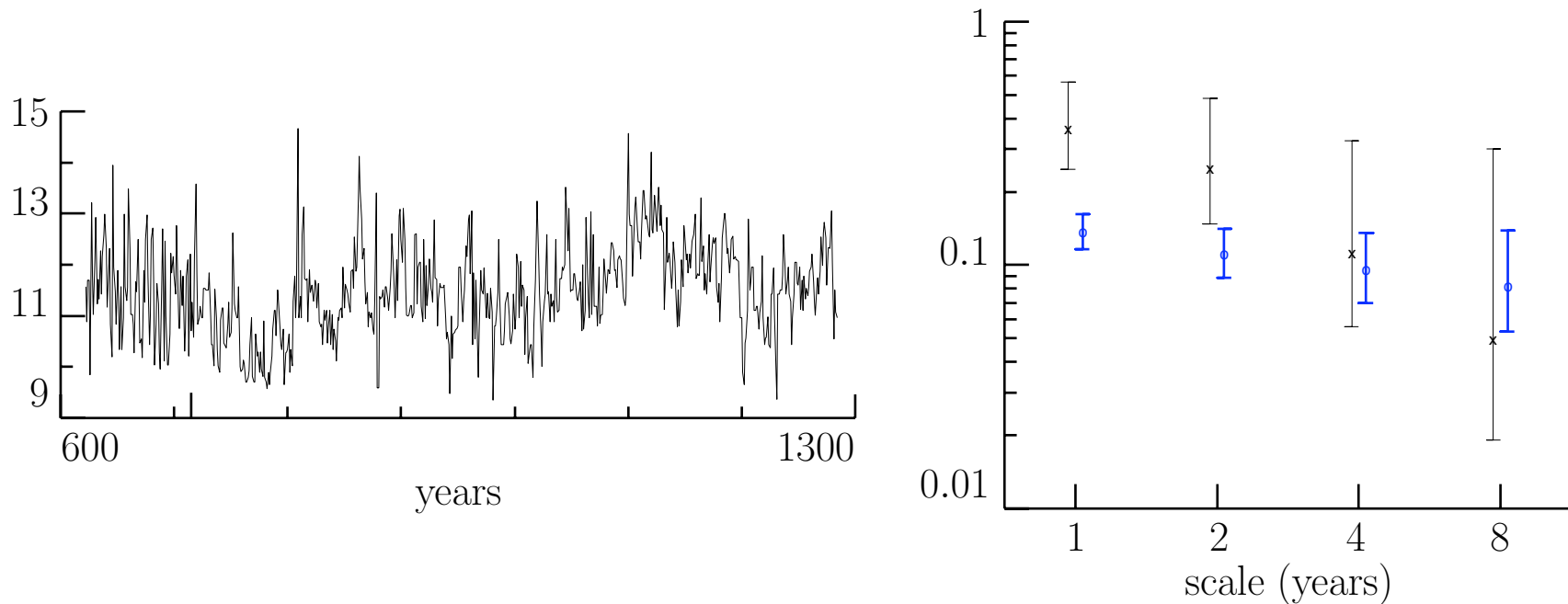
$$\mathcal{P}_k \equiv \frac{\sum_{t=L'_j}^k W_{j,t}^2}{\sum_{t=L'_j}^{N_j-1} W_{j,t}^2}, \quad k = L'_j, \dots, N_j - 2$$

- if null hypothesis rejected at level j , can use nonboundary MODWT coefficients to locate change point based on

$$\tilde{\mathcal{P}}_k \equiv \frac{\sum_{t=L_j-1}^k \tilde{W}_{j,t}^2}{\sum_{t=L_j-1}^{N-1} \tilde{W}_{j,t}^2}, \quad k = L_j - 1, \dots, N - 2$$

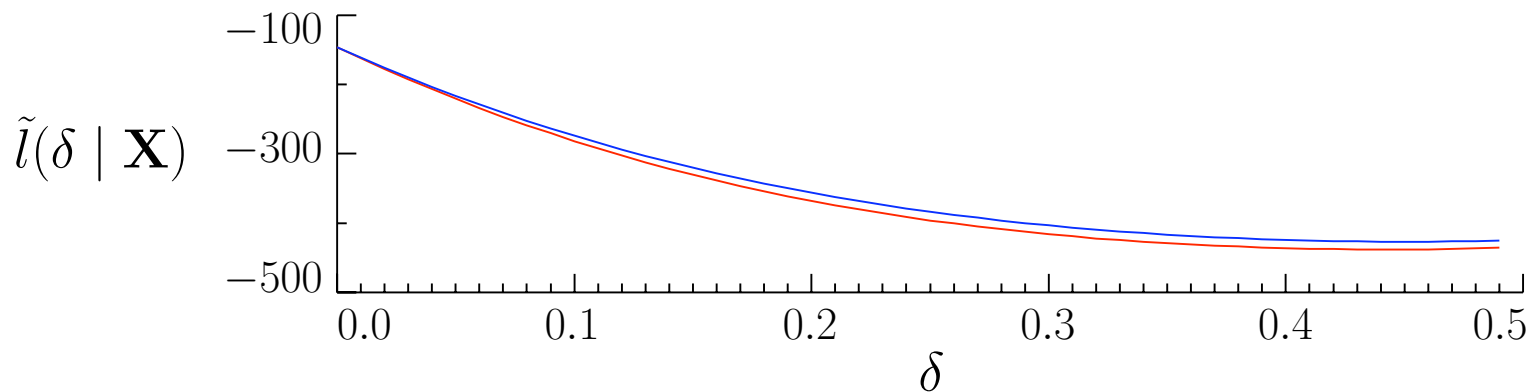
along with analogs \tilde{D}_k^+ and \tilde{D}_k^- of D_k^+ and D_k^-

