## Wavelet Methods for Time Series Analysis

Part II: Wavelet-Based Statistical Analysis of Time Series

- topics to covered:
  - wavelet variance (analysis phase of MODWT)
  - wavelet-based signal extraction (synthesis phase of DWT)
  - wavelet-based decorrelation of time series (analysis phase of DWT, but synthesis phase plays a role also)

## Wavelet Variance: Overview

- review of decomposition of sample variance using wavelets
- theoretical wavelet variance for stochastic processes
  - stationary processes
  - nonstationary processes with stationary differences
- sampling theory for Gaussian processes
- real-world examples
- extensions and summary

## Decomposing Sample Variance of Time Series

- let $X_0, X_1, \ldots, X_{N-1}$ represent time series with $N$ values
- let $\overline{X}$ denote sample mean of $X_t$'s: $\overline{X} \equiv \frac{1}{N} \sum_{t=0}^{N-1} X_t$
- let $\hat{\sigma}_X^2$ denote sample variance of $X_t$'s:

$$\hat{\sigma}_X^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} \left( X_t - \overline{X} \right)^2$$

- idea is to decompose (analyze, break up) $\hat{\sigma}_X^2$ into pieces that quantify how one time series might differ from another
- wavelet variance does analysis based upon differences between (possibly weighted) adjacent averages over scales

## Empirical Wavelet Variance

- define empirical wavelet variance for scale $\tau_j \equiv 2^{j-1}$ as

$$\tilde{\nu}_X^2(\tau_j) \equiv \frac{1}{N} \sum_{t=0}^{N-1} \widetilde{W}_{j,t}^2, \quad \text{where} \quad \widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}$$

- if $N = 2^J$, obtain analysis (decomposition) of sample variance:

$$\hat{\sigma}_X^2 = \frac{1}{N} \sum_{t=0}^{N-1} \left( X_t - \overline{X} \right)^2 = \sum_{j=1}^{J} \tilde{\nu}_X^2(\tau_j)$$

(if $N$ not a power of 2, can analyze variance to any level $J_0$, but need additional component involving scaling coefficients)
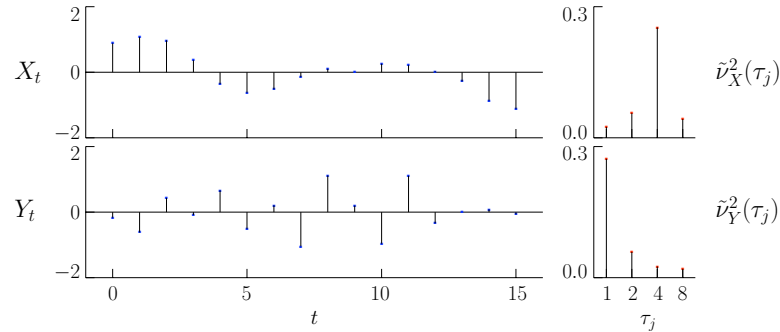
- interpretation: $\tilde{\nu}_X^2(\tau_j)$ is portion of $\hat{\sigma}_X^2$ due to changes in averages over scale $\tau_j$; i.e., 'scale by scale' analysis of variance

## Example of Empirical Wavelet Variance

- wavelet variances for time series $X_t$ and $Y_t$ of length $N = 16$, each with zero sample mean and same sample variance

## Theoretical Wavelet Variance: I

- now assume $X_t$ is a real-valued random variable (RV)
- let $\{X_t, t \in \mathbb{Z}\}$ denote a stochastic process, i.e., collection of RVs indexed by 'time' $t$ (here $\mathbb{Z}$ denotes the set of all integers)
- apply $j$th level equivalent MODWT filter $\{\tilde{h}_{j,l}\}$ to $\{X_t\}$ to create a new stochastic process:

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t \in \mathbb{Z},$$

which should be contrasted with

$$\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \ldots, N-1$$

## Theoretical Wavelet Variance: II

- if $Y$ is any RV, let $E\{Y\}$ denote its expectation
- let var $\{Y\}$ denote its variance: var $\{Y\} \equiv E\{(Y - E\{Y\})^2\}$
- definition of time dependent wavelet variance:

$$\nu_{X,t}^2(\tau_j) \equiv \text{var} \{\overline{W}_{j,t}\},$$

with conditions on $X_t$ so that var $\{\overline{W}_{j,t}\}$ exists and is finite
- $\nu_{X,t}^2(\tau_j)$ depends on $\tau_j$ and $t$
- will focus on time independent wavelet variance

$$\nu_X^2(\tau_j) \equiv \text{var} \{\overline{W}_{j,t}\}$$

(can adapt theory to handle time varying situation)
- $\nu_X^2(\tau_j)$ well-defined for stationary processes and certain related processes, so let's review concept of stationarity

## Definition of a Stationary Process

- if $U$ and $V$ are two RVs, denote their covariance by

$$\text{cov} \{U, V\} = E\{(U - E\{U\})(V - E\{V\})\}$$

- stochastic process $X_t$ called stationary if
  - $E\{X_t\} = \mu_X$ for all $t$, i.e., constant independent of $t$
  - $\text{cov}\{X_t, X_{t+\tau}\} = s_{X,\tau}$, i.e., depends on lag $\tau$, but not $t$
- $s_{X,\tau}$, $\tau \in \mathbb{Z}$, is autocovariance sequence (ACVS)
- $s_{X,0} = \text{cov}\{X_t, X_t\} = \text{var}\{X_t\}$; i.e., variance same for all $t$

## Wavelet Variance for Stationary Processes

- for stationary processes, wavelet variance decomposes var $\{X_t\}$:

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var}\,\{X_t\},$$

which is similar to

$$\sum_{j=1}^{J} \tilde{\nu}_X^2(\tau_j) = \hat{\sigma}_X^2$$

- $\nu_X^2(\tau_j)$ is thus contribution to var $\{X_t\}$ due to scale $\tau_j$
- note: $\nu_X^2(\tau_j)$ and $X_t^2$ have same units (can be important for interpretability)

## White Noise Process

- simplest example of a stationary process is 'white noise'
- process $X_t$ said to be white noise if
  - it has a constant mean $E\{X_t\} = \mu_X$
  - it has a constant variance var $\{X_t\} = \sigma_X^2$
  - cov $\{X_t, X_{t+\tau}\} = 0$ for all $t$ and nonzero $\tau$; i.e., distinct RVs in the process are uncorrelated
- ACVS for white noise takes a very simple form:

$$s_{X,\tau} = \text{cov}\,\{X_t, X_{t+\tau}\} = \begin{cases} \sigma_X^2, & \tau = 0; \\ 0, & \text{otherwise.} \end{cases}$$

## Wavelet Variance for White Noise Process: I

- for a white noise process, can show that

$$\nu_X^2(\tau_j) = \frac{\text{var}\,\{X_t\}}{2^j} \propto \tau_j^{-1} \ \text{ since } \ \tau_j = 2^{j-1}$$

- note that

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var}\,\{X_t\} \left( \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \right) = \text{var}\,\{X_t\},$$
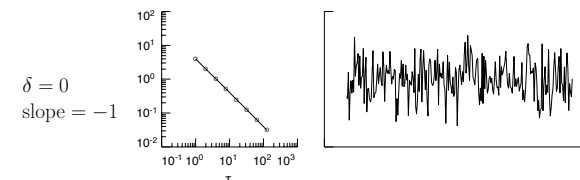
as required
- note also that

$$\log\left(\nu_X^2(\tau_j)\right) \propto -\log\left(\tau_j\right),$$

so plot of $\log\left(\nu_X^2(\tau_j)\right)$ vs. $\log\left(\tau_j\right)$ is linear with a slope of $-1$

## Wavelet Variance for White Noise Process: II



$\delta = 0$
slope $= -1$

- $\nu_X^2(\tau_j)$ versus $\tau_j$ for $j = 1, \ldots, 8$ (left-hand plot), along with sample of length $N = 256$ of Gaussian white noise
- largest contribution to var $\{X_t\}$ is at smallest scale $\tau_1$
- note: later on, we will discuss fractionally differenced (FD) processes that are characterized by a parameter $\delta$; when $\delta = 0$, an FD process is the same as a white noise process

## Generalization to Certain Nonstationary Processes

- if wavelet filter is properly chosen, $\nu_X^2(\tau_j)$ well-defined for certain processes with stationary backward differences (increments); these are also known as intrinsically stationary processes

- first order backward difference of $X_t$ is process defined by
$$X_t^{(1)} = X_t - X_{t-1}$$

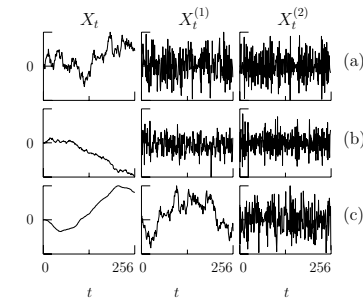- second order backward difference of $X_t$ is process defined by
$$X_t^{(2)} = X_t^{(1)} - X_{t-1}^{(1)} = X_t - 2X_{t-1} + X_{t-2}$$

- $X_t$ said to have $d$th order stationary backward differences if
$$Y_t \equiv \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k}$$
forms a stationary process ($d$ is a nonnegative integer)

## Examples of Processes with Stationary Increments



- 1st column shows, from top to bottom, realizations from
  (a) random walk: $X_t = \sum_{u=1}^{t} \epsilon_u$, & $\epsilon_t$ is zero mean white noise
  (b) like (a), but now $\epsilon_t$ has mean of $-0.2$
  (c) random run: $X_t = \sum_{u=1}^{t} Y_u$, where $Y_t$ is a random walk

- 2nd & 3rd columns show 1st & 2nd differences $X_t^{(1)}$ and $X_t^{(2)}$

## Wavelet Variance for Processes with Stationary Backward Differences: I

- let $\{X_t\}$ be nonstationary with $d$th order stationary differences

- if we use a Daubechies wavelet filter of width $L$ satisfying $L \geq 2d$, then $\nu_X^2(\tau_j)$ is well-defined and finite for all $\tau_j$, but now
$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \infty$$

- works because there is a backward difference operator of order $d = L/2$ embedded within $\{\tilde{h}_{j,l}\}$, so this filter reduces $X_t$ to
$$\sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k} = Y_t$$
and then creates localized weighted averages of $Y_t$'s

## Wavelet Variance for Random Walk Process: I

- random walk process $X_t = \sum_{u=1}^{t} \epsilon_u$ has first order ($d = 1$) stationary differences since $X_t - X_{t-1} = \epsilon_t$ (i.e., white noise)

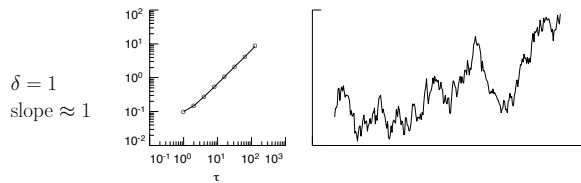- $L \geq 2d$ holds for all wavelets when $d = 1$; for Haar ($L = 2$),
$$\nu_X^2(\tau_j) = \frac{\text{var}\{\epsilon_t\}}{6}\left(\tau_j + \frac{1}{2\tau_j}\right) \approx \frac{\text{var}\{\epsilon_t\}}{6}\tau_j,$$
with the approximation becoming better as $\tau_j$ increases

- note that $\nu_X^2(\tau_j)$ increases as $\tau_j$ increases

- $\log(\nu_X^2(\tau_j)) \propto \log(\tau_j)$ approximately, so plot of $\log(\nu_X^2(\tau_j))$ vs. $\log(\tau_j)$ is approximately linear with a slope of $+1$

- as required, also have
$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \frac{\text{var}\{\epsilon_t\}}{6}\left(1 + \frac{1}{2} + 2 + \frac{1}{4} + 4 + \frac{1}{8} + \cdots\right) = \infty$$

## Wavelet Variance for Random Walk Process: II



$\delta = 1$
slope $\approx 1$

- $\nu_X^2(\tau_j)$ versus $\tau_j$ for $j = 1, \ldots, 8$ (left-hand plot), along with sample of length $N = 256$ of a Gaussian random walk process

- smallest contribution to var $\{X_t\}$ is at smallest scale $\tau_1$

- note: a fractionally differenced process with parameter $\delta = 1$ is the same as a random walk process

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## Fractionally Differenced (FD) Processes: I

- can create a continuum of processes that 'interpolate' between white noise and random walks and 'extrapolate' beyond them using notion of 'fractional differencing' (Granger and Joyeux, 1980; Hosking, 1981)

- FD($\delta$) process is determined by 2 parameters $\delta$ and $\sigma_\epsilon^2$, where $-\infty < \delta < \infty$ and $\sigma_\epsilon^2 > 0$ ($\sigma_\epsilon^2$ is less important than $\delta$)

- if $\delta < 1/2$, FD process $\{X_t\}$ is stationary, and, in particular,

  – reduces to white noise if $\delta = 0$

  – has 'long memory' or 'long range dependence' if $\delta > 0$

  – is 'antipersistent' if $\delta < 0$ (i.e., cov $\{X_t, X_{t+1}\} < 0$)

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## Fractionally Differenced (FD) Processes: II

- if $\delta \geq 1/2$, FD process $\{X_t\}$ is nonstationary with $d$th order stationary backward differences $\{Y_t\}$

  – here $d = \lfloor \delta + 1/2 \rfloor$, where $\lfloor x \rfloor$ is integer part of $x$

  – $\{Y_t\}$ is stationary FD($\delta - d$) process

- if $\delta = 1$, FD process is the same as a random walk process

- except possibly for two or three smallest scales, have
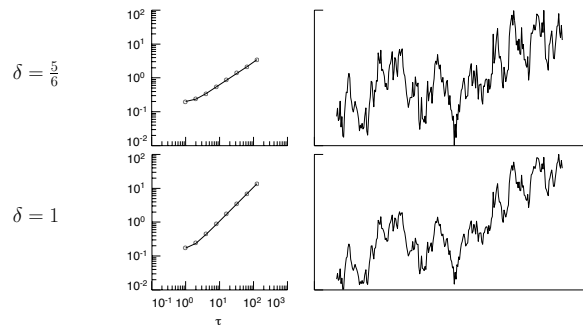
$$\nu_X^2(\tau_j) \approx C \tau_j^{2\delta - 1}$$

