Wavelet Methods for Time Series Analysis

Part I: Introduction to Wavelets and Wavelet Transforms

- wavelets are analysis tools for time series and images (mostly)
- following work on continuous wavelet transform by Morlet and co-workers in 1983, Daubechies, Mallat and others introduced discrete wavelet transform (DWT) in 1988
- begin with qualitative description of the DWT
- discuss two key descriptive capabilities of the DWT:
  - multiresolution analysis (an additive decomposition)
  - wavelet variance or spectrum (decomposition of sum of squares)
- look at how DWT is formed based on a wavelet filter
- discuss maximal overlap DWT (MODWT)

Qualitative Description of DWT

- let $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ be a vector of N time series values (note: 'T' denotes transpose; i.e., $\mathbf{X}$ is a column vector)
- assume initially $N = 2^J$ for some positive integer J (will relax this restriction later on)
- DWT is a linear transform of $\mathbf{X}$ yielding N DWT coefficients
- notation: $\mathbf{W} = \mathcal{W}\mathbf{X}$
  - $\mathbf{W}$ is vector of DWT coefficients (jth component is W_j)
  - $\mathcal{W}$ is $N \times N$ *orthonormal* transform matrix
- orthonormality says $\mathcal{W}^T\mathcal{W} = I_N$ ($N \times N$ identity matrix)
- inverse of $\mathcal{W}$ is just its transpose, so $\mathcal{W}\mathcal{W}^T = I_N$ also

Implications of Orthonormality

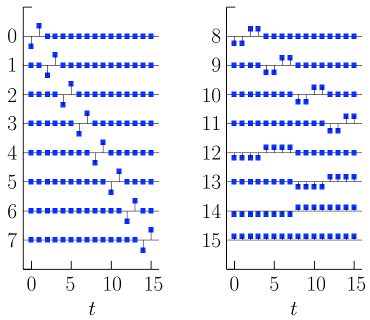
- let $\mathcal{W}_{j\bullet}^T$ denote the jth row of $\mathcal{W}$, where $j = 0, 1, \ldots, N-1$
- let $\mathcal{W}_{j,l}$ denote lth element of $\mathcal{W}_{j\bullet}$
- consider two rows, say, $\mathcal{W}_{j\bullet}^T$ and $\mathcal{W}_{k\bullet}^T$
- orthonormality says

$$\langle \mathcal{W}_{j\bullet}, \mathcal{W}_{k\bullet} \rangle \equiv \sum_{l=0}^{N-1} \mathcal{W}_{j,l}\mathcal{W}_{k,l} = \begin{cases} 1, & \text{when } j = k, \\ 0, & \text{when } j \neq k \end{cases}$$

  - $\langle \mathcal{W}_{j\bullet}, \mathcal{W}_{k\bullet} \rangle$ is inner product of jth & kth rows
  - $\langle \mathcal{W}_{j\bullet}, \mathcal{W}_{j\bullet} \rangle = \|\mathcal{W}_{j\bullet}\|^2$ is squared norm (energy) for $\mathcal{W}_{j\bullet}$

Example: the Haar DWT

- $N = 16$ example of Haar DWT matrix $\mathcal{W}$



- note that rows are orthogonal to each other

Haar DWT Coefficients: I

- obtain Haar DWT coefficients $\mathbf{W}$ by premultiplying $\mathbf{X}$ by $\mathcal{W}$:

$$\mathbf{W} = \mathcal{W}\mathbf{X}$$

- jth coefficient W_j is inner product of jth row $\mathcal{W}_{j\bullet}^T$ and $\mathbf{X}$:

$$W_j = \langle \mathcal{W}_{j\bullet}, \mathbf{X} \rangle$$

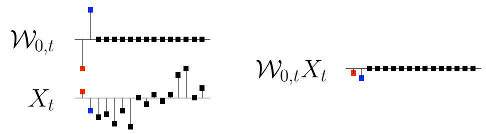
- can interpret coefficients as difference of averages
- to see this, let

$$\overline{X}_t(\lambda) \equiv \frac{1}{\lambda} \sum_{l=0}^{\lambda-1} X_{t-l} = \text{‘scale } \lambda\text{’ average}$$

  – note: $\overline{X}_t(1) = X_t$ = scale 1 ‘average’
  – note: $\overline{X}_{N-1}(N) = \overline{X}$ = sample average

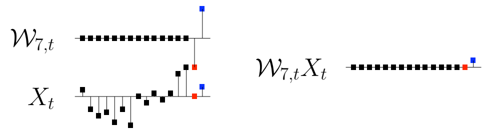
Haar DWT Coefficients: II

- consider form $W_0 = \langle \mathcal{W}_{0\bullet}, \mathbf{X} \rangle$ takes in $N = 16$ example:



sum $\propto \overline{X}_1(1) - \overline{X}_0(1)$

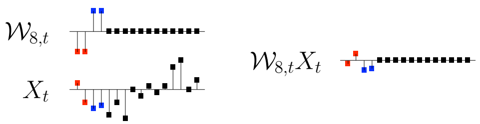
- similar interpretation for $W_1, \ldots, W_{\frac{N}{2}-1} = W_7 = \langle \mathcal{W}_{7\bullet}, \mathbf{X} \rangle$:



sum $\propto \overline{X}_{15}(1) - \overline{X}_{14}(1)$

Haar DWT Coefficients: III

- now consider form of $W_{\frac{N}{2}} = W_8 = \langle \mathcal{W}_{8\bullet}, \mathbf{X} \rangle$:

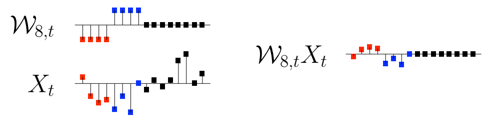


sum $\propto \overline{X}_3(2) - \overline{X}_1(2)$

- similar interpretation for $W_{\frac{N}{2}+1}, \ldots, W_{\frac{3N}{4}-1}$

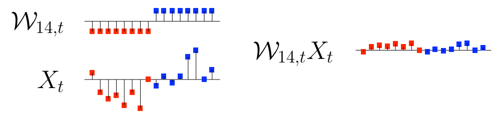
Haar DWT Coefficients: IV

- $W_{\frac{3N}{4}} = W_{12} = \langle \mathcal{W}_{12\bullet}, \mathbf{X} \rangle$ takes the following form:



sum $\propto \overline{X}_7(4) - \overline{X}_3(4)$

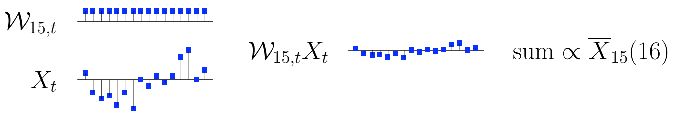
- continuing in this manner, come to $W_{N-2} = \langle \mathcal{W}_{14\bullet}, \mathbf{X} \rangle$:



sum $\propto \overline{X}_{15}(8) - \overline{X}_7(8)$

Haar DWT Coefficients: V

- final coefficient $W_{N-1} = W_{15}$ has a different interpretation:



- structure of rows in $\mathcal{W}$
  - first $\frac{N}{2}$ rows yield W_j's $\propto$ *changes* on scale 1
  - next $\frac{N}{4}$ rows yield W_j's $\propto$ *changes* on scale 2
  - next $\frac{N}{8}$ rows yield W_j's $\propto$ *changes* on scale 4
  - next to last row yields $W_j \propto$ *change* on scale $\frac{N}{2}$
  - last row yields $W_j \propto$ *average* on scale N

Structure of DWT Matrices

- $\frac{N}{2\tau_j}$ wavelet coefficients for scale $\tau_j \equiv 2^{j-1}$, $j = 1, \ldots, J$
  - $\tau_j \equiv 2^{j-1}$ is standardized scale
  - $\tau_j \, \Delta$ is physical scale, where Δ is sampling interval
- each W_j localized in time: as scale $\uparrow$, localization $\downarrow$
- rows of $\mathcal{W}$ for given scale τ_j:
  - circularly shifted with respect to each other
  - shift between adjacent rows is $2\tau_j = 2^j$
- similar structure for DWTs other than the Haar
- differences of averages common theme for DWTs
  - simple differencing replaced by higher order differences
  - simple averages replaced by weighted averages

Two Basic Decompositions Derivable from DWT

- additive decomposition
  - reexpresses $\mathbf{X}$ as the sum of $J+1$ new time series, each of which is associated with a particular scale τ_j
  - called multiresolution analysis (MRA)
- energy decomposition
  - yields analysis of variance across J scales
  - called wavelet spectrum or wavelet variance

Partitioning of DWT Coefficient Vector W

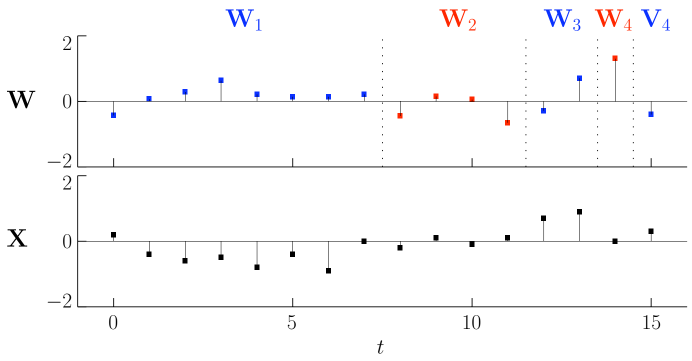
- decompositions are based on partitioning of $\mathbf{W}$ and $\mathcal{W}$
- partition $\mathbf{W}$ into subvectors associated with scale:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_j \\ \vdots \\ \mathbf{W}_J \\ \mathbf{V}_J \end{bmatrix}$$

- $\mathbf{W}_j$ has $N/2^j$ elements (scale $\tau_j = 2^{j-1}$ changes)
  note: $\sum_{j=1}^{J} \frac{N}{2^j} = \frac{N}{2} + \frac{N}{4} + \cdots + 2 + 1 = 2^J - 1 = N - 1$
- $\mathbf{V}_J$ has 1 element, which is equal to $\sqrt{N} \cdot \overline{X}$ (scale N average)

Example of Partitioning of W

- consider time series $\mathbf{X}$ of length $N = 16$ & its Haar DWT $\mathbf{W}$



Partitioning of DWT Matrix $\mathcal{W}$

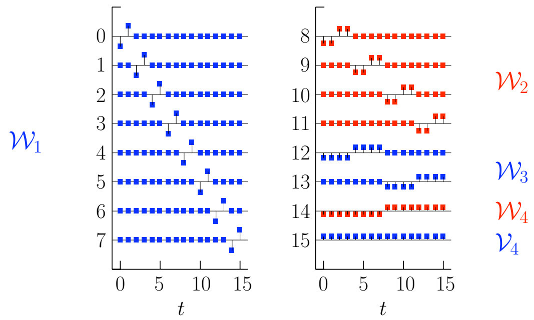
- partition $\mathcal{W}$ commensurate with partitioning of $\mathbf{W}$:

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{W}_2 \\ \vdots \\ \mathcal{W}_j \\ \vdots \\ \mathcal{W}_J \\ \mathcal{V}_J \end{bmatrix}$$

- $\mathcal{W}_j$ is $\frac{N}{2^j} \times N$ matrix (related to scale $\tau_j = 2^{j-1}$ changes)
- $\mathcal{V}_J$ is $1 \times N$ row vector (each element is $\frac{1}{\sqrt{N}}$)

Example of Partitioning of $\mathcal{W}$

- $N = 16$ example of Haar DWT matrix $\mathcal{W}$



- two properties: (a) $\mathbf{W}_j = \mathcal{W}_j \mathbf{X}$ and (b) $\mathcal{W}_j \mathcal{W}_j^T = I_{\frac{N}{2^j}}$

DWT Analysis and Synthesis Equations

- recall the DWT analysis equation $\mathbf{W} = \mathcal{W}\mathbf{X}$
- $\mathcal{W}^T\mathcal{W} = I_N$ because $\mathcal{W}$ is an orthonormal transform
- implies that $\mathcal{W}^T\mathbf{W} = \mathcal{W}^T\mathcal{W}\mathbf{X} = \mathbf{X}$
- yields DWT synthesis equation:

$$\mathbf{X} = \mathcal{W}^T\mathbf{W} = \left[\mathcal{W}_1^T, \mathcal{W}_2^T, \ldots, \mathcal{W}_J^T, \mathcal{V}_J^T\right] \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_J \\ \mathbf{V}_J \end{bmatrix}$$

$$= \sum_{j=1}^{J} \mathcal{W}_j^T \mathbf{W}_j + \mathcal{V}_J^T \mathbf{V}_J$$

Multiresolution Analysis: I

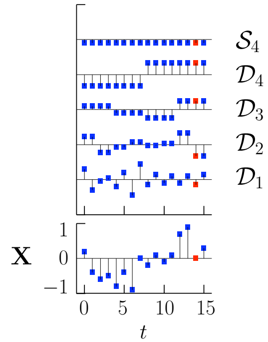
- synthesis equation leads to additive decomposition:

$$\mathbf{X} = \sum_{j=1}^{J} \mathcal{W}_j^T \mathbf{W}_j + \mathcal{V}_J^T \mathbf{V}_J \equiv \sum_{j=1}^{J} \mathcal{D}_j + \mathcal{S}_J$$