Example – Annual Minima of Nile River: I



- left-hand plot: annual minima of Nile River
- new measuring device introduced around year 715
- right: Haar $\hat{\nu}_X^2(\tau_j)$ before (x's) and after (o's) year 715.5, with 95% confidence intervals based upon $\chi_{\eta_3}^2$ approximation

Example – Annual Minima of Nile River: II



- based upon last 512 values (years 773 to 1284), plot shows $\tilde{l}(\delta | \mathbf{X})$ versus δ for the first wavelet-based approximate MLE using the LA(8) wavelet (**upper curve**) and corresponding curve for exact MLE (**lower**)
 - wavelet-based approximate MLE is value minimizing **upper curve**: $\tilde{\delta} \doteq 0.4532$
 - exact MLE is value minimizing **lower** curve: $\hat{\delta} \doteq 0.4452$

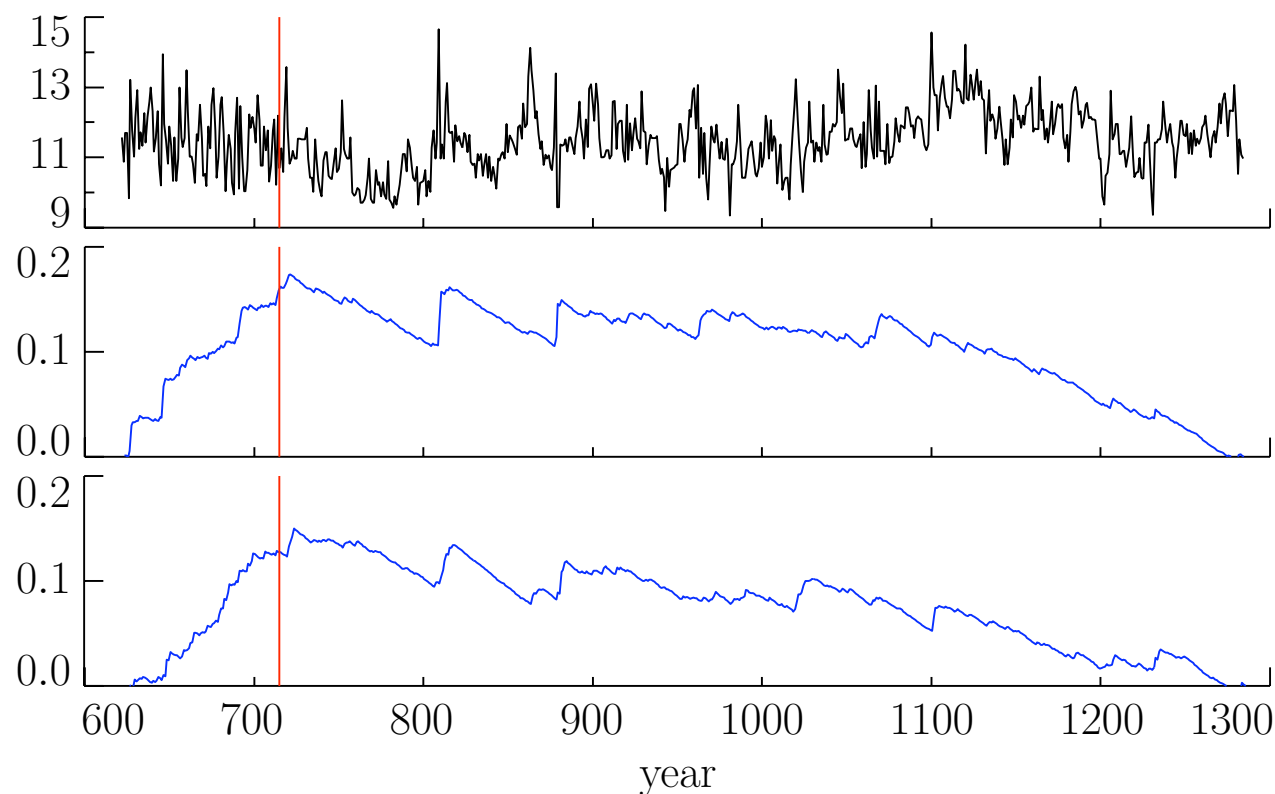
Example – Annual Minima of Nile River: III

- results of testing all Nile River minima for homogeneity of variance using the Haar wavelet filter with critical values determined by computer simulations

τ_j	M'_j	D	critical levels		
			10%	5%	1%
1 year	331	0.1559	0.0945	0.1051	0.1262
2 years	165	0.1754	0.1320	0.1469	0.1765
4 years	82	0.1000	0.1855	0.2068	0.2474
8 years	41	0.2313	0.2572	0.2864	0.3436

- can reject null hypothesis of homogeneity of variance at level of significance 0.05 for scales τ_1 & τ_2 , but not at larger scales

Example – Annual Minima of Nile River: IV



- Nile River minima (top plot) along with curves (constructed per Equation (382)) for scales τ_1 & τ_2 (middle & bottom) to identify change point via time of maximum deviation (vertical lines denote year 715)

Summary

- DWT approximately decorrelate certain time series, including ones coming from FD and related processes
- leads to schemes for simulating time series and bootstrapping
- also leads to schemes for estimating parameters of FD process
 - approximate maximum likelihood estimators (two varieties)
 - weighted least squares estimator
- can also devise wavelet-based tests for
 - homogeneity of variance
 - trends (see Craigmile *et al.*, 2004, for details)

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