- thus $\log(\nu_X^2(\tau_j)) \approx \log(C) + (2\delta - 1)\log(\tau_j)$, so a log/log plot of $\nu_X^2(\tau_j)$ vs. $\tau_j$ looks approximately linear with slope $2\delta - 1$ for $\tau_j$ large enough

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## LA(8) Wavelet Variance for 2 FD Processes



$\delta = \frac{1}{4}$

$\delta = \frac{1}{2}$
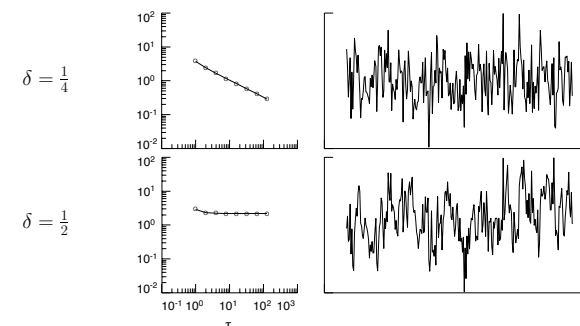
- see overhead 12 for $\delta = 0$ (white noise), which has slope $= -1$

- $\delta = \frac{1}{4}$ has slope $-\frac{1}{2}$

- $\delta = \frac{1}{2}$ has slope 0 (related to so-called 'pink noise')

## LA(8) Wavelet Variance for 2 More FD Processes



$\delta = \frac{5}{6}$

$\delta = 1$

- $\delta = \frac{5}{6}$ has slope $\frac{2}{3}$ (related to Kolmogorov turbulence)
- $\delta = 1$ has slope $1$ (random walk)
- nonnegative slopes indicate nonstationarity, while negative slopes indicate stationarity

## Wavelet Variance for Processes with Stationary Backward Differences: II

- summary: $\nu_X^2(\tau_j)$ well-defined for process $\{X_t\}$ that is
  - stationary
  - nonstationary with $d$th order stationary increments, but width of wavelet filter must satisfy $L \geq 2d$
- if $\{X_t\}$ is stationary, then

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \operatorname{var}\{X_t\} < \infty$$

(recall that each RV in a stationary process must have the same finite variance)

## Wavelet Variance for Processes with Stationary Backward Differences: III

- if $\{X_t\}$ is nonstationary, then

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \infty$$

- with a suitable construction, we can take variance of nonstationary process with $d$th order stationary increments to be $\infty$
- using this construction, we have

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \operatorname{var}\{X_t\}$$

for both the stationary and nonstationary cases

## Background on Gaussian Random Variables

- $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian (normal) RV with mean $\mu$ and variance $\sigma^2$
- will write

$$X \stackrel{\mathrm{d}}{=} \mathcal{N}(\mu, \sigma^2)$$

to mean 'RV $X$ has same distribution as Gaussian RV'
- RV $\mathcal{N}(0,1)$ often written as $Z$ (called standard Gaussian or standard normal)
- let $\Phi(\cdot)$ be Gaussian cumulative distribution function

$$\Phi(z) \equiv \mathbf{P}[Z \leq z] = \int_{-\infty}^{z} \frac{1}{\sqrt{(2\pi)}} e^{-x^2/2}\, dx$$

- inverse $\Phi^{-1}(\cdot)$ of $\Phi(\cdot)$ is such that $\mathbf{P}[Z \leq \Phi^{-1}(p)] = p$
- $\Phi^{-1}(p)$ called $p \times 100\%$ percentage point

## Background on Chi-Square Random Variables

- $X$ said to be a chi-square RV with $\eta$ degrees of freedom if its probability density function (PDF) is given by
$$f_X(x;\eta) = \frac{1}{2^{\eta/2}\Gamma(\eta/2)} x^{(\eta/2)-1} e^{-x/2}, \quad x \geq 0, \ \eta > 0$$

- $\chi_\eta^2$ denotes RV with above PDF

- two important facts: $E\{\chi_\eta^2\} = \eta$ and $\operatorname{var}\{\chi_\eta^2\} = 2\eta$

- let $Q_\eta(p)$ denote the $p$th percentage point for the RV $\chi_\eta^2$:
$$\mathbf{P}[\chi_\eta^2 \leq Q_\eta(p)] = p$$

## Expected Value of Wavelet Coefficients

- in preparation for considering problem of estimating $\nu_X^2(\tau_j)$ given an observed time series, need to consider $E\{\overline{W}_{j,t}\}$

- if $\{X_t\}$ is nonstationary but has $d$th order stationary increments, let $\{Y_t\}$ bestationary process obtained by differencing $\{X_t\}$ $d$ times; if $\{X_t\}$ is stationary ($d = 0$ case), let $Y_t = X_t$

- with $\mu_Y \equiv E\{Y_t\}$, have
  - $E\{\overline{W}_{j,t}\} = 0$ if either (i) $L > 2d$ or (ii) $L = 2d$ and $\mu_Y = 0$
  - $E\{\overline{W}_{j,t}\} \neq 0$ if $\mu_Y \neq 0$ and $L = 2d$

- thus have $E\{\overline{W}_{j,t}\} = 0$ if $L$ is picked large enough ($L > 2d$ is sufficient, but might not be necessary)

- knowing $E\{\overline{W}_{j,t}\} = 0$ eases job of estimating $\nu_X^2(\tau_j)$ considerably

## Unbiased Estimator of Wavelet Variance: I

- given a realization of $X_0, X_1, \ldots, X_{N-1}$ from a process with $d$th order stationary differences, want to estimate $\nu_X^2(\tau_j)$

- for wavelet filter such that $L \geq 2d$ and $E\{\overline{W}_{j,t}\} = 0$, have
$$\nu_X^2(\tau_j) = \operatorname{var}\{\overline{W}_{j,t}\} = E\{\overline{W}_{j,t}^2\}$$

- can base estimator on squares of
$$\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \ldots, N-1$$

- recall that
$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \qquad t \in \mathbb{Z}$$

## Unbiased Estimator of Wavelet Variance: II

- comparing
$$\widetilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N} \quad \text{with} \quad \overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}$$
says that $\widetilde{W}_{j,t} = \overline{W}_{j,t}$ if 'mod $N$' not needed; this happens when $L_j - 1 \leq t < N$ (recall that $L_j = (2^j - 1)(L - 1) + 1$)

- if $N - L_j \geq 0$, unbiased estimator of $\nu_X^2(\tau_j)$ is
$$\hat{\nu}_X^2(\tau_j) \equiv \frac{1}{N - L_j + 1} \sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2 = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \overline{W}_{j,t}^2,$$
where $M_j \equiv N - L_j + 1$

## Statistical Properties of $\hat{\nu}_X^2(\tau_j)$

- assume that $\{\overline{W}_{j,t}\}$ is Gaussian stationary process with mean zero and ACVS $\{s_{j,\tau}\}$

- suppose $\{s_{j,\tau}\}$ is such that

$$A_j \equiv \sum_{\tau=-\infty}^{\infty} s_{j,\tau}^2 < \infty$$

(if $A_j = \infty$, can make it finite usually by just increasing $L$)

- can show that $\hat{\nu}_X^2(\tau_j)$ is asymptotically Gaussian with mean $\nu_X^2(\tau_j)$ and large sample variance $2A_j/M_j$; i.e.,

$$\frac{\hat{\nu}_X^2(\tau_j) - \nu_X^2(\tau_j)}{(2A_j/M_j)^{1/2}} = \frac{M_j^{1/2}(\hat{\nu}_X^2(\tau_j) - \nu_X^2(\tau_j))}{(2A_j)^{1/2}} \overset{\mathrm{d}}{=} \mathcal{N}(0,1)$$

approximately for large $M_j \equiv N - L_j + 1$

## Estimation of $A_j$

- in practical applications, need to estimate $A_j = \sum_\tau s_{j,\tau}^2$

- can argue that, for large $M_j$, the estimator

$$\hat{A}_j \equiv \frac{\left(\hat{s}_{j,0}^{(p)}\right)^2}{2} + \sum_{\tau=1}^{M_j-1} \left(\hat{s}_{j,\tau}^{(p)}\right)^2,$$

is approximately unbiased, where

$$\hat{s}_{j,\tau}^{(p)} \equiv \frac{1}{M_j} \sum_{t=L_j-1}^{N-1-|\tau|} \widetilde{W}_{j,t}\widetilde{W}_{j,t+|\tau|}, \quad 0 \le |\tau| \le M_j - 1$$

- Monte Carlo results: $\hat{A}_j$ reasonably good for $M_j \ge 128$

## Confidence Intervals for $\nu_X^2(\tau_j)$: I

- based upon large sample theory, can form a $100(1-2p)\%$ confidence interval (CI) for $\nu_X^2(\tau_j)$:

$$\left[\hat{\nu}_X^2(\tau_j) - \Phi^{-1}(1-p)\frac{\sqrt{2A_j}}{\sqrt{M_j}}, \hat{\nu}_X^2(\tau_j) + \Phi^{-1}(1-p)\frac{\sqrt{2A_j}}{\sqrt{M_j}}\right];$$

i.e., random interval traps unknown $\nu_X^2(\tau_j)$ with probability $1-2p$

- if $A_j$ replaced by $\hat{A}_j$, get approximate $100(1-2p)\%$ CI

- critique: lower limit of CI can very well be negative even though $\nu_X^2(\tau_j) \ge 0$ always

- can avoid this problem by using a $\chi^2$ approximation

## Confidence Intervals for $\nu_X^2(\tau_j)$: II

- $\chi_\eta^2$ useful for approximating distribution of sum of squared Gaussian RVs, which is what we are dealing with here:

$$\hat{\nu}_X^2(\tau_j) = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \overline{W}_{j,t}^2$$

- idea is to assume $\hat{\nu}_X^2(\tau_j) \overset{\mathrm{d}}{=} a\chi_\eta^2$, where $a$ and $\eta$ are constants to be set via moment matching

- because $E\{\chi_\eta^2\} = \eta$ and $\mathrm{var}\{\chi_\eta^2\} = 2\eta$, we have $E\{a\chi_\eta^2\} = a\eta$ and $\mathrm{var}\{a\chi_\eta^2\} = 2a^2\eta$

- can equate $E\{\hat{\nu}_X^2(\tau_j)\}$ & $\mathrm{var}\{\hat{\nu}_X^2(\tau_j)\}$ to $a\eta$ & $2a^2\eta$ to determine $a$ & $\eta$

## Confidence Intervals for $\nu_X^2(\tau_j)$: III

- obtain

$$\eta = \frac{2\left(E\{\hat{\nu}_X^2(\tau_j)\}\right)^2}{\operatorname{var}\{\hat{\nu}_X^2(\tau_j)\}} = \frac{2\nu_X^4(\tau_j)}{\operatorname{var}\{\hat{\nu}_X^2(\tau_j)\}} \text{ and } a = \frac{\nu_X^2(\tau_j)}{\nu}$$

- after $\eta$ has been determined, can obtain a CI for $\nu_X^2(\tau_j)$: with probability $1 - 2p$, the random interval

$$\left[\frac{\eta\hat{\nu}_X^2(\tau_j)}{Q_\eta(1-p)}, \frac{\eta\hat{\nu}_X^2(\tau_j)}{Q_\eta(p)}\right]$$

traps the true unknown $\nu_X^2(\tau_j)$

- lower limit is now nonnegative
- as $N \to \infty$, above CI and Gaussian-based CI converge

## Three Ways to Set $\eta$

1. use large sample theory with appropriate estimates:

$$\eta = \frac{2\nu_X^4(\tau_j)}{\operatorname{var}\{\hat{\nu}_X^2(\tau_j)\}} \approx \frac{2\nu_X^4(\tau_j)}{2A_j/M_j} \text{ suggests } \hat{\eta}_1 = \frac{M_j\hat{\nu}_X^4(\tau_j)}{\hat{A}_j}$$

2. make an assumption about the effect of wavelet filter on $\{X_t\}$ to obtain simple approximation

$$\eta_3 = \max\{M_j/2^j, 1\}$$

(this effective – but conservative – approach is valuable if there are insufficient data to reliably estimate $A_j$)

3. third way requires assuming shape of spectral density function associated with $\{X_t\}$ (questionable assumption, but common practice in, e.g., atomic clock literature)
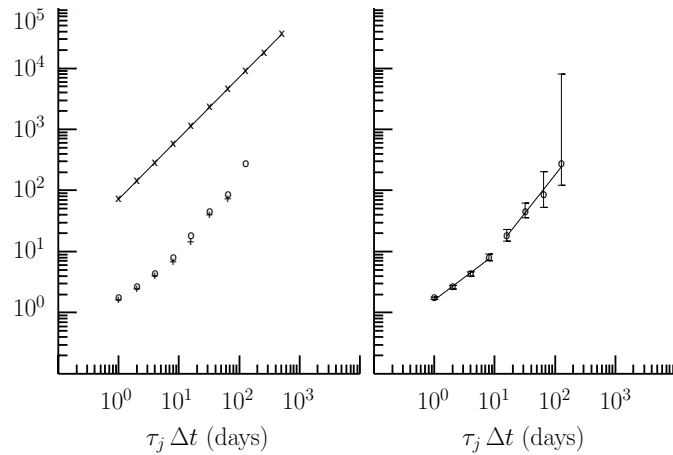
## Atomic Clock Deviates: I
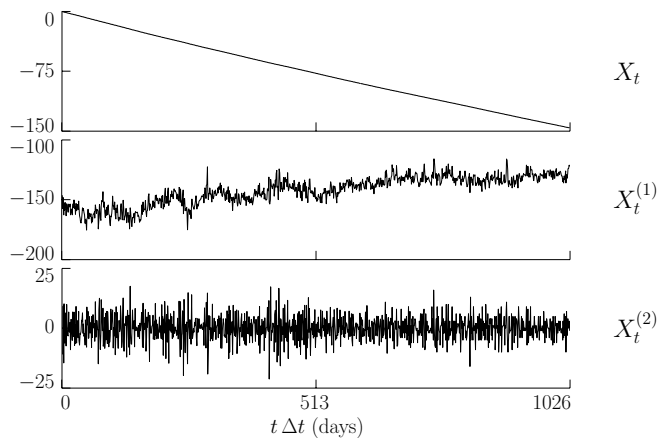
## Atomic Clock Deviates: II