- $\mathcal{D}_j \equiv \mathcal{W}_j^T \mathbf{W}_j$ is portion of synthesis due to scale τ_j
- $\mathcal{D}_j$ is vector of length N and is called jth 'detail'
- $\mathcal{S}_J \equiv \mathcal{V}_J^T \mathbf{V}_J = \overline{X}\mathbf{1}$, where $\mathbf{1}$ is a vector containing N ones (later on we will call this the 'smooth' of Jth order)
- additive decomposition called multiresolution analysis (MRA)

Multiresolution Analysis: II

- example of MRA for time series of length $N = 16$



- adding values for, e.g., $t = 14$ in $\mathcal{D}_1, \ldots, \mathcal{D}_4$ & $\mathcal{S}_4$ yields X_{14}

Energy Preservation Property of DWT Coefficients

- define 'energy' in $\mathbf{X}$ as its squared norm:

$$\|\mathbf{X}\|^2 = \langle \mathbf{X}, \mathbf{X} \rangle = \mathbf{X}^T \mathbf{X} = \sum_{t=0}^{N-1} X_t^2$$

- energy of $\mathbf{X}$ is preserved in its DWT coefficients $\mathbf{W}$ because

$$\begin{aligned} \|\mathbf{W}\|^2 = \mathbf{W}^T \mathbf{W} &= (\mathcal{W}\mathbf{X})^T \mathcal{W}\mathbf{X} \\ &= \mathbf{X}^T \mathcal{W}^T \mathcal{W} \mathbf{X} \\ &= \mathbf{X}^T I_N \mathbf{X} = \mathbf{X}^T \mathbf{X} = \|\mathbf{X}\|^2 \end{aligned}$$

- note: same argument holds for *any* orthonormal transform

Wavelet Spectrum (Variance Decomposition): I

- let $\overline{X}$ denote sample mean of X_t's: $\overline{X} \equiv \frac{1}{N} \sum_{t=0}^{N-1} X_t$
- let $\hat{\sigma}_X^2$ denote sample variance of X_t's:

$$\begin{aligned} \hat{\sigma}_X^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} \left(X_t - \overline{X}\right)^2 &= \frac{1}{N} \sum_{t=0}^{N-1} X_t^2 - \overline{X}^2 \\ &= \frac{1}{N} \|\mathbf{X}\|^2 - \overline{X}^2 \\ &= \frac{1}{N} \|\mathbf{W}\|^2 - \overline{X}^2 \end{aligned}$$

- since $\|\mathbf{W}\|^2 = \sum_{j=1}^{J} \|\mathbf{W}_j\|^2 + \|\mathbf{V}_J\|^2$ and $\frac{1}{N}\|\mathbf{V}_J\|^2 = \overline{X}^2$,

$$\hat{\sigma}_X^2 = \frac{1}{N} \sum_{j=1}^{J} \|\mathbf{W}_j\|^2$$

Wavelet Spectrum (Variance Decomposition): II

- define discrete wavelet power spectrum:

$$P_X(\tau_j) \equiv \frac{1}{N}\|\mathbf{W}_j\|^2, \text{ where } \tau_j = 2^{j-1}$$

- gives us a scale-based decomposition of the sample variance:

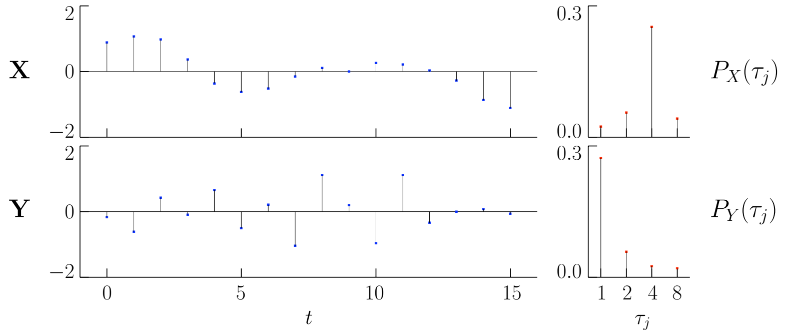
$$\hat{\sigma}_X^2 = \sum_{j=1}^{J} P_X(\tau_j)$$

- in addition, each $W_{j,t}$ in $\mathbf{W}_j$ associated with a portion of $\mathbf{X}$; i.e., $W_{j,t}^2$ offers scale- & time-based decomposition of $\hat{\sigma}_X^2$

WMTSA: 62

Wavelet Spectrum (Variance Decomposition): III

- wavelet spectra for time series $\mathbf{X}$ and $\mathbf{Y}$ of length $N = 16$, each with zero sample mean and same sample variance



Defining the Discrete Wavelet Transform (DWT)

- can formulate DWT via elegant 'pyramid' algorithm
- *defines* $\mathcal{W}$ for non-Haar wavelets (consistent with Haar)
- computes $\mathbf{W} = \mathcal{W}\mathbf{X}$ using $O(N)$ multiplications
  - 'brute force' method uses $O(N^2)$ multiplications
  - faster than celebrated algorithm for fast Fourier transform! (this uses $O(N \cdot \log_2(N))$ multiplications)
- can formulate algorithm using linear filters or matrices (two approaches are complementary)
- need to review ideas from theory of linear (time-invariant) filters

WMTSA: 68

Fourier Theory for Sequences: I

- let $\{a_t\}$ denote a real-valued sequence such that $\sum_t a_t^2 < \infty$
- discrete Fourier transform (DFT) of $\{a_t\}$:

$$A(f) \equiv \sum_t a_t e^{-i2\pi ft}$$

- f called frequency: $e^{-i2\pi ft} = \cos(2\pi ft) - i\sin(2\pi ft)$
- $A(f)$ defined for all f, but $0 \le f \le 1/2$ is of main interest:
  - $A(\cdot)$ periodic with unit period, i.e., $A(f+1) = A(f)$, all f
  - $A(-f) = A^*(f)$, complex conjugate of $A(f)$
  - need only know $A(f)$ for $0 \le f \le 1/2$ to know it for all f
- 'low frequencies' are those in lower range of $[0, 1/2]$
- 'high frequencies' are those in upper range of $[0, 1/2]$