- top plot: errors $\{X_t\}$ in time kept by atomic clock 571 (measured in microseconds: 1,000,000 microseconds = 1 second)

- middle: 1st backward differences $\{X_t^{(1)}\}$ in nanoseconds (1000 nanoseconds = 1 microsecond)

- bottom: 2nd backward differences $\{X_t^{(2)}\}$, also in nanoseconds

- if $\{X_t\}$ nonstationary with $d$th order stationary increments, need $L \geq 2d$, but might need $L > 2d$ to get $E\{\overline{W}_{j,t}\} = 0$

- might regard $\{X_t^{(1)}\}$ as realization of stationary process, but, if so, with a mean value far from 0; $\{X_t^{(2)}\}$ resembles realization of stationary process, but mean value still might not be 0 if we believe there is a linear trend in $\{X_t^{(1)}\}$; thus might need $L \geq 6$, but could get away with $L \geq 4$

## Atomic Clock Deviates: III



$\tau_j \, \Delta t$ (days)       $\tau_j \, \Delta t$ (days)

## Atomic Clock Deviates: IV

- square roots of wavelet variance estimates for atomic clock time errors $\{X_t\}$ based upon unbiased MODWT estimator with
  - Haar wavelet (**x**'s in left-hand plot, with linear fit)
  - D(4) wavelet (circles in left- and right-hand plots)
  - D(6) wavelet (pluses in left-hand plot).
- Haar wavelet inappropriate
  - need $\{X_t^{(1)}\}$ to be a realization of a stationary process with mean 0 (stationarity might be OK, but mean 0 is way off)
  - linear appearance can be explained in terms of nonzero mean
- 95% confidence intervals in the right-hand plot are the square roots of intervals computed using the chi-square approximation with $\eta$ given by $\hat{\eta}_1$ for $j = 1, \ldots, 6$ and by $\eta_3$ for $j = 7 \,\&\, 8$

## Annual Minima of Nile River



years      scale (years)
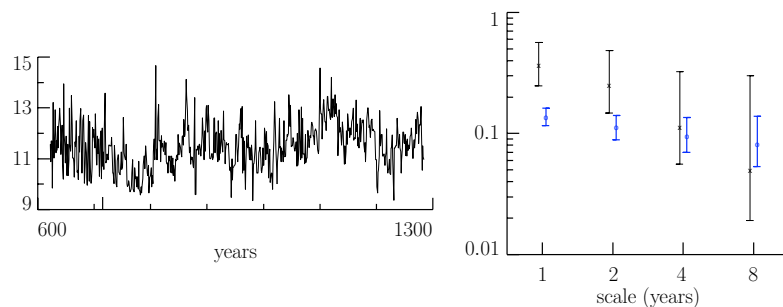
- left-hand plot: annual minima of Nile River
- right: Haar $\hat{\nu}_X^2(\tau_j)$ before (**x**'s) and after (**o**'s) year 715.5, with 95% confidence intervals based upon $\chi^2_{\eta_3}$ approximation

## Wavelet Variance Analysis of Time Series with Time-Varying Statistical Properties

- each wavelet coefficient $\widetilde{W}_{j,t}$ formed using portion of $X_t$
- suppose $X_t$ associated with actual time $t_0 + t \, \Delta t$
  - $*$ $t_0$ is actual time of first observation $X_0$
  - $*$ $\Delta t$ is spacing between adjacent observations
- suppose $\tilde{h}_{j,l}$ is least asymmetric Daubechies wavelet
- can associate $\widetilde{W}_{j,t}$ with an interval of width $2\tau_j \, \Delta t$ centered at
$$t_0 + (2^j(t+1) - 1 - |\nu_j^{(H)}| \bmod N) \, \Delta t,$$
where, e.g., $|\nu_j^{(H)}| = [7(2^j - 1) + 1]/2$ for LA(8) wavelet
- can thus form 'localized' wavelet variance analysis (implicitly assumes stationarity or stationary increments locally)

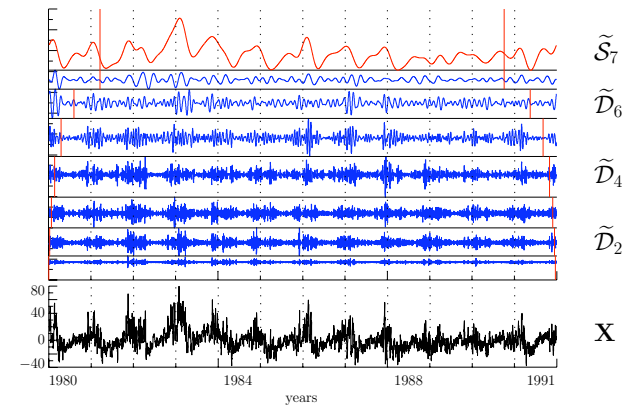## Subtidal Sea Level Fluctuations: I
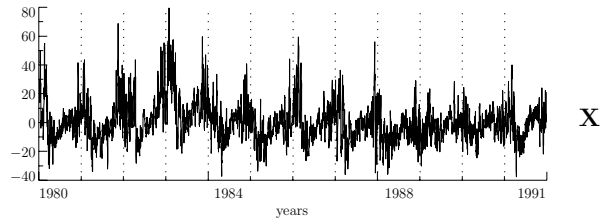


- subtidal sea level fluctuations **X** for Crescent City, CA, collected by National Ocean Service with permanent tidal gauge
- $N = 8746$ values from Jan 1980 to Dec 1991 (almost 12 years)
- one value every 12 hours, so $\Delta t = 1/2$ day
- 'subtidal' is what remains after diurnal & semidiurnal tides are removed by low-pass filter (filter seriously distorts frequency band corresponding to first physical scale $\tau_1 \Delta t = 1/2$ day)

## Subtidal Sea Level Fluctuations: II



- level $J_0 = 7$ LA(8) MODWT multiresolution analysis

## Subtidal Sea Level Fluctuations: III
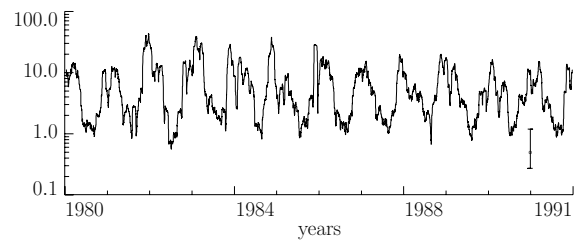


- estimated time-dependent LA(8) wavelet variances for physical scale $\tau_2 \Delta t = 1$ day based upon averages over monthly blocks (30.5 days, i.e., 61 data points)
- plot also shows a representative 95% confidence interval based upon a hypothetical wavelet variance estimate of $1/2$ and a chi-square distribution with $\nu = 15.25$

## Subtidal Sea Level Fluctuations: IV
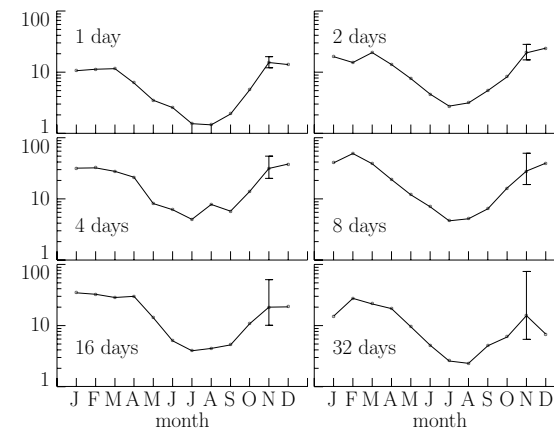


- estimated LA(8) wavelet variances for physical scales $\tau_j \Delta t = 2^{j-2}$ days, $j = 2, \ldots, 7$, grouped by calendar month

## Some Extensions

- wavelet cross-covariance and cross-correlation (Whitcher, Guttorp and Percival, 2000; Serroukh and Walden, 2000a, 2000b)
- asymptotic theory for non-Gaussian processes satisfying a certain 'mixing' condition (Serroukh, Walden and Percival, 2000)
- biased estimators of wavelet variance (Aldrich, 2005)
- unbiased estimator of wavelet variance for 'gappy' time series (Mondal and Percival, 2010a)
- robust estimation (Mondal and Percival, 2010b)
- wavelet variance for random fields (Mondal and Percival, 2010c)
- wavelet-based characteristic scales (Keim and Percival, 2010)

## Summary

- wavelet variance gives scale-based analysis of variance
- presented statistical theory for Gaussian processes with stationary increments
- in addition to the applications we have considered, the wavelet variance has been used to analyze
  - genome sequences
  - changes in variance of soil properties
  - canopy gaps in forests
  - accumulation of snow fields in polar regions
  - boundary layer atmospheric turbulence
  - regular and semiregular variable stars

## Wavelet-Based Signal Extraction: Overview

- outline key ideas behind wavelet-based approach
- description of four basic models for signal estimation
- discussion of why wavelets can help estimate certain signals
- simple thresholding & shrinkage schemes for signal estimation
- wavelet-based thresholding and shrinkage
- discuss some extensions to basic approach

## Wavelet-Based Signal Estimation: I

- DWT analysis of $\mathbf{X}$ yields $\mathbf{W} = \mathcal{W}\mathbf{X}$
- DWT synthesis $\mathbf{X} = \mathcal{W}^T\mathbf{W}$ yields multiresolution analysis by splitting $\mathcal{W}^T\mathbf{W}$ into pieces associated with different scales
- DWT synthesis can also estimate 'signal' hidden in $\mathbf{X}$ if we can modify $\mathbf{W}$ to get rid of noise in the wavelet domain
- if $\mathbf{W}'$ is a 'noise reduced' version of $\mathbf{W}$, can form signal estimate via $\mathcal{W}^T\mathbf{W}'$

## Wavelet-Based Signal Estimation: II

- key ideas behind simple wavelet-based signal estimation
  - certain signals can be efficiently described by the DWT using
    * all of the scaling coefficients
    * a small number of 'large' wavelet coefficients
  - noise is manifested in a large number of 'small' wavelet coefficients
  - can either 'threshold' or 'shrink' wavelet coefficients to eliminate noise in the wavelet domain
- key ideas led to wavelet thresholding and shrinkage proposed by Donoho, Johnstone and coworkers in 1990s

## Models for Signal Estimation: I

- will consider two types of signals:
  1. $\mathbf{D}$, an $N$ dimensional deterministic signal
  2. $\mathbf{C}$, an $N$ dimensional stochastic signal; i.e., a vector of random variables (RVs) with covariance matrix $\Sigma_{\mathbf{C}}$
- will consider two types of noise:
  1. $\boldsymbol{\epsilon}$, an $N$ dimensional vector of independent and identically distributed (IID) RVs with mean 0 and covariance matrix $\Sigma_{\boldsymbol{\epsilon}} = \sigma_{\epsilon}^2 I_N$
  2. $\boldsymbol{\eta}$, an $N$ dimensional vector of non-IID RVs with mean 0 and covariance matrix $\Sigma_{\boldsymbol{\eta}}$
     * one form: RVs independent, but have different variances
     * another form of non-IID: RVs are correlated

## Models for Signal Estimation: II

- leads to four basic 'signal + noise' models for $\mathbf{X}$
  1. $\mathbf{X} = \mathbf{D} + \boldsymbol{\epsilon}$
  2. $\mathbf{X} = \mathbf{D} + \boldsymbol{\eta}$
  3. $\mathbf{X} = \mathbf{C} + \boldsymbol{\epsilon}$
  4. $\mathbf{X} = \mathbf{C} + \boldsymbol{\eta}$
- in the latter two cases, the stochastic signal $\mathbf{C}$ is assumed to be independent of the associated noise

## Signal Representation via Wavelets: I

- consider $\mathbf{X} = \mathbf{D} + \boldsymbol{\epsilon}$ first
- signal estimation problem is simplified if we can assume that the important part of $\mathbf{D}$ is in its large values
- assumption is not usually viable in the original (i.e., time domain) representation $\mathbf{D}$, but might be true in another domain
- an orthonormal transform $\mathcal{O}$ might be useful because
  - $\mathbf{d} = \mathcal{O}\mathbf{D}$ is equivalent to $\mathbf{D}$ (since $\mathbf{D} = \mathcal{O}^T\mathbf{d}$)
  - we might be able to find $\mathcal{O}$ such that the signal is isolated in $M \ll N$ large transform coefficients
- Q: how can we judge whether a particular $\mathcal{O}$ might be useful for representing $\mathbf{D}$?

## Signal Representation via Wavelets: II

- let $d_j$ be the $j$th transform coefficient in $\mathbf{d} = \mathcal{O}\mathbf{D}$

- let $d_{(0)}, d_{(1)}, \ldots, d_{(N-1)}$ be the $d_j$'s reordered by magnitude:
$$|d_{(0)}| \geq |d_{(1)}| \geq \cdots \geq |d_{(N-1)}|$$

- example: if $\mathbf{d} = [-3, 1, 4, -7, 2, -1]^T$, then
$d_{(0)} = d_3 = -7$, $d_{(1)} = d_2 = 4$, $d_{(2)} = d_0 = -3$ etc.

- define a normalized partial energy sequence (NPES):
$$C_{M-1} \equiv \frac{\sum_{j=0}^{M-1} |d_{(j)}|^2}{\sum_{j=0}^{N-1} |d_{(j)}|^2} = \frac{\text{energy in largest } M \text{ terms}}{\text{total energy in signal}}$$

- let $\mathcal{I}_M$ be $N \times N$ diagonal matrix whose $j$th diagonal term is 1 if $|d_j|$ is one of the $M$ largest magnitudes and is 0 otherwise

## Signal Representation via Wavelets: III

- form $\widehat{\mathbf{D}}_M \equiv \mathcal{O}^T \mathcal{I}_M \mathbf{d}$, which is an approximation to $\mathbf{D}$

- when $\mathbf{d} = [-3, 1, 4, -7, 2, -1]^T$ and $M = 3$, we have
$$\mathcal{I}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and thus } \widehat{\mathbf{D}}_M = \mathcal{O}^T \begin{bmatrix} -3 \\ 0 \\ 4 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

- one interpretation for NPES:
$$C_{M-1} = 1 - \frac{\|\mathbf{D} - \widehat{\mathbf{D}}_M\|^2}{\|\mathbf{D}\|^2} = 1 - \text{relative approximation error}$$

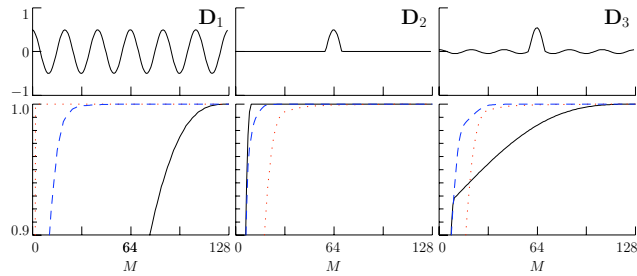## Signal Representation via Wavelets: IV
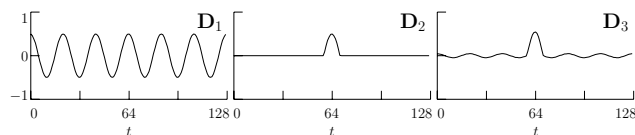


- consider three signals plotted above

- $\mathbf{D}_1$ is a sinusoid, which can be represented succinctly by the discrete Fourier transform (DFT)

- $\mathbf{D}_2$ is a bump (only a few nonzero values in the time domain)

- $\mathbf{D}_3$ is a linear combination of $\mathbf{D}_1$ and $\mathbf{D}_2$

## Signal Representation via Wavelets: V

- consider three different orthonormal transforms
  - identity transform $I$ (time)
  - the orthonormal DFT $\mathcal{F}$ (frequency), where $\mathcal{F}$ has $(k, t)$th element $\exp(-i2\pi tk/N)/\sqrt{N}$ for $0 \leq k, t \leq N - 1$
  - the LA(8) DWT $\mathcal{W}$ (wavelet)
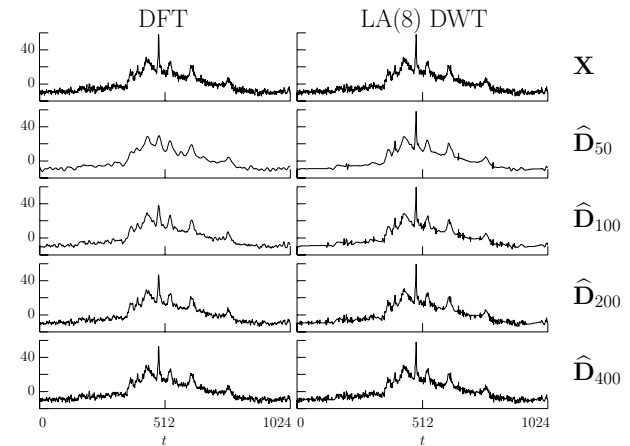
- # of terms $M$ needed to achieve relative error $< 1\%$:

| | $\mathbf{D}_1$ | $\mathbf{D}_2$ | $\mathbf{D}_3$ |
|---|---|---|---|
| DFT | 2 | 29 | 28 |
| identity | 105 | 9 | 75 |
| LA(8) wavelet | 22 | 14 | 21 |

## Signal Representation via Wavelets: VI



- use NPESs to see how well these three signals are represented in the time, frequency (DFT) and wavelet (LA(8)) domains

- time (solid curves), frequency (dotted) and wavelet (dashed)

## Signal Representation via Wavelets: IX



- example: DFT $\widehat{\mathbf{D}}_M$ (left-hand column) & $J_0 = 6$ LA(8) DWT $\widehat{\mathbf{D}}_M$ (right) for NMR series $\mathbf{X}$ (A. Maudsley, UCSF)

## Signal Estimation via Thresholding: I

- thresholding schemes involve

  1. computing $\mathbf{O} \equiv \mathcal{O}\mathbf{X}$
  2. defining $\mathbf{O}^{(t)}$ as vector with $l$th element

$$O_l^{(t)} = \begin{cases} 0, & \text{if } |O_l| \leq \delta; \\ \text{some nonzero value,} & \text{otherwise,} \end{cases}$$

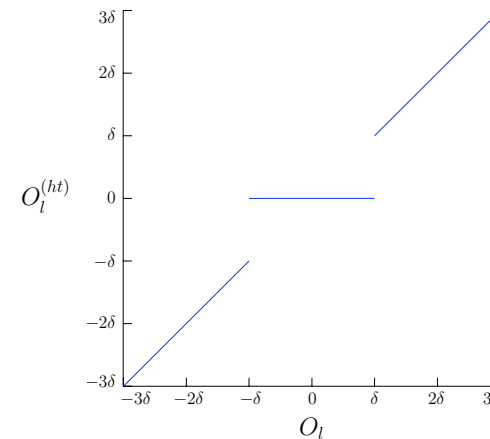  where nonzero values are yet to be defined
  3. estimating $\mathbf{D}$ via $\widehat{\mathbf{D}}^{(t)} \equiv \mathcal{O}^T \mathbf{O}^{(t)}$

- simplest scheme is 'hard thresholding' ('kill/keep' strategy):

$$O_l^{(ht)} = \begin{cases} 0, & \text{if } |O_l| \leq \delta; \\ O_l, & \text{otherwise.} \end{cases}$$

## Hard Thresholding Function

- plot shows mapping from $O_l$ to $O_l^{(ht)}$

## Signal Estimation via Thresholding: II

- hard thresholding is strategy that arises from solution to simple optimization problem, namely, find $\widehat{\mathbf{D}}_M$ such that

$$\gamma_m \equiv \|\mathbf{X} - \widehat{\mathbf{D}}_m\|^2 + m\delta^2$$

  is minimized over all possible $\widehat{\mathbf{D}}_m = \mathcal{O}^T \mathcal{I}_m \mathbf{O}$, $m = 0, \ldots, N$
- $\delta$ is a fixed parameter that is set *a priori* (we assume $\delta > 0$)
- $\|\mathbf{X} - \widehat{\mathbf{D}}_m\|^2$ is a measure of 'fidelity'
  - rationale for this term: $\widehat{\mathbf{D}}_m$ shouldn't stray too far from $\mathbf{X}$ (particularly if signal-to-noise ratio is high)
  - fidelity increases (the measure decreases) as $m$ increases
  - in minimizing $\gamma_m$, consideration of this term alone suggests that $m$ should be large

## Signal Estimation via Thresholding: III

- $m\delta^2$ is a penalty for too many terms
  - rationale: heuristic says $\mathbf{d} = \mathcal{O}\mathbf{D}$ consists of just a few large coefficients
  - penalty increases as $m$ increases
  - in minimizing $\gamma_m$, consideration of this term alone suggests that $m$ should be small
- optimization problem: balance off fidelity & parsimony
- can show that $\gamma_m = \|\mathbf{X} - \widehat{\mathbf{D}}_m\|^2 + m\delta^2$ is minimized when $m$ is set such that $\mathcal{I}_m$ picks out all coefficients satisfying $O_j^2 > \delta^2$

## Signal Estimation via Thresholding: IV

- alternative scheme is 'soft thresholding:'

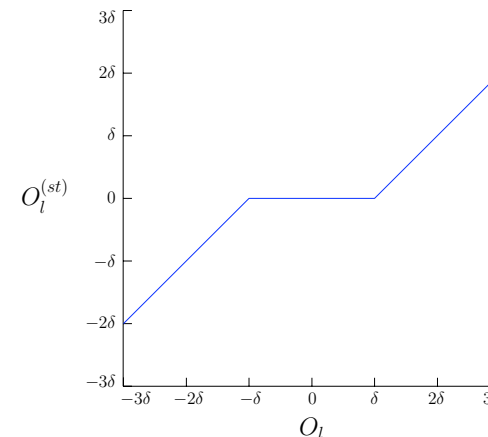$$O_l^{(st)} = \text{sign}\,\{O_l\}\,(|O_l| - \delta)_+ ,$$

  where

$$\text{sign}\,\{O_l\} \equiv \begin{cases} +1, & \text{if } O_l > 0; \\ 0, & \text{if } O_l = 0; \\ -1, & \text{if } O_l < 0. \end{cases} \quad \text{and} \quad (x)_+ \equiv \begin{cases} x, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

- one rationale for soft thresholding: fits into Stein's class of estimators, for which unbiased estimation of risk is possible

## Soft Thresholding Function

- here is the mapping from $O_l$ to $O_l^{(st)}$

## Signal Estimation via Thresholding: V

- third scheme is 'mid thresholding:'

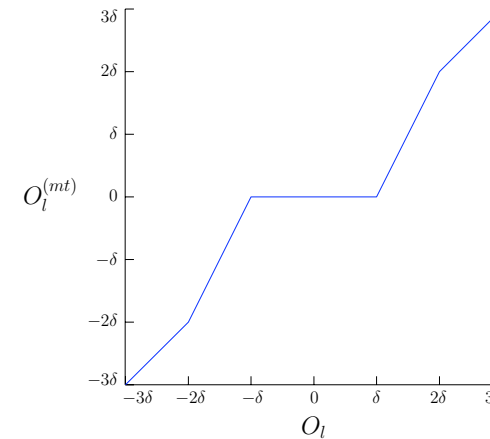$$O_l^{(mt)} = \text{sign}\{O_l\}\,(|O_l| - \delta)_{++}\,,$$

where

$$(|O_l| - \delta)_{++} \equiv \begin{cases} 2(|O_l| - \delta)_+, & \text{if } |O_l| < 2\delta; \\ |O_l|, & \text{otherwise} \end{cases}$$

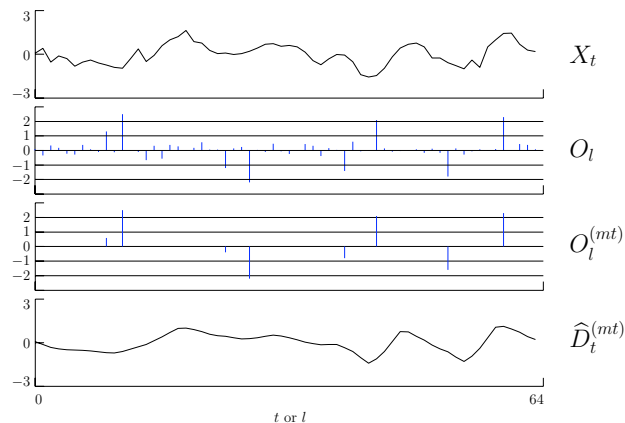- provides compromise between hard and soft thresholding

## Mid Thresholding Function

- here is the mapping from $O_l$ to $O_l^{(mt)}$

## Signal Estimation via Thresholding: VI

- example of mid thresholding with $\delta = 1$

## Universal Threshold

- Q: how do we go about setting $\delta$?
- specialize to IID Gaussian noise $\boldsymbol{\epsilon}$ with covariance $\sigma_\epsilon^2 I_N$
- can argue $\mathbf{e} \equiv \mathcal{O}\boldsymbol{\epsilon}$ is also IID Gaussian with covariance $\sigma_\epsilon^2 I_N$
- Donoho & Johnstone (1995) proposed $\delta^{(u)} \equiv \sqrt{[2\sigma_\epsilon^2 \log(N)]}$ ('log' here is 'log base $e$')
- rationale for $\delta^{(u)}$: because of Gaussianity, can argue that

$$\mathbf{P}\big[\max_l\{|e_l|\} > \delta^{(u)}\big] \leq \frac{1}{\sqrt{[4\pi \log(N)]}} \to 0 \text{ as } N \to \infty$$

and hence $\mathbf{P}\big[\max_l\{|e_l|\} \leq \delta^{(u)}\big] \to 1$ as $N \to \infty$, so no noise will exceed threshold in the limit

## Wavelet-Based Thresholding

- assume model of deterministic signal plus IID Gaussian noise with mean 0 and variance $\sigma_\epsilon^2$: $\mathbf{X} = \mathbf{D} + \boldsymbol{\epsilon}$
- using a DWT matrix $\mathcal{W}$, form $\mathbf{W} = \mathcal{W}\mathbf{X} = \mathcal{W}\mathbf{D} + \mathcal{W}\boldsymbol{\epsilon} \equiv \mathbf{d} + \mathbf{e}$
- because $\boldsymbol{\epsilon}$ IID Gaussian, so is $\mathbf{e}$
- Donoho & Johnstone (1994) advocate the following:
  - form partial DWT of level $J_0$: $\mathbf{W}_1, \ldots, \mathbf{W}_{J_0}$ and $\mathbf{V}_{J_0}$
  - threshold $\mathbf{W}_j$'s but leave $\mathbf{V}_{J_0}$ alone (i.e., administratively, all $N/2^{J_0}$ scaling coefficients assumed to be part of $\mathbf{d}$)
  - use universal threshold $\delta^{(u)} = \sqrt{[2\sigma_\epsilon^2 \log(N)]}$
  - use thresholding rule to form $\mathbf{W}_j^{(t)}$ (hard, etc.)
  - estimate $\mathbf{D}$ by inverse transforming $\mathbf{W}_1^{(t)}, \ldots, \mathbf{W}_{J_0}^{(t)}$ and $\mathbf{V}_{J_0}$

## MAD Scale Estimator: I

- procedure assumes $\sigma_\epsilon$ is know, which is not usually the case
- if unknown, use median absolute deviation (MAD) scale estimator to estimate $\sigma_\epsilon$ using $\mathbf{W}_1$

$$\hat{\sigma}_{(\text{mad})} \equiv \frac{\text{median}\left\{|W_{1,0}|, |W_{1,1}|, \ldots, |W_{1,\frac{N}{2}-1}|\right\}}{0.6745}$$

  - heuristic: bulk of $W_{1,t}$'s should be due to noise
  - '0.6745' yields estimator such that $E\{\hat{\sigma}_{(\text{mad})}\} = \sigma_\epsilon$ when $W_{1,t}$'s are IID Gaussian with mean 0 and variance $\sigma_\epsilon^2$
  - designed to be robust against large $W_{1,t}$'s due to signal