WMTSA: 21–22

Fourier Theory for Sequences: II

- can recover (synthesize) $\{a_t\}$ from its DFT:
$$\int_{-1/2}^{1/2} A(f)e^{i2\pi ft}\,df = a_t;$$
left-hand side called inverse DFT of $A(\cdot)$
- $\{a_t\}$ and $A(\cdot)$ are two representations for one 'thingy'
- large $|A(f)|$ says $e^{i2\pi ft}$ important in synthesizing $\{a_t\}$; i.e., $\{a_t\}$ resembles some combination of $\cos(2\pi ft)$ and $\sin(2\pi ft)$

Convolution of Sequences

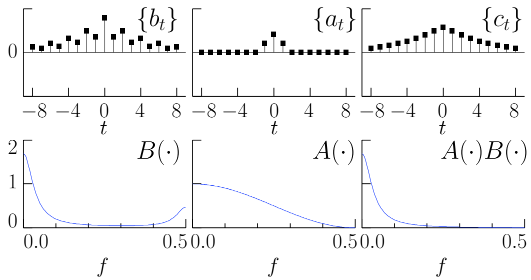
- given two sequences $\{a_t\}$ and $\{b_t\}$, define their convolution by
$$c_t \equiv \sum_{u=-\infty}^{\infty} a_u b_{t-u}$$
- DFT of $\{c_t\}$ has a simple form, namely,
$$\sum_{t=-\infty}^{\infty} c_t e^{-i2\pi ft} = A(f)B(f),$$
where $A(\cdot)$ is the DFT of $\{a_t\}$, and $B(\cdot)$ is the DFT of $\{b_t\}$; i.e., just multiply two DFTs together!!!

Basic Concepts of Filtering

- convolution & linear time-invariant filtering are same concepts:
  - $\{b_t\}$ is input to filter
  - $\{a_t\}$ represents the filter
  - $\{c_t\}$ is filter output
- flow diagram for filtering: $\{b_t\} \longrightarrow \boxed{\{a_t\}} \longrightarrow \{c_t\}$
- $\{a_t\}$ is called impulse response sequence for filter
- its DFT $A(\cdot)$ is called transfer function
- in general $A(\cdot)$ is complex-valued, so write $A(f) = |A(f)|e^{i\theta(f)}$
  - $|A(f)|$ defines gain function
  - $\mathcal{A}(f) \equiv |A(f)|^2$ defines squared gain function
  - $\theta(\cdot)$ called phase function (well-defined at f if $|A(f)| > 0$)

Example of a Low-Pass Filter

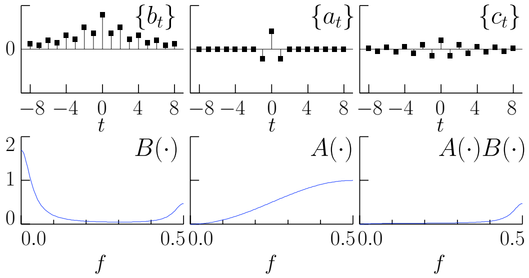
- consider $b_t = \frac{3}{16}\left(\frac{4}{5}\right)^{|t|} + \frac{1}{20}\left(-\frac{4}{5}\right)^{|t|}$ & $a_t = \begin{cases} \frac{1}{2}, & t = 0 \\ \frac{1}{4}, & t = -1 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$



- note: $A(\cdot)$ & $B(\cdot)$ both real-valued ($A(\cdot)$ = its gain function)

Example of a High-Pass Filter

- consider same $\{b_t\}$, but now let $a_t = \begin{cases} \frac{1}{2}, & t = 0 \\ -\frac{1}{4}, & t = -1 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$



- note: $\{a_t\}$ resembles some wavelet filters we'll see later

The Wavelet Filter: I

- precise definition of DWT begins with notion of wavelet filter
- let $\{h_l : l = 0, \ldots, L-1\}$ be a real-valued filter of width L
  - both h_0 and h_{L-1} must be nonzero
  - for convenience, will define $h_l = 0$ for $l < 0$ and $l \geq L$
  - L must be even $(2, 4, 6, 8, \ldots)$ for technical reasons (hence ruling out $\{a_t\}$ on the previous overhead)

The Wavelet Filter: II

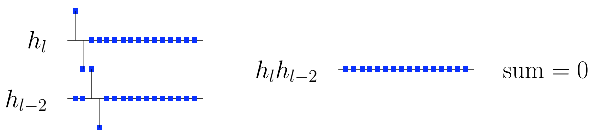
- $\{h_l\}$ called a wavelet filter if it has these 3 properties
  1. summation to zero:
  $$\sum_{l=0}^{L-1} h_l = 0$$
  2. unit energy:
  $$\sum_{l=0}^{L-1} h_l^2 = 1$$
  3. orthogonality to even shifts: for all nonzero integers n, have
  $$\sum_{l=0}^{L-1} h_l h_{l+2n} = 0$$
- 2 and 3 together are called the *orthonormality property*

The Wavelet Filter: III

- summation to zero and unit energy relatively easy to achieve
- orthogonality to even shifts is key property & hardest to satisfy
- define transfer and squared gain functions for wavelet filter:
$$H(f) \equiv \sum_{l=0}^{L-1} h_l e^{-i2\pi f l} \text{ and } \mathcal{H}(f) \equiv |H(f)|^2$$
- orthonormality property is equivalent to
$$\mathcal{H}(f) + \mathcal{H}(f + \tfrac{1}{2}) = 2 \quad \text{for all } f$$
(an elegant – but not obvious! – result)

Haar Wavelet Filter

- simplest wavelet filter is Haar ($L = 2$): $h_0 = \frac{1}{\sqrt{2}}$ & $h_1 = -\frac{1}{\sqrt{2}}$
- note that $h_0 + h_1 = 0$ and $h_0^2 + h_1^2 = 1$, as required
- orthogonality to even shifts also readily apparent

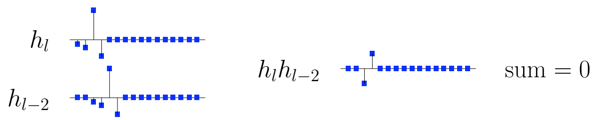


D(4) Wavelet Filter: I

- next simplest wavelet filter is D(4), for which $L = 4$:

$$h_0 = \frac{1-\sqrt{3}}{4\sqrt{2}},\ \ h_1 = \frac{-3+\sqrt{3}}{4\sqrt{2}},\ \ h_2 = \frac{3+\sqrt{3}}{4\sqrt{2}},\ \ h_3 = \frac{-1-\sqrt{3}}{4\sqrt{2}}$$

  - 'D' stands for Daubechies
  - $L = 4$ width member of her 'extremal phase' wavelets
- computations show $\sum_l h_l = 0$ & $\sum_l h_l^2 = 1$, as required
- orthogonality to even shifts apparent except for ± 2 case:



D(4) Wavelet Filter: II

- Q: what is rationale for D(4) filter?
- consider $X_t^{(1)} \equiv X_t - X_{t-1} = a_0 X_t + a_1 X_{t-1}$, where $\{a_0 = 1, a_1 = -1\}$ defines 1st difference filter:

$$\{X_t\} \longrightarrow \boxed{\{1, -1\}} \longrightarrow \{X_t^{(1)}\}$$

  - Haar wavelet filter is normalized 1st difference filter
  - $X_t^{(1)}$ is difference between two '1 point averages'
- consider filter 'cascade' with two 1st difference filters:

$$\{X_t\} \longrightarrow \boxed{\{1, -1\}} \longrightarrow \boxed{\{1, -1\}} \longrightarrow \{X_t^{(2)}\}$$

- by considering convolution of $\{1, -1\}$ with itself, can reexpress the above using a single 'equivalent' (2nd difference) filter:

$$\{X_t\} \longrightarrow \boxed{\{1, -2, 1\}} \longrightarrow \{X_t^{(2)}\}$$

D(4) Wavelet Filter: III

- renormalizing and shifting 2nd difference filter yields high-pass filter considered earlier:

$$a_t = \begin{cases} \frac{1}{2}, & t = 0 \\ -\frac{1}{4}, & t = -1 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$$

- consider '2 point weighted average' followed by 2nd difference:

$$\{X_t\} \longrightarrow \boxed{\{a, b\}} \longrightarrow \boxed{\{1, -2, 1\}} \longrightarrow \{Y_t\}$$

- convolution of $\{a, b\}$ and $\{1, -2, 1\}$ yields an equivalent filter, which is how the D(4) wavelet filter arises:

$$\{X_t\} \longrightarrow \boxed{\{h_0, h_1, h_2, h_3\}} \longrightarrow \{Y_t\}$$

D(4) Wavelet Filter: IV

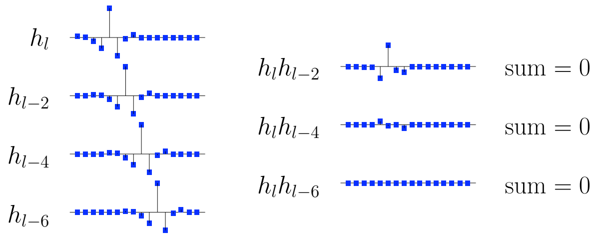
- using conditions

  1. summation to zero: $h_0 + h_1 + h_2 + h_3 = 0$
  2. unit energy: $h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1$
  3. orthogonality to even shifts: $h_0 h_2 + h_1 h_3 = 0$

  can solve for feasible values of a and b
- one solution is $a = \frac{1+\sqrt{3}}{4\sqrt{2}} \doteq 0.48$ and $b = \frac{-1+\sqrt{3}}{4\sqrt{2}} \doteq 0.13$ (other solutions yield essentially the same filter)
- interpret D(4) filtered output as changes in weighted averages
  - ‘change’ now measured by 2nd difference (1st for Haar)
  - average is now 2 point weighted average (1 point for Haar)
  - can argue that effective scale of weighted average is one

Another Popular Daubechies Wavelet Filter

- LA(8) wavelet filter (‘LA’ stands for ‘least asymmetric’)



- resembles three-point high-pass filter $\{-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}\}$ (somewhat)
- can interpret this filter as cascade consisting of
  - 4th difference filter
  - weighted average filter of width 4, but effective width 1
- filter output can be interpreted as changes in weighted averages

First Level Wavelet Coefficients: I

- given wavelet filter $\{h_l\}$ of width L & time series of length $N = 2^J$, obtain first level wavelet coefficients as follows
- *circularly* filter $\mathbf{X}$ with wavelet filter to yield output

$$\sum_{l=0}^{L-1} h_l X_{t-l} = \sum_{l=0}^{L-1} h_l X_{t-l \bmod N}, \quad t = 0, \ldots, N-1;$$

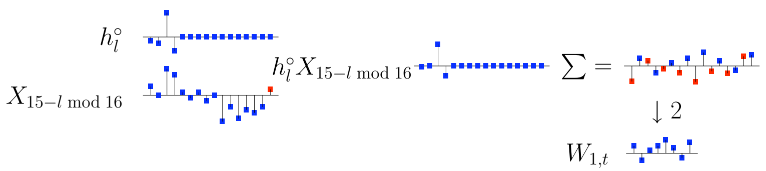
  i.e., if $t-l$ does not satisfy $0 \le t-l \le N-1$, interpret X_{t-l} as $X_{t-l \bmod N}$; e.g., $X_{-1} = X_{N-1}$ and $X_{-2} = X_{N-2}$
- take every other value of filter output to define

$$W_{1,t} \equiv \sum_{l=0}^{L-1} h_l X_{2t+1-l \bmod N}, \quad t = 0, \ldots, \frac{N}{2} - 1;$$

  $\{W_{1,t}\}$ formed by *downsampling* filter output by a factor of 2

First Level Wavelet Coefficients: II

- example of formation of $\{W_{1,t}\}$



- $\{W_{1,t}\}$ are unit scale wavelet coefficients – these are the elements of $\mathbf{W}_1$ and first $N/2$ elements of $\mathbf{W} = \mathcal{W}\mathbf{X}$
- also have $\mathbf{W}_1 = \mathcal{W}_1 \mathbf{X}$, with $\mathcal{W}_1$ being first $N/2$ rows of $\mathcal{W}$
- hence elements of $\mathcal{W}_1$ dictated by wavelet filter

Upper Half $\mathcal{W}_1$ of Haar DWT Matrix $\mathcal{W}$

- consider Haar wavelet filter ($L = 2$): $h_0 = \frac{1}{\sqrt{2}}$ & $h_1 = -\frac{1}{\sqrt{2}}$
- when $N = 16$, $\mathcal{W}_1$ looks like

$$\begin{bmatrix}
h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_0
\end{bmatrix}$$

- rows obviously orthogonal to each other

Upper Half $\mathcal{W}_1$ of D(4) DWT Matrix $\mathcal{W}$

- when $L = 4$ & $N = 16$, $\mathcal{W}_1$ looks like

$$\begin{bmatrix}
h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2\\
h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0
\end{bmatrix}$$

- rows orthogonal because $h_0 h_2 + h_1 h_3 = 0$
- note: $\langle \mathcal{W}_{0\bullet}, \mathbf{X} \rangle$ yields $W_0 = h_1 X_0 + h_0 X_1 + h_3 X_{14} + h_2 X_{15}$
- unlike other coefficients from above, this 'boundary' coefficient depends on circular treatment of $\mathbf{X}$ (a curse, not a feature!)

Orthonormality of Upper Half of DWT Matrix: I

- can show that, for all L and even N,

$$W_{1,t} = \sum_{l=0}^{L-1} h_l X_{2t+1-l \bmod N}, \text{ or, equivalently, } \mathbf{W}_1 = \mathcal{W}_1 \mathbf{X}$$

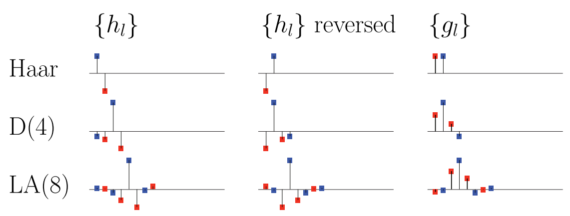
forms *half* an orthonormal transform; i.e.,

$$\mathcal{W}_1 \mathcal{W}_1^T = I_{\frac{N}{2}}$$

- Q: how can we construct the other half of $\mathcal{W}$?

The Scaling Filter: I

- create scaling (or 'father wavelet') filter $\{g_l\}$ by reversing $\{h_l\}$ and then changing sign of coefficients with even indices



- 2 filters related by $g_l \equiv (-1)^{l+1} h_{L-1-l}$ & $h_l = (-1)^l g_{L-1-l}$

The Scaling Filter: II

- $\{g_l\}$ is 'quadrature mirror' filter corresponding to $\{h_l\}$
- properties 2 and 3 of $\{h_l\}$ are shared by $\{g_l\}$:

  2. unit energy:
$$\sum_{l=0}^{L-1} g_l^2 = 1$$
  3. orthogonality to even shifts: for all nonzero integers n, have
$$\sum_{l=0}^{L-1} g_l g_{l+2n} = 0$$
- scaling & wavelet filters both satisfy orthonormality property

First Level Scaling Coefficients: I

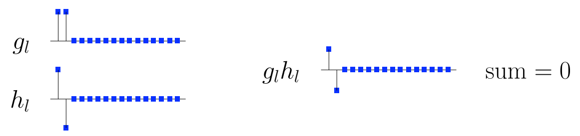
- orthonormality property of $\{h_l\}$ is all that is needed to prove $\mathcal{W}_1$ is half of an orthonormal transform (never used $\sum_l h_l = 0$)
- going back and replacing h_l with g_l everywhere yields another half of an orthonormal transform
- circularly filter $\mathbf{X}$ using $\{g_l\}$ and downsample to define
$$V_{1,t} \equiv \sum_{l=0}^{L-1} g_l X_{2t+1-l \bmod N}, \quad t = 0, \ldots, \tfrac{N}{2} - 1$$
- $\{V_{1,t}\}$ called scaling coefficients for level $j = 1$
- place these $N/2$ coefficients in vector called $\mathbf{V}_1$

First Level Scaling Coefficients: III

- define $\mathcal{V}_1$ in a manner analogous to $\mathcal{W}_1$ so that $\mathbf{V}_1 = \mathcal{V}_1 \mathbf{X}$
- when $L = 4$ and $N = 16$, $\mathcal{V}_1$ looks like
$$\begin{bmatrix}
g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 \\
g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0
\end{bmatrix}$$
- $\mathcal{V}_1$ obeys same orthonormality property as $\mathcal{W}_1$:
$$\text{similar to } \mathcal{W}_1 \mathcal{W}_1^T = I_{\frac{N}{2}}, \text{ have } \mathcal{V}_1 \mathcal{V}_1^T = I_{\frac{N}{2}}$$

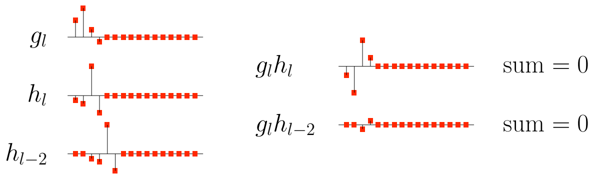
Orthonormality of $\mathcal{V}_1$ and $\mathcal{W}_1$: I

- Q: how does $\mathcal{V}_1$ help us?
- A: rows of $\mathcal{V}_1$ and $\mathcal{W}_1$ are pairwise orthogonal!
- readily apparent in Haar case:



Orthonormality of $\mathcal{V}_1$ and $\mathcal{W}_1$: II

- let's check that orthogonality holds for D(4) case also:



Orthonormality of $\mathcal{V}_1$ and $\mathcal{W}_1$: III

- implies that

$$\mathcal{P}_1 \equiv \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{V}_1 \end{bmatrix}$$

is an $N \times N$ orthonormal matrix since

$$\mathcal{P}_1 \mathcal{P}_1^T = \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{V}_1 \end{bmatrix} \begin{bmatrix} \mathcal{W}_1^T, \mathcal{V}_1^T \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{W}_1 \mathcal{W}_1^T & \mathcal{W}_1 \mathcal{V}_1^T \\ \mathcal{V}_1 \mathcal{W}_1^T & \mathcal{V}_1 \mathcal{V}_1^T \end{bmatrix} = \begin{bmatrix} I_{\frac{N}{2}} & 0_{\frac{N}{2}} \\ 0_{\frac{N}{2}} & I_{\frac{N}{2}} \end{bmatrix} = I_N$$

- if $N = 2$ (not of too much interest!), in fact $\mathcal{P}_1 = \mathcal{W}$
- if $N > 2$, $\mathcal{P}_1$ is an intermediate step: $\mathcal{V}_1$ spans same subspace as lower half of $\mathcal{W}$ and will be further manipulated

Interpretation of Scaling Coefficients: I

- consider Haar scaling filter ($L = 2$): $g_0 = g_1 = \frac{1}{\sqrt{2}}$
- when $N = 16$, matrix $\mathcal{V}_1$ looks like

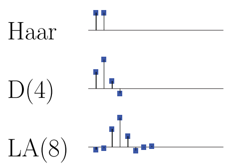
$$\begin{bmatrix}
g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_0
\end{bmatrix}$$