## MAD Scale Estimator: II

- example: suppose $\mathbf{W}_1$ has 7 small 'noise' coefficients & 2 large 'signal' coefficients (say, $a$ & $b$, with $2 \ll |a| < |b|$):

$$\mathbf{W}_1 = [1.23, -1.72, -0.80, -0.01, a, 0.30, 0.67, b, -1.33]^T$$

- ordering these by their magnitudes yields

$$0.01, 0.30, 0.67, 0.80, 1.23, 1.33, 1.72, |a|, |b|$$

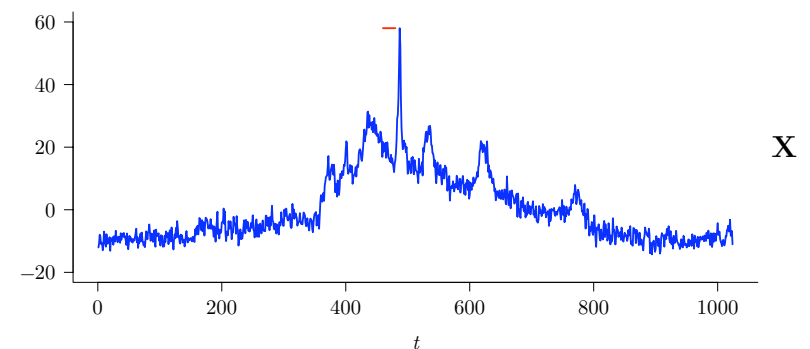- median of these absolute deviations is $1.23$, so

$$\hat{\sigma}_{(\text{mad})} = 1.23/0.6745 \doteq 1.82$$

- $\hat{\sigma}_{(\text{mad})}$ not influenced adversely by $a$ and $b$; i.e., scale estimate depends largely on the many small coefficients due to noise
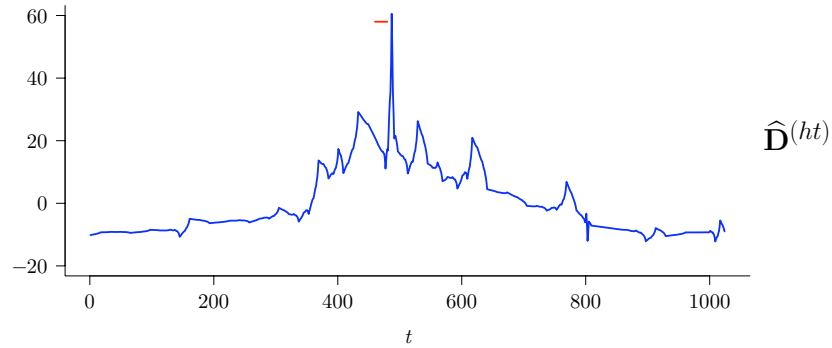
## Examples of DWT-Based Thresholding: I



**X**

- NMR spectrum

## Examples of DWT-Based Thresholding: II
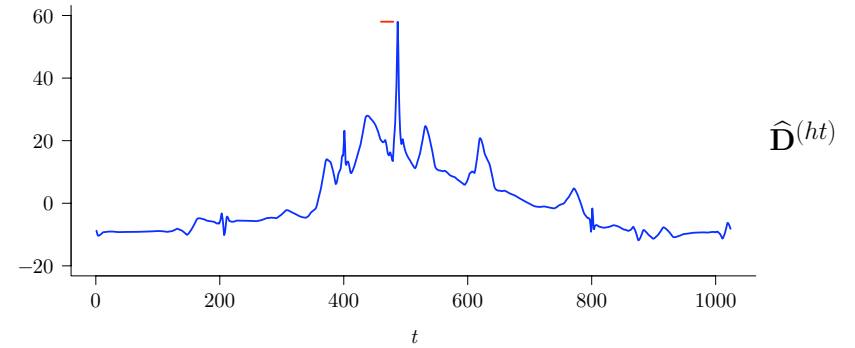


$\widehat{\mathbf{D}}^{(ht)}$

- signal estimate using $J_0 = 6$ partial D(4) DWT with hard thresholding and universal threshold level estimated by $\hat{\delta}^{(u)} = \sqrt{[2\hat{\sigma}^2_{(\mathrm{mad})}\log(N)]} \doteq 6.49$
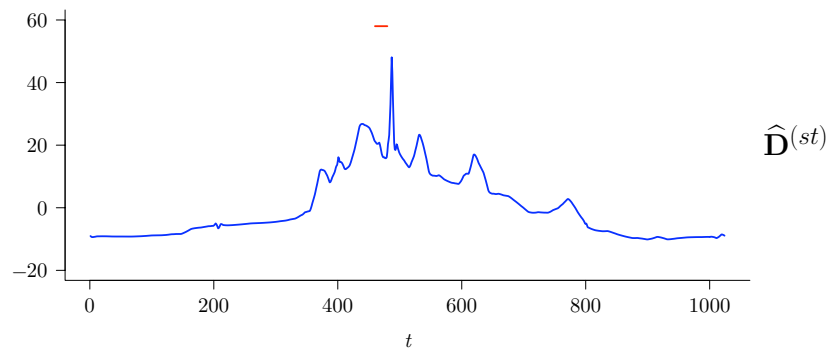
## Examples of DWT-Based Thresholding: III



$\widehat{\mathbf{D}}^{(ht)}$

- same as before, but now using LA(8) DWT with $\hat{\delta}^{(u)} \doteq 6.13$
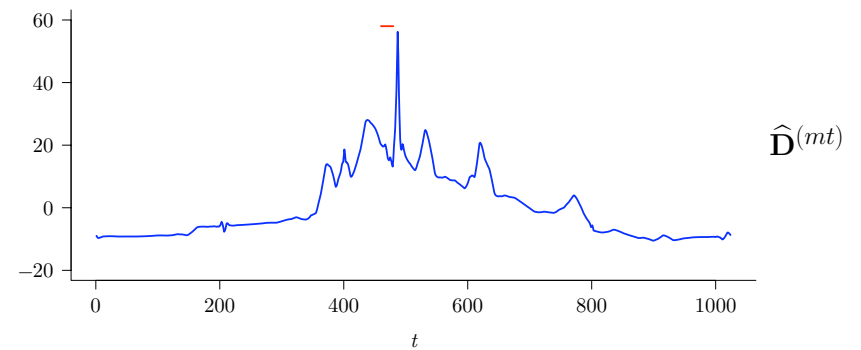
## Examples of DWT-Based Thresholding: IV



$\widehat{\mathbf{D}}^{(st)}$

- signal estimate using $J_0 = 6$ partial LA(8) DWT, but now with soft thresholding

## Examples of DWT-Based Thresholding: V



$\widehat{\mathbf{D}}^{(mt)}$

- signal estimate using $J_0 = 6$ partial LA(8) DWT, but now with mid thresholding
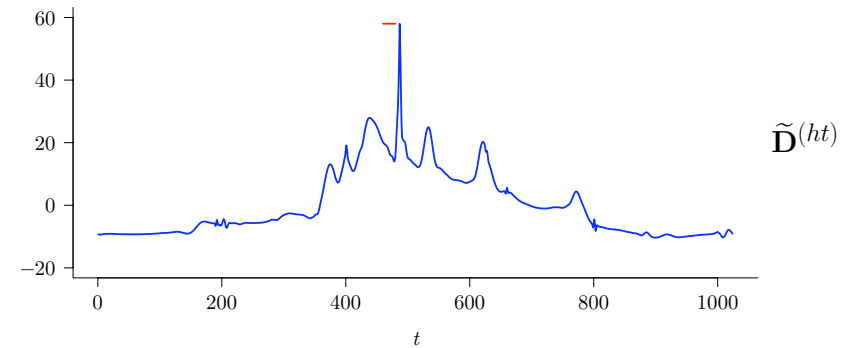
## MODWT-Based Thresholding

- can base thresholding procedure on MODWT rather than DWT, yielding signal estimators $\widetilde{\mathbf{D}}^{(ht)}$, $\widetilde{\mathbf{D}}^{(st)}$ and $\widetilde{\mathbf{D}}^{(mt)}$

- because MODWT filters are normalized differently, universal threshold must be adjusted for each level:

$$\tilde{\delta}_j^{(u)} \equiv \sqrt{[2\tilde{\sigma}_{(\text{mad})}^2 \log{(N)}/2^j]},$$

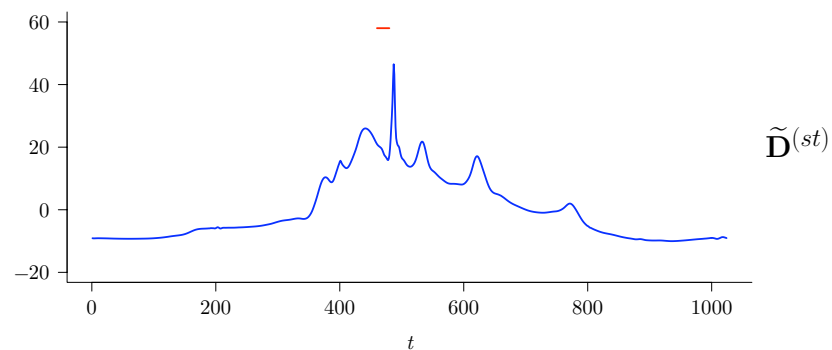where now MAD scale estimator is based on unit scale MODWT wavelet coefficients

- results are identical to what 'cycle spinning' would yield

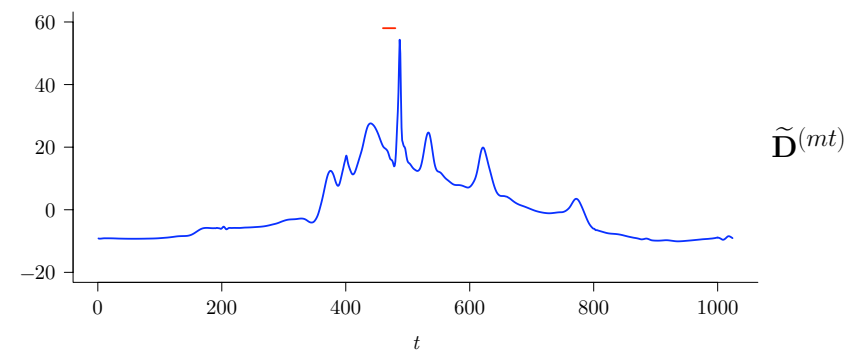## Examples of MODWT-Based Thresholding: I



$\widetilde{\mathbf{D}}^{(ht)}$

- signal estimate using $J_0 = 6$ LA(8) MODWT with hard thresholding

## Examples of MODWT-Based Thresholding: II



$\widetilde{\mathbf{D}}^{(st)}$

- same as before, but now with soft thresholding

## Examples of MODWT-Based Thresholding: III



$\widetilde{\mathbf{D}}^{(mt)}$

- same as before, but now with mid thresholding

## Signal Estimation via Shrinkage: I

- so far, we have only considered signal estimation via thresholding rules, which will map some $O_l$ to zeros

- will now consider shrinkage rules, which differ from thresholding only in that nonzero coefficients are mapped to nonzero values rather than exactly zero (but values can be *very* close to zero!)

- several ways in which shrinkage rules arise – will consider a conditional mean approach (identical to a Bayesian approach)

## Background on Conditional PDFs: I

- let $X$ and $Y$ be RVs with marginal probability density functions (PDFs) $f_X(\cdot)$ and $f_Y(\cdot)$

- let $f_{X,Y}(x, y)$ be their joint PDF at the point $(x, y)$

- conditional PDF of $Y$ given $X = x$ is defined as

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

- $f_{Y|X=x}(\cdot)$ is a PDF, so its mean value is

$$E\{Y|X = x\} = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)\, dy;$$

the above is called the conditional mean of $Y$, given $X$

## Background on Conditional PDFs: II

- suppose RVs $X$ and $Y$ are related, but we can only observe $X$

- want to approximate unobservable $Y$ based on function of $X$

- example: $X$ represents a stochastic signal $Y$ buried in noise

- suppose we want our approximation to be the function of $X$, say $U_2(X)$, such that the mean square difference between $Y$ and $U_2(X)$ is as small as possible; i.e., we want
$$E\{(Y - U_2(X))^2\}$$
to be as small as possible

- solution is to use $U_2(X) = E\{Y|X\}$; i.e., the conditional mean of $Y$ given $X$ is our best guess at $Y$ in the sense of minimizing the mean square error (related to fact that $E\{(Y - a)^2\}$ is smallest when $a = E\{Y\}$)

## Conditional Mean Approach: I

- assume model of stochastic signal plus non-IID noise:
  $\mathbf{X} = \mathbf{C} + \boldsymbol{\eta}$ so that $\mathbf{O} = \mathcal{O}\mathbf{X} = \mathcal{O}\mathbf{C} + \mathcal{O}\boldsymbol{\eta} \equiv \mathbf{R} + \mathbf{n}$

- component-wise, have $O_l = R_l + n_l$

- because $\mathbf{C}$ and $\boldsymbol{\eta}$ are independent, $\mathbf{R}$ and $\mathbf{n}$ must be also

- suppose we approximate $R_l$ via $\widehat{R}_l \equiv U_2(O_l)$, where $U_2(O_l)$ is selected to minimize $E\{(R_l - U_2(O_l))^2\}$

- solution is to set $U_2(O_l)$ equal to $E\{R_l|O_l\}$, so let's work out what form this conditional mean takes

- to get $E\{R_l|O_l\}$, need the PDF of $R_l$ given $O_l$, which is
$$f_{R_l|O_l=o_l}(r_l) = \frac{f_{R_l,O_l}(r_l, o_l)}{f_{O_l}(o_l)} = \frac{f_{R_l}(r_l)f_{n_l}(o_l - r_l)}{\int_{-\infty}^{\infty} f_{R_l}(r_l)f_{n_l}(o_l - r_l)\, dr_l}$$

## Conditional Mean Approach: II

- mean value of $f_{R_l|O_l=o_l}(\cdot)$ yields estimator $\widehat{R}_l = E\{R_l|O_l\}$:

$$E\{R_l|O_l = o_l\} = \int_{-\infty}^{\infty} r_l f_{R_l|O_l=o_l}(r_l)\,dr_l$$
$$= \frac{\int_{-\infty}^{\infty} r_l f_{R_l}(r_l) f_{n_l}(o_l - r_l) dr_l}{\int_{-\infty}^{\infty} f_{R_l}(r_l) f_{n_l}(o_l - r_l)\,dr_l}$$

- to make further progress, we need a model for the wavelet-domain representation $R_l$ of the signal

- heuristic that signal in the wavelet domain has a few large values and lots of small values suggests a Gaussian mixture model
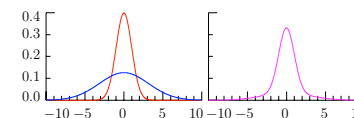
## Conditional Mean Approach: III

- let $\mathcal{I}_l$ be an RV such that $\mathbf{P}\,[\mathcal{I}_l = 1] = p_l$ & $\mathbf{P}\,[\mathcal{I}_l = 0] = 1 - p_l$
- under Gaussian mixture model, $R_l$ has same distribution as
$$\mathcal{I}_l \mathcal{N}(0, \gamma_l^2 \sigma_{G_l}^2) + (1 - \mathcal{I}_l)\mathcal{N}(0, \sigma_{G_l}^2)$$
  where $\mathcal{N}(0, \sigma^2)$ is a Gaussian RV with mean 0 and variance $\sigma^2$
  - 2nd component models small # of large signal coefficients
  - 1st component models large # of small coefficients ($\gamma_l^2 \ll 1$)
- example: PDFs for case $\sigma_{G_l}^2 = 10$, $\gamma_l^2 \sigma_{G_l}^2 = 1$ and $p_l = 0.75$

## Conditional Mean Approach: VI

- let's simplify to a 'sparse' signal model by setting $\gamma_l = 0$; i.e., large # of small coefficients are all zero

- distribution for $R_l$ same as $(1 - \mathcal{I}_l)\mathcal{N}(0, \sigma_{G_l}^2)$

- to complete model, let $n_l$ obey a Gaussian distribution with mean 0 and variance $\sigma_{n_l}^2$

- conditional mean estimator becomes $E\{R_l|O_l = o_l\} = \frac{b_l}{1+c_l} o_l$, where

$$c_l = \frac{p_l \sqrt{(\sigma_{G_l}^2 + \sigma_{n_l}^2)}}{(1 - p_l)\sigma_{n_l}} e^{-o_l^2 b_l/(2\sigma_{n_l}^2)}$$

## Conditional Mean Approach: VII



$E\{R_l|O_l = o_l\}$

$o_l$
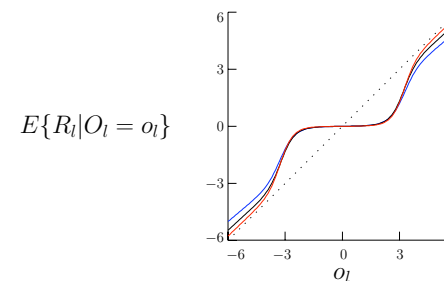
- conditional mean shrinkage rule for $p_l = 0.95$ (i.e., $\approx 95\%$ of signal coefficients are 0); $\sigma_{n_l}^2 = 1$; and $\sigma_{G_l}^2 = 5$ (curve furthest from dotted diagonal), 10 and 25 (curve nearest to diagonal)

- as $\sigma_{G_l}^2$ gets large (i.e., large signal coefficients increase in size), shrinkage rule starts to resemble mid thresholding rule

## Wavelet-Based Shrinkage: I

- assume model of stochastic signal plus Gaussian IID noise:
  $\mathbf{X} = \mathbf{C} + \boldsymbol{\epsilon}$ so that $\mathbf{W} = \mathcal{W}\mathbf{X} = \mathcal{W}\mathbf{C} + \mathcal{W}\boldsymbol{\epsilon} \equiv \mathbf{R} + \mathbf{e}$

- component-wise, have $W_{j,t} = R_{j,t} + e_{j,t}$

- form partial DWT of level $J_0$, shrink $\mathbf{W}_j$'s, but leave $\mathbf{V}_{J_0}$ alone

- assume $E\{R_{j,t}\} = 0$ (reasonable for $\mathbf{W}_j$, but not for $\mathbf{V}_{J_0}$)

- use a conditional mean approach with the sparse signal model
  - $R_{j,t}$ has distribution dictated by $(1 - \mathcal{I}_{j,t})\mathcal{N}(0, \sigma_G^2)$, where
  $$\mathbf{P}\left[\mathcal{I}_{j,t} = 1\right] = p \text{ and } \mathbf{P}\left[\mathcal{I}_{j,t} = 0\right] = 1 - p$$
  - $R_{j,t}$'s are assumed to be IID
  - model for $e_{j,t}$ is Gaussian with mean 0 and variance $\sigma_\epsilon^2$
  - note: parameters do not vary with $j$ or $t$

## Wavelet-Based Shrinkage: II

- model has three parameters $\sigma_G^2$, $p$ and $\sigma_\epsilon^2$, which need to be set

- let $\sigma_R^2$ and $\sigma_W^2$ be variances of RVs $R_{j,t}$ and $W_{j,t}$

- have relationships $\sigma_R^2 = (1 - p)\sigma_G^2$ and $\sigma_W^2 = \sigma_R^2 + \sigma_\epsilon^2$
  - set $\hat{\sigma}_\epsilon^2 = \hat{\sigma}_{(\text{mad})}^2$ using $\mathbf{W}_1$
  - let $\hat{\sigma}_W^2$ be sample mean of all $W_{j,t}^2$
  - given $p$, let $\hat{\sigma}_G^2 = (\hat{\sigma}_W^2 - \hat{\sigma}_\epsilon^2)/(1 - p)$
  - $p$ usually chosen subjectively, keeping in mind that $p$ is proportion of noise-dominated coefficients (can set based on rough estimate of proportion of 'small' coefficients)
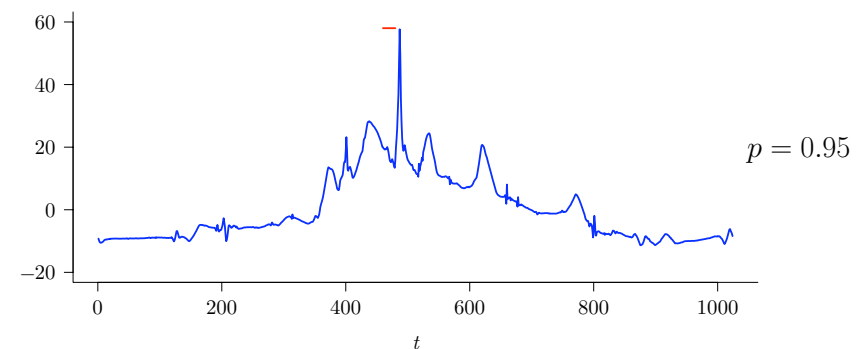
## Examples of Wavelet-Based Shrinkage: I



$p = 0.9$
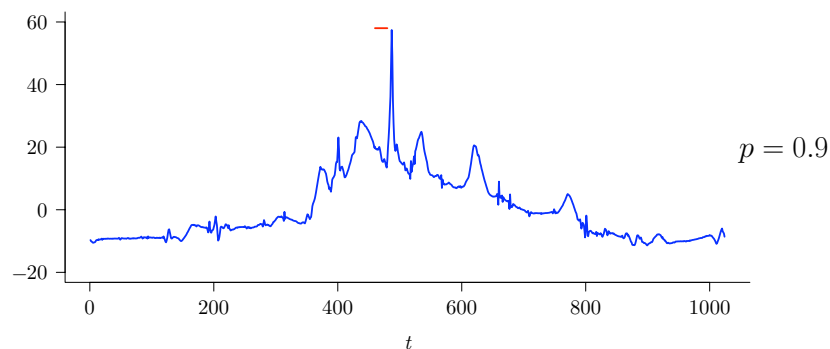
- shrinkage signal estimates of NMR spectrum based upon level $J_0 = 6$ partial LA(8) DWT and conditional mean with $p = 0.9$

## Examples of Wavelet-Based Shrinkage: II



$p = 0.95$
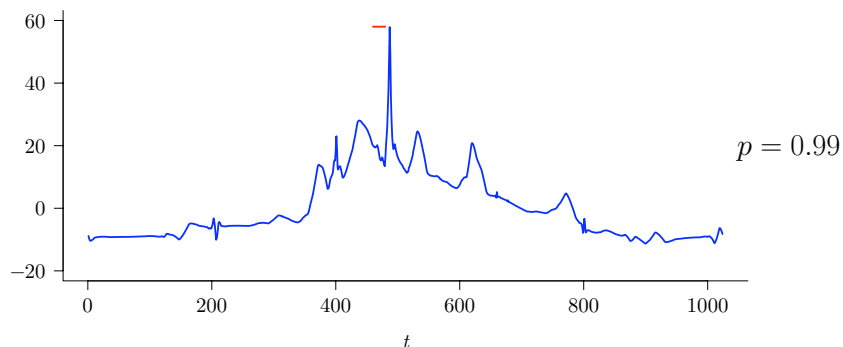
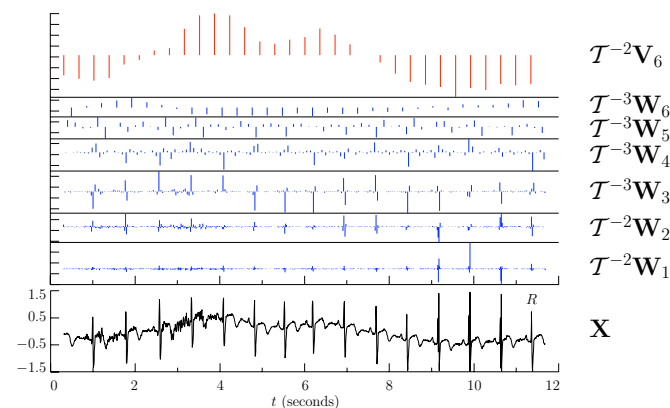- same as before, but now with $p = 0.95$

## Examples of Wavelet-Based Shrinkage: III



$$p = 0.99$$

- same as before, but now with $p = 0.99$ (as $p \to 1$, we declare there are proportionately fewer significant signal coefficients, implying need for heavier shrinkage)

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## Comments on 'Next Generation' Denoising: I



- 'classical' denoising looks at each $W_{j,t}$ alone; for 'real world' signals, coefficients often cluster within a given level and persist across adjacent levels (ECG series offers an example)

---

## Comments on 'Next Generation' Denoising: II

- here are some 'next generation' approaches that exploit these 'real world' properties:

  – Crouse *et al.* (1998) use hidden Markov models for stochastic signal DWT coefficients to handle clustering, persistence and non-Gaussianity

  – Huang and Cressie (2000) consider scale-dependent multi-scale graphical models to handle clustering and persistence

  – Cai and Silverman (2001) consider 'block' thesholding in which coefficients are thresholded in blocks rather than individually (handles clustering)

  – Dragotti and Vetterli (2003) introduce the notion of 'wavelet footprints' to track discontinuities in a signal across different scales (handles persistence)

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## Comments on 'Next Generation' Denoising: III

- 'classical' denoising also suffers from problem of overall significance of multiple hypothesis tests

- 'next generation' work integrates idea of 'false discovery rate' (Benjamini and Hochberg, 1995) into denoising (see Wink and Roerdink, 2004, for an applications-oriented discussion)

- for more recent developments (there are a lot!!!), see

  – review article by Antoniadis (2007)

  – Chapters 3 and 4 of book by Nason (2008)

  – October 2009 issue of *Statistica Sinica*, which has a special section entitled 'Multiscale Methods and Statistics: A Productive Marriage'

## Wavelet-Based Decorrelation of Time Series: Overview

- DWT well-suited for decorrelating certain time series, including ones generated from a fractionally differenced (FD) process
- on synthesis side, leads to
  - DWT-based simulation of FD processes
  - wavelet-based bootstrapping
- on analysis side, leads to
  - wavelet-based estimators for FD parameters
  - test for homogeneity of variance
  - test for trends (won't discuss – see Craigmile *et al.*, 2004, for details)

## DWT of an FD Process: I
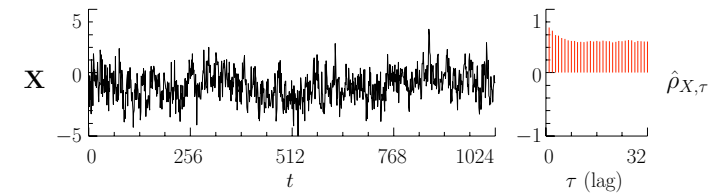


- realization of an FD(0.4) time series $\mathbf{X}$ along with its sample autocorrelation sequence (ACS): for $\tau \geq 0$,

$$\hat{\rho}_{X,\tau} \equiv \frac{\sum_{t=0}^{N-1-\tau} X_t X_{t+\tau}}{\sum_{t=0}^{N-1} X_t^2}$$
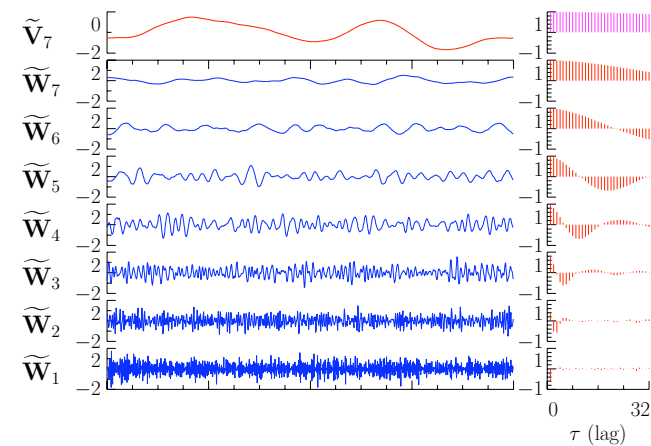
- note that ACS dies down slowly

## DWT of an FD Process: II
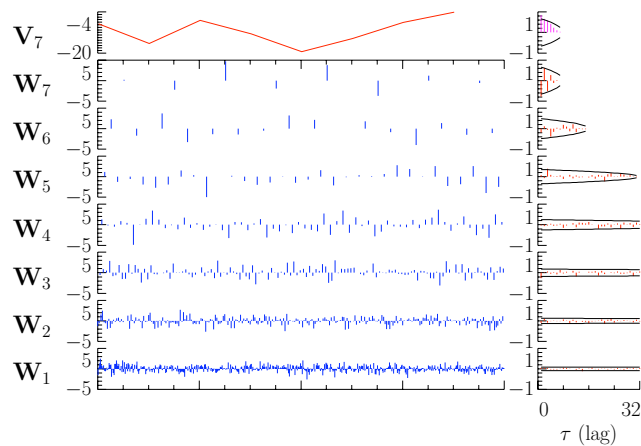


- LA(8) DWT of FD(0.4) series and sample ACSs for each $\mathbf{W}_j$ & $\mathbf{V}_7$, along with 95% confidence intervals for white noise