- since $\mathbf{V}_1 = \mathcal{V}_1 \mathbf{X}$, each $V_{1,t}$ is proportional to a 2 point average:

$$V_{1,0} = g_1 X_0 + g_0 X_1 = \tfrac{1}{\sqrt{2}} X_0 + \tfrac{1}{\sqrt{2}} X_1 \propto \overline{X}_1(2) \text{ and so forth}$$

Interpretation of Scaling Coefficients: II

- reconsider shapes of $\{g_l\}$ seen so far:



- for $L > 2$, can regard $V_{1,t}$ as proportional to weighted average
- can argue that effective width of $\{g_l\}$ is 2 in each case; thus scale associated with $V_{1,t}$ is 2, whereas scale is 1 for $W_{1,t}$

Frequency Domain Properties of Scaling Filter

- define transfer and squared gain functions for $\{g_l\}$

$$G(f) \equiv \sum_{l=0}^{L-1} g_l e^{-i2\pi fl} \;\; \& \;\; \mathcal{G}(f) \equiv |G(f)|^2$$

- can argue that $\mathcal{G}(f) = \mathcal{H}(f + \frac{1}{2})$, which, combined with

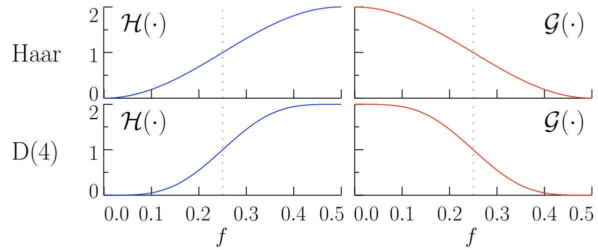
$$\mathcal{H}(f) + \mathcal{H}(f + \tfrac{1}{2}) = 2,$$

yields

$$\mathcal{H}(f) + \mathcal{G}(f) = 2$$

Frequency Domain Properties of $\{h_l\}$ and $\{g_l\}$

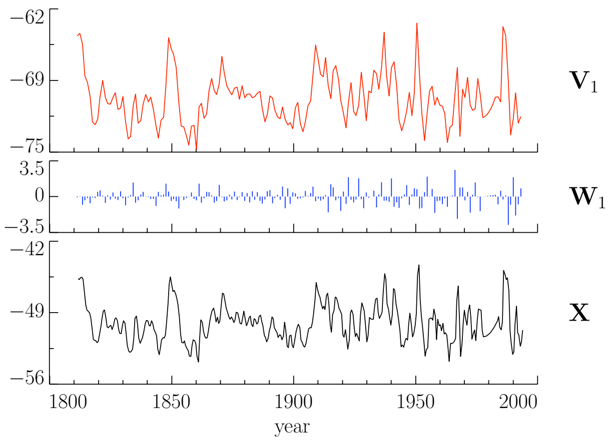
- since $\mathbf{W}_1$ & $\mathbf{V}_1$ contain output from filters, consider their squared gain functions, recalling that $\mathcal{H}(f) + \mathcal{G}(f) = 2$
- example: $\mathcal{H}(\cdot)$ and $\mathcal{G}(\cdot)$ for Haar & D(4) filters



- $\{h_l\}$ is high-pass filter with nominal pass-band $[1/4, 1/2]$
- $\{g_l\}$ is low-pass filter with nominal pass-band $[0, 1/4]$

Example of Decomposing X into $\mathbf{W}_1$ and $\mathbf{V}_1$: I

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core



Example of Decomposing X into $\mathbf{W}_1$ and $\mathbf{V}_1$: II

- oxygen isotope record series $\mathbf{X}$ has $N = 352$ observations
- spacing between observations is $\Delta \doteq 0.5$ years
- used Haar DWT, obtaining 176 scaling and wavelet coefficients
- scaling coefficients $\mathbf{V}_1$ related to averages on scale of 2Δ
- wavelet coefficients $\mathbf{W}_1$ related to changes on scale of Δ
- coefficients $V_{1,t}$ and $W_{1,t}$ plotted against mid-point of years associated with X_{2t} and X_{2t+1}
- note: variability in wavelet coefficients increasing with time (thought to be due to diffusion)
- data courtesy of Lars Karlöf, Norwegian Polar Institute, Polar Environmental Centre, Tromsø, Norway

Reconstructing $\mathbf{X}$ from $\mathbf{W}_1$ and $\mathbf{V}_1$

- in matrix notation, form wavelet & scaling coefficients via

$$\begin{bmatrix}\mathbf{W}_1\\ \mathbf{V}_1\end{bmatrix} = \begin{bmatrix}\mathcal{W}_1\mathbf{X}\\ \mathcal{V}_1\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathcal{W}_1\\ \mathcal{V}_1\end{bmatrix}\mathbf{X} = \mathcal{P}_1\mathbf{X}$$

- recall that $\mathcal{P}_1^T\mathcal{P}_1 = I_N$ because $\mathcal{P}_1$ is orthonormal
- since $\mathcal{P}_1^T\mathcal{P}_1\mathbf{X} = \mathbf{X}$, premultiplying both sides by $\mathcal{P}_1^T$ yields

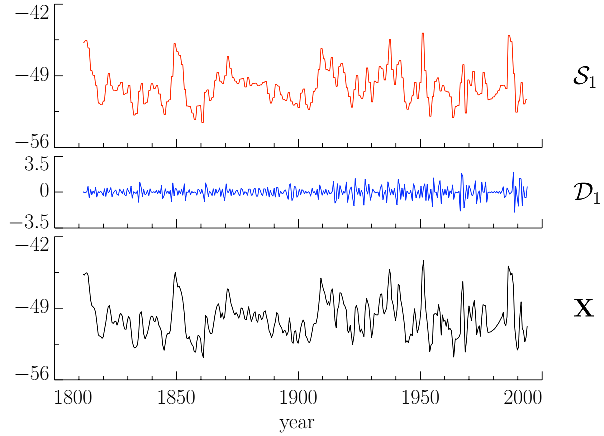
$$\mathcal{P}_1^T\begin{bmatrix}\mathbf{W}_1\\ \mathbf{V}_1\end{bmatrix} = [\mathcal{W}_1^T\ \mathcal{V}_1^T]\begin{bmatrix}\mathbf{W}_1\\ \mathbf{V}_1\end{bmatrix} = \mathcal{W}_1^T\mathbf{W}_1 + \mathcal{V}_1^T\mathbf{V}_1 = \mathbf{X}$$

- $\mathcal{D}_1 \equiv \mathcal{W}_1^T\mathbf{W}_1$ is the first level detail
- $\mathcal{S}_1 \equiv \mathcal{V}_1^T\mathbf{V}_1$ is the first level 'smooth'
- $\mathbf{X} = \mathcal{D}_1 + \mathcal{S}_1$ in this notation

WMTSA: 80–81

Example of Synthesizing $\mathbf{X}$ from $\mathcal{D}_1$ and $\mathcal{S}_1$

- Haar-based decomposition for oxygen isotope records $\mathbf{X}$



First Level Variance Decomposition: I

- recall that 'energy' in $\mathbf{X}$ is its squared norm $\|\mathbf{X}\|^2$
- because $\mathcal{P}_1$ is orthonormal, have $\mathcal{P}_1^T\mathcal{P}_1 = I_N$ and hence

$$\|\mathcal{P}_1\mathbf{X}\|^2 = (\mathcal{P}_1\mathbf{X})^T\mathcal{P}_1\mathbf{X} = \mathbf{X}^T\mathcal{P}_1^T\mathcal{P}_1\mathbf{X} = \mathbf{X}^T\mathbf{X} = \|\mathbf{X}\|^2$$

- can conclude that $\|\mathbf{X}\|^2 = \|\mathbf{W}_1\|^2 + \|\mathbf{V}_1\|^2$ because

$$\mathcal{P}_1\mathbf{X} = \begin{bmatrix}\mathbf{W}_1\\ \mathbf{V}_1\end{bmatrix} \text{ and hence } \|\mathcal{P}_1\mathbf{X}\|^2 = \|\mathbf{W}_1\|^2 + \|\mathbf{V}_1\|^2$$

- leads to a decomposition of the sample variance for $\mathbf{X}$:

$$\hat{\sigma}_X^2 \equiv \frac{1}{N}\sum_{t=0}^{N-1}\left(X_t - \overline{X}\right)^2 = \frac{1}{N}\|\mathbf{X}\|^2 - \overline{X}^2$$
$$= \frac{1}{N}\|\mathbf{W}_1\|^2 + \frac{1}{N}\|\mathbf{V}_1\|^2 - \overline{X}^2$$

First Level Variance Decomposition: II

- breaks up $\hat{\sigma}_X^2$ into two pieces:
  1. $\frac{1}{N}\|\mathbf{W}_1\|^2$, attributable to changes in averages over scale 1
  2. $\frac{1}{N}\|\mathbf{V}_1\|^2 - \overline{X}^2$, attributable to averages over scale 2
- Haar-based example for oxygen isotope records
  - first piece: $\frac{1}{N}\|\mathbf{W}_1\|^2 \doteq 0.295$
  - second piece: $\frac{1}{N}\|\mathbf{V}_1\|^2 - \overline{X}^2 \doteq 2.909$
  - sample variance: $\hat{\sigma}_X^2 \doteq 3.204$
  - changes on scale of $\Delta \doteq 0.5$ years account for 9% of $\hat{\sigma}_X^2$ (standardized scale 1 corresponds to physical scale Δ)

Summary of First Level of Basic Algorithm

- transforms $\{X_t : t = 0, \ldots, N-1\}$ into 2 types of coefficients
- $N/2$ wavelet coefficients $\{W_{1,t}\}$ associated with:
  - $\mathbf{W}_1$, a vector consisting of first $N/2$ elements of $\mathbf{W}$
  - changes on scale 1 and nominal frequencies $\frac{1}{4} \leq |f| \leq \frac{1}{2}$
  - first level detail $\mathcal{D}_1$
  - $\mathcal{W}_1$, an $\frac{N}{2} \times N$ matrix consisting of first $\frac{N}{2}$ rows of $\mathcal{W}$
- $N/2$ scaling coefficients $\{V_{1,t}\}$ associated with:
  - $\mathbf{V}_1$, a vector of length $N/2$
  - averages on scale 2 and nominal frequencies $0 \leq |f| \leq \frac{1}{4}$
  - first level smooth $\mathcal{S}_1$
  - $\mathcal{V}_1$, an $\frac{N}{2} \times N$ matrix spanning same subspace as last $N/2$ rows of $\mathcal{W}$

Constructing Remaining DWT Coefficients: I

- have regarded time series X_t as 'one point' averages $\overline{X}_t(1)$ over scale of 1
- first level of basic algorithm transforms $\mathbf{X}$ of length N into
  - $N/2$ wavelet coefficients $\mathbf{W}_1 \propto$ changes on a scale of 1
  - $N/2$ scaling coefficients $\mathbf{V}_1 \propto$ averages of X_t on a scale of 2
- in essence basic algorithm takes length N series $\mathbf{X}$ related to scale 1 averages and produces
  - length $N/2$ series $\mathbf{W}_1$ associated with the same scale
  - length $N/2$ series $\mathbf{V}_1$ related to averages on double the scale

Constructing Remaining DWT Coefficients: II

- Q: what if we now treat $\mathbf{V}_1$ in the same manner as $\mathbf{X}$?
- basic algorithm will transform length $N/2$ series $\mathbf{V}_1$ into
  - length $N/4$ series $\mathbf{W}_2$ associated with the same scale (2)
  - length $N/4$ series $\mathbf{V}_2$ related to averages on twice the scale
- by definition, $\mathbf{W}_2$ contains the level 2 wavelet coefficients
- Q: what if we treat $\mathbf{V}_2$ in the same way?
- basic algorithm will transform length $N/4$ series $\mathbf{V}_2$ into
  - length $N/8$ series $\mathbf{W}_3$ associated with the same scale (4)
  - length $N/8$ series $\mathbf{V}_3$ related to averages on twice the scale
- by definition, $\mathbf{W}_3$ contains the level 3 wavelet coefficients

Constructing Remaining DWT Coefficients: III

- continuing in this manner defines remaining subvectors of $\mathbf{W}$ (recall that $\mathbf{W} = \mathcal{W}\mathbf{X}$ is the vector of DWT coefficients)
- at each level j, outputs $\mathbf{W}_j$ and $\mathbf{V}_j$ from the basic algorithm are each half the length of the input $\mathbf{V}_{j-1}$
- length of $\mathbf{V}_j$ given by $N/2^j$
- since $N = 2^J$, length of $\mathbf{V}_J$ is 1, at which point we must stop
- J applications of the basic algorithm *defines* the remaining subvectors $\mathbf{W}_2, \ldots, \mathbf{W}_J, \mathbf{V}_J$ of DWT coefficient vector $\mathbf{W}$
- overall scheme is known as the 'pyramid' algorithm