## MODWT of an FD Process



- LA(8) MODWT of FD(0.4) series & sample ACSs for MODWT coefficients, none of which are approximately uncorrelated

## DWT of an FD Process: III

- in contrast to $\mathbf{X}$, ACSs for $\mathbf{W}_j$ consistent with white noise
- variance of RVs in $\mathbf{W}_j$ increases with $j$: for FD process,

$$\text{var}\{W_{j,t}\} \approx c\tau_j^{2\delta} \equiv C_j,$$

  where $c$ is a constant depending on $\delta$ but not $j$, and $\tau_j = 2^{j-1}$ is scale associated with $\mathbf{W}_j$
- for white noise ($\delta = 0$), $\text{var}\{W_{j,t}\}$ is the same for all $j$
- dependence in $\mathbf{X}$ thus manifests itself in wavelet domain by different variances for wavelet coefficients at different scales

## Correlations Within a Scale and Between Two Scales

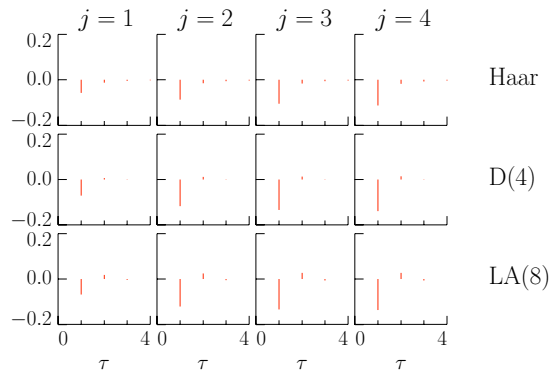- let $\{s_{X,\tau}\}$ denote autocovariance sequence (ACVS) for $\{X_t\}$; i.e., $s_{X,\tau} = \text{cov}\{X_t, X_{t+\tau}\}$
- let $\{h_{j,l}\}$ denote equivalent wavelet filter for $j$th level
- to quantify decorrelation, can write

$$\text{cov}\{W_{j,t}, W_{j',t'}\} = \sum_{l=0}^{L_j-1}\sum_{l'=0}^{L_{j'}-1} h_{j,l}h_{j',l'}s_{X,2^j(t+1)-l-2^{j'}(t'+1)+l'},$$

  from which we can get ACVS (and hence within-scale correlations) for $\{W_{j,t}\}$:

$$\text{cov}\{W_{j,t}, W_{j,t+\tau}\} = \sum_{m=-(L_j-1)}^{L_j-1} s_{X,2^j\tau+m} \sum_{l=0}^{L_j-|m|-1} h_{j,l}h_{j,l+|m|}$$

## Correlations Within a Scale



- correlations between $W_{j,t}$ and $W_{j,t+\tau}$ for an FD(0.4) process
- correlations within scale are slightly smaller for Haar
- maximum magnitude of correlation is less than 0.2

## Correlations Between Two Scales: I
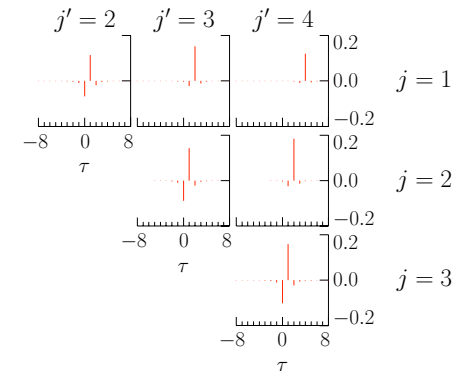


- correlation between Haar wavelet coefficients $W_{j,t}$ and $W_{j',t'}$ from FD(0.4) process and for levels satisfying $1 \le j < j' \le 4$

## Correlations Between Two Scales: II



- same as before, but now for LA(8) wavelet coefficients
- correlations between scales decrease as $L$ increases

---

## Wavelet Domain Description of FD Process

- DWT acts as a decorrelating transform for FD processes and other (but not all!) intrinsically stationary processes
- wavelet domain description is simple
  - wavelet coefficients within a given scale approximately uncorrelated (refinement: assume 1st order autoregressive model)
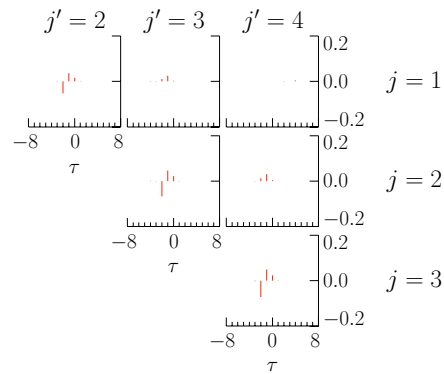  - wavelet coefficients have scale-dependent variance controlled by the two FD parameters ($\delta$ and $\sigma_\varepsilon^2$)
  - wavelet coefficients between scales also approximately uncorrelated (approximation improves as filter width $L$ increases)

---

## DWT-Based Simulation

- properties of DWT of FD processes lead to schemes for simulating time series $\mathbf{X} \equiv [X_0, \ldots, X_{N-1}]^T$ with zero mean and with a multivariate Gaussian distribution
- with $N = 2^J$, recall that $\mathbf{X} = \mathcal{W}^T \mathbf{W}$, where

$$
\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_j \\ \vdots \\ \mathbf{W}_J \\ \mathbf{V}_J \end{bmatrix}
$$

---

## Basic DWT-Based Simulation Scheme

- assume $\mathbf{W}$ to contain $N$ uncorrelated Gaussian (normal) random variables (RVs) with zero mean
- assume $\mathbf{W}_j$ to have variance $C_j = c\tau_j^{2\delta}$
- assume single RV in $\mathbf{V}_J$ to have variance $C_{J+1}$ (see Percival and Walden, 2000, for details on how to set $C_{J+1}$)
- approximate FD time series $\mathbf{X}$ via $\mathbf{Y} \equiv \mathcal{W}^T \Lambda^{1/2} \mathbf{Z}$, where
  - $\Lambda^{1/2}$ is $N \times N$ diagonal matrix with diagonal elements

$$
\underbrace{C_1^{1/2}, \ldots, C_1^{1/2}}_{\frac{N}{2} \text{ of these}}, \underbrace{C_2^{1/2}, \ldots, C_2^{1/2}}_{\frac{N}{4} \text{ of these}}, \ldots, \underbrace{C_{J-1}^{1/2}, C_{J-1}^{1/2}}_{2 \text{ of these}}, C_J^{1/2}, C_{J+1}^{1/2}
$$

  - $\mathbf{Z}$ is vector of deviations drawn from a Gaussian distribution with zero mean and unit variance

## Refinements to Basic Scheme: I

- covariance matrix for approximation $\mathbf{Y}$ does not correspond to that of a stationary process
- recall $\mathcal{W}$ treats $\mathbf{X}$ as if it were circular
- let $\mathcal{T}$ be $N \times N$ 'circular shift' matrix:

$$\mathcal{T} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_0 \end{bmatrix}; \quad \mathcal{T}^2 \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_2 \\ Y_3 \\ Y_0 \\ Y_1 \end{bmatrix}; \quad \text{etc.}$$

- let $\kappa$ be uniformily distributed over $0, \ldots, N-1$
- define $\widetilde{\mathbf{Y}} \equiv \mathcal{T}^{\kappa} \mathbf{Y}$
- $\widetilde{\mathbf{Y}}$ is stationary with ACVS given by, say, $s_{\widetilde{Y},\tau}$

## Refinements to Basic Scheme: II

- Q: how well does $\{s_{\widetilde{Y},\tau}\}$ match $\{s_{X,\tau}\}$?
- due to circularity, find that $s_{\widetilde{Y},N-\tau} = s_{\widetilde{Y},\tau}$ for $\tau = 1, \ldots, N/2$
- implies $s_{\widetilde{Y},\tau}$ cannot approximate $s_{X,\tau}$ well for $\tau$ close to $N$
- can patch up by simulating $\widetilde{\mathbf{Y}}$ with $M > N$ elements and then extracting first $N$ deviates ($M = 4N$ works well)

## Refinements to Basic Scheme: III
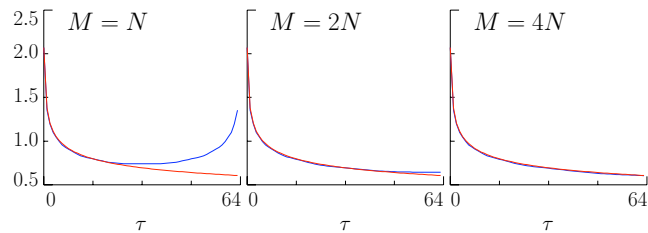


- plot shows true ACVS $\{s_{X,\tau}\}$ (thick curves) for FD(0.4) process and wavelet-based approximate ACVSs $\{s_{\widetilde{Y},\tau}\}$ (thin curves) based on an LA(8) DWT in which an $N = 64$ series is extracted from $M = N$, $M = 2N$ and $M = 4N$ series

## Example and Some Notes
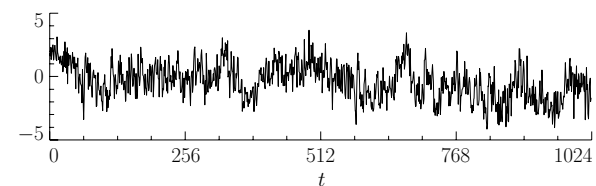


- simulated FD(0.4) series (LA(8), $N = 1024$ and $M = 4N$)
- notes:
  - can form realizations faster than best exact method
  - can efficiently simulate extremely long time series in 'real-time' (e.g, $N = 2^{30} = 1,073,741,824$ or even longer!)
  - effect of random circular shifting is to render time series slightly non-Gaussian (a Gaussian mixture model)

## Wavelet-Domain Bootstrapping

- for many (but not all!) time series, DWT acts as a decorrelating transform: to a good approximation, each $\mathbf{W}_j$ is a sample of a white noise process, and coefficients from different sub-vectors $\mathbf{W}_j$ and $\mathbf{W}_{j'}$ are also pairwise uncorrelated

- variance of coefficients in $\mathbf{W}_j$ depends on $j$

- scaling coefficients $\mathbf{V}_{J_0}$ are still autocorrelated, but there will be just a few of them if $J_0$ is selected to be large

- decorrelating property holds particularly well for FD and other processes with long-range dependence

- above suggests the following recipe for wavelet-domain bootstrapping of a statistic of interest, e.g., sample autocorrelation sequence $\hat{\rho}_{X,\tau}$ at unit lag $\tau = 1$
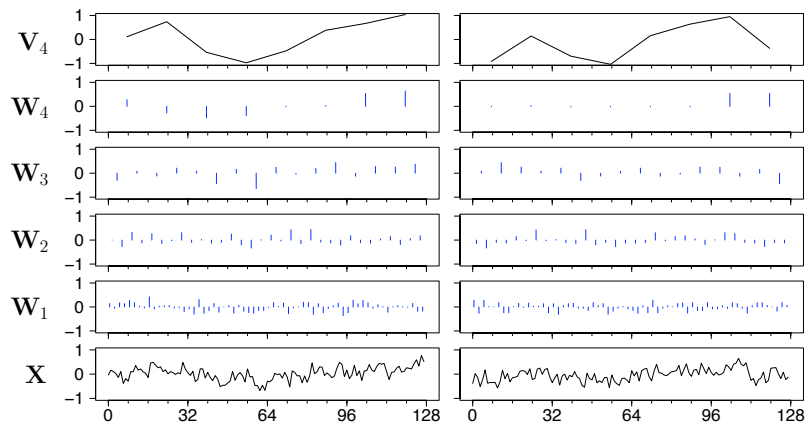
## Recipe for Wavelet-Domain Bootstrapping

1. given $\mathbf{X}$ of length $N = 2^J$, compute level $J_0$ DWT (the choice $J_0 = J - 3$ yields 8 coefficients in $\mathbf{W}_{J_0}$ and $\mathbf{V}_{J_0}$)

2. randomly sample with replacement from $\mathbf{W}_j$ to create bootstrapped vector $\mathbf{W}_j^{(b)}$, $j = 1, \ldots, J_0$

3. create $\mathbf{V}_{J_0}^{(b)}$ using 1st-order autoregressive parametric bootstrap

4. apply $\mathcal{W}^T$ to $\mathbf{W}_1^{(b)}, \ldots, \mathbf{W}_{J_0}^{(b)}$ and $\mathbf{V}_{J_0}^{(b)}$ to obtain bootstrapped time series $\mathbf{X}^{(b)}$ and then form $\hat{\rho}_{X,1}^{(b)}$

- repeat above many times to build up sample distribution of bootstrapped autocorrelations

## Illustration of Wavelet-Domain Bootstrapping



- Haar DWT of FD(0.45) series $\mathbf{X}$ (left-hand column) and wavelet-domain bootstrap thereof (right-hand)

## Wavelet-Domain Bootstrapping of FD Series

- approximations to true PDF using (a) Haar & (b) LA(8) wavelets



vertical line indicates $\hat{\rho}_{X,1}$
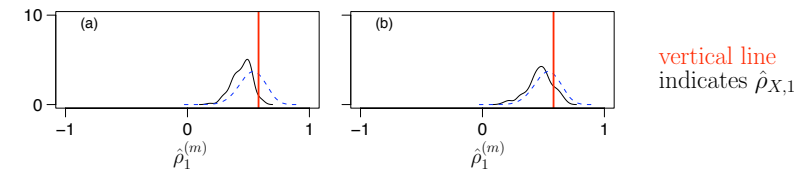
- using 50 FD time series and the Haar DWT yields:

  average of 50 sample means $\doteq 0.35$    (truth $\doteq 0.53$)
  average of 50 sample SDs $\doteq 0.096$    (truth $\doteq 0.107$)

- using 50 FD time series and the LA(8) DWT yields:

  average of 50 sample means $\doteq 0.43$    (truth $\doteq 0.53$)
  average of 50 sample SDs $\doteq 0.098$    (truth $\doteq 0.107$)

## MLEs of FD Parameters: I

- FD process depends on 2 parameters, namely, $\delta$ and $\sigma_\varepsilon^2$

- given $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ with $N = 2^J$, suppose we want to estimate $\delta$ and $\sigma_\varepsilon^2$

- if $\mathbf{X}$ is stationary (i.e. $\delta < 1/2$) and multivariate Gaussian, can use the maximum likelihood (ML) method

## MLEs of FD Parameters: II

- definition of Gaussian likelihood function:
$$L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X}) \equiv \frac{1}{(2\pi)^{N/2}|\Sigma_{\mathbf{X}}|^{1/2}} e^{-\mathbf{X}^T \Sigma_{\mathbf{X}}^{-1} \mathbf{X}/2}$$

  where $\Sigma_{\mathbf{X}}$ is covariance matrix for $\mathbf{X}$, with $(s,t)$th element given by $s_{X,s-t}$, and $|\Sigma_{\mathbf{X}}|$ & $\Sigma_{\mathbf{X}}^{-1}$ denote determinant & inverse

- ML estimators of $\delta$ and $\sigma_\varepsilon^2$ maximize $L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})$ or, equivalently, mininize
$$-2\log\left(L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})\right) = N\log\left(2\pi\right) + \log\left(|\Sigma_{\mathbf{X}}|\right) + \mathbf{X}^T \Sigma_{\mathbf{X}}^{-1}\mathbf{X}$$

- exact MLEs computationally intensive, mainly because of the need to deal with $|\Sigma_{\mathbf{X}}|$ and $\Sigma_{\mathbf{X}}^{-1}$

- good approximate MLEs of considerable interest

## MLEs of FD Parameters: III

- key ideas behind first wavelet-based approximate MLEs

  – have seen that we can approximate FD time series $\mathbf{X}$ by $\mathbf{Y} = \mathcal{W}^T \Lambda^{1/2} \mathbf{Z}$, where $\Lambda^{1/2}$ is a diagonal matrix, all of whose diagonal elements are positive

  – since covariance matrix for $\mathbf{Z}$ is $I_N$, the one for $\mathbf{Y}$ is
$$\mathcal{W}^T \Lambda^{1/2} I_N (\mathcal{W}^T \Lambda^{1/2})^T = \mathcal{W}^T \Lambda^{1/2}\Lambda^{1/2}\mathcal{W} = \mathcal{W}^T \Lambda \mathcal{W} \equiv \widetilde{\Sigma}_{\mathbf{X}},$$

  where $\Lambda \equiv \Lambda^{1/2}\Lambda^{1/2}$ is also diagonal

  – can consider $\widetilde{\Sigma}_{\mathbf{X}}$ to be an approximation to $\Sigma_{\mathbf{X}}$

- leads to approximation of log likelihood:
$$-2\log\left(L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})\right) \approx N\log\left(2\pi\right) + \log\left(|\widetilde{\Sigma}_{\mathbf{X}}|\right) + \mathbf{X}^T \widetilde{\Sigma}_{\mathbf{X}}^{-1}\mathbf{X}$$

## MLEs of FD Parameters: IV

- Q: so how does this help us?

  – easy to invert $\widetilde{\Sigma}_{\mathbf{X}}$:
$$\widetilde{\Sigma}_{\mathbf{X}}^{-1} = \left(\mathcal{W}^T \Lambda \mathcal{W}\right)^{-1} = (\mathcal{W})^{-1} \Lambda^{-1} \left(\mathcal{W}^T\right)^{-1} = \mathcal{W}^T \Lambda^{-1}\mathcal{W},$$

  where $\Lambda^{-1}$ is another diagonal matrix, leading to
$$\mathbf{X}^T \widetilde{\Sigma}_{\mathbf{X}}^{-1}\mathbf{X} = \mathbf{X}^T \mathcal{W}^T \Lambda^{-1}\mathcal{W}\mathbf{X} = \mathbf{W}^T \Lambda^{-1}\mathbf{W}$$

  – easy to compute the determinant of $\widetilde{\Sigma}_{\mathbf{X}}$:
$$|\widetilde{\Sigma}_{\mathbf{X}}| = |\mathcal{W}^T \Lambda \mathcal{W}| = |\Lambda \mathcal{W}\mathcal{W}^T| = |\Lambda I_N| = |\Lambda|,$$

  and the determinant of a diagonal matrix is just the product of its diagonal elements

## MLEs of FD Parameters: V

- define the following three functions of $\delta$:

$$C_j'(\delta) \equiv \int_{1/2^{j+1}}^{1/2^j} \frac{2^{j+1}}{[4\sin^2(\pi f)]^\delta} \, df \approx \int_{1/2^{j+1}}^{1/2^j} \frac{2^{j+1}}{[2\pi f]^{2\delta}} \, df$$

$$C_{J+1}'(\delta) \equiv \frac{N\Gamma(1-2\delta)}{\Gamma^2(1-\delta)} - \sum_{j=1}^{J} \frac{N}{2^j} C_j'(\delta)$$

$$\sigma_\varepsilon^2(\delta) \equiv \frac{1}{N}\left( \frac{V_{J,0}^2}{C_{J+1}'(\delta)} + \sum_{j=1}^{J} \frac{1}{C_j'(\delta)} \sum_{t=0}^{\frac{N}{2^j}-1} W_{j,t}^2 \right)$$

## MLEs of FD Parameters: VI

- wavelet-based approximate MLE $\tilde{\delta}$ for $\delta$ is the value that minimizes the following function of $\delta$:

$$\tilde{l}(\delta \mid \mathbf{X}) \equiv N \log(\sigma_\varepsilon^2(\delta)) + \log(C_{J+1}'(\delta)) + \sum_{j=1}^{J} \frac{N}{2^j} \log(C_j'(\delta))$$

- once $\tilde{\delta}$ has been determined, MLE for $\sigma_\varepsilon^2$ is given by $\sigma_\varepsilon^2(\tilde{\delta})$
- computer experiments indicate scheme works quite well

## Other Wavelet-Based Estimators of FD Parameters

- second MLE approach: formulate likelihood directly in terms of nonboundary wavelet coefficients
  - handles stationary or nonstationary FD processes (i.e., need not assume $\delta < 1/2$)
  - handles certain deterministic trends
- alternative to MLEs are least square estimators (LSEs)
  - recall that, for large $\tau$ and for $\beta = 2\delta - 1$, have
  $$\log\left(\nu_X^2(\tau_j)\right) \approx \zeta + \beta \log\left(\tau_j\right)$$
  - suggests determining $\delta$ by regressing $\log\left(\hat{\nu}_X^2(\tau_j)\right)$ on $\log\left(\tau_j\right)$ over range of $\tau_j$
  - weighted LSE takes into account fact that variance of $\log\left(\hat{\nu}_X^2(\tau_j)\right)$ depends upon scale $\tau_j$ (increases as $\tau_j$ increases)

## Homogeneity of Variance: I

- because DWT decorrelates FD and related processes, nonboundary coefficients in $\mathbf{W}_j$ should resemble white noise; i.e.,
$$\mathrm{cov}\left\{W_{j,t}, W_{j,t'}\right\} \approx 0$$
when $t \neq t'$, and $\mathrm{var}\left\{W_{j,t}\right\}$ should not depend upon $t$
- can test for homogeneity of variance in $\mathbf{X}$ using $\mathbf{W}_j$ over a range of levels $j$
- suppose $U_0, \ldots, U_{N-1}$ are independent normal RVs with $E\{U_t\} = 0$ and $\mathrm{var}\{U_t\} = \sigma_t^2$
- want to test null hypothesis $H_0 : \sigma_0^2 = \sigma_1^2 = \cdots = \sigma_{N-1}^2$
- can test $H_0$ versus a variety of alternatives, e.g.,
$$H_1 : \sigma_0^2 = \cdots = \sigma_k^2 \neq \sigma_{k+1}^2 = \cdots = \sigma_{N-1}^2$$
using normalized cumulative sum of squares

## Homogeneity of Variance: II

- to define test statistic $D$, start with

$$\mathcal{P}_k \equiv \frac{\sum_{j=0}^{k} U_j^2}{\sum_{j=0}^{N-1} U_j^2}, \quad k = 0, \ldots, N-2$$

and then compute $D \equiv \max(D^+, D^-)$, where

$$D^+ \equiv \max_{0 \le k \le N-2} \left( \frac{k+1}{N-1} - \mathcal{P}_k \right) \ \& \ D^- \equiv \max_{0 \le k \le N-2} \left( \mathcal{P}_k - \frac{k}{N-1} \right)$$

- can reject $H_0$ if observed $D$ is 'too large,' where 'too large' is quantified by considering distribution of $D$ under $H_0$

- need to find critical value $x_\alpha$ such that $\mathbf{P}[D \ge x_\alpha] = \alpha$ for, e.g., $\alpha = 0.01$, $0.05$ or $0.1$

## Homogeneity of Variance: III

- once determined, can perform $\alpha$ level test of $H_0$:
  - compute $D$ statistic from data $U_0, \ldots, U_{N-1}$
  - reject $H_0$ at level $\alpha$ if $D \ge x_\alpha$
  - fail to reject $H_0$ at level $\alpha$ if $D < x_\alpha$
- can determine critical values $x_\alpha$ in two ways
  - Monte Carlo simulations
  - large sample approximation to distribution of $D$:

$$\mathbf{P}[(N/2)^{1/2} D \ge x] \approx 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-2l^2 x^2}$$

(reasonable approximation for $N \ge 128$)

## Homogeneity of Variance: IV

- idea: given time series $\{X_t\}$, compute $D$ using nonboundary wavelet coefficients $W_{j,t}$ (there are $M_j' \equiv N_j - L_j'$ of these):
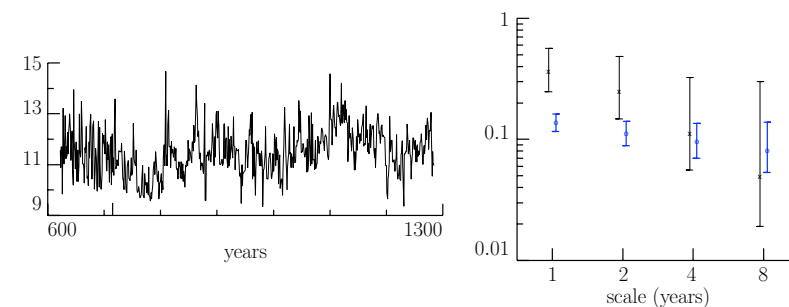
$$\mathcal{P}_k \equiv \frac{\sum_{t=L_j'}^{k} W_{j,t}^2}{\sum_{t=L_j'}^{N_j-1} W_{j,t}^2}, \quad k = L_j', \ldots, N_j - 2$$

- if null hypothesis rejected at level $j$, can use nonboundary MODWT coefficients to locate change point based on

$$\widetilde{\mathcal{P}}_k \equiv \frac{\sum_{t=L_j-1}^{k} \widetilde{W}_{j,t}^2}{\sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2}, \quad k = L_j - 1, \ldots, N - 2$$
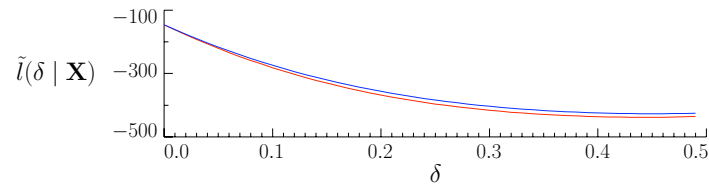
along with analogs $\widetilde{D}_k^+$ and $\widetilde{D}_k^-$ of $D_k^+$ and $D_k^-$

## Example – Annual Minima of Nile River: I



- left-hand plot: annual minima of Nile River

- new measuring device introduced around year 715

- right: Haar $\hat{\nu}_X^2(\tau_j)$ before (x's) and after (o's) year 715.5, with 95% confidence intervals based upon $\chi_{\eta_3}^2$ approximation

## Example – Annual Minima of Nile River: II



- based upon last 512 values (years 773 to 1284), plot shows $\tilde{l}(\delta \mid \mathbf{X})$ versus $\delta$ for the first wavelet-based approximate MLE using the LA(8) wavelet (upper curve) and corresponding curve for exact MLE (lower)

  – wavelet-based approximate MLE is value minimizing upper curve: $\tilde{\delta} \doteq 0.4532$

  – exact MLE is value minimizing lower curve: $\hat{\delta} \doteq 0.4452$

## Example – Annual Minima of Nile River: III

- results of testing all Nile River minima for homogeneity of variance using the Haar wavelet filter with critical values determined by computer simulations

| $\tau_j$ | $M_j'$ | $D$ | critical levels | | |
|---|---|---|---|---|---|
| | | | 10% | 5% | 1% |
| 1 year | 331 | 0.1559 | 0.0945 | 0.1051 | 0.1262 |
| 2 years | 165 | 0.1754 | 0.1320 | 0.1469 | 0.1765 |
| 4 years | 82 | 0.1000 | 0.1855 | 0.2068 | 0.2474 |
| 8 years | 41 | 0.2313 | 0.2572 | 0.2864 | 0.3436 |

- can reject null hypothesis of homogeneity of variance at level of significance 0.05 for scales $\tau_1$ & $\tau_2$, but not at larger scales

## Example – Annual Minima of Nile River: IV
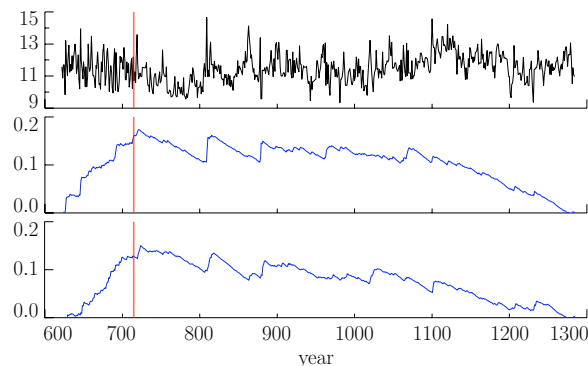


- Nile River minima (top plot) along with curves (constructed per Equation (382)) for scales $\tau_1$ & $\tau_2$ (middle & bottom) to identify change point via time of maximum deviation (vertical lines denote year 715)

## Summary

- DWT approximately decorrelate certain time series, including ones coming from FD and related processes

- leads to schemes for simulating time series and bootstrapping

- also leads to schemes for estimating parameters of FD process

  – approximate maximum likelihood estimators (two varieties)

  – weighted least squares estimator

- can also devise wavelet-based tests for

  – homogeneity of variance

  – trends (see Craigmile *et al.*, 2004, for details)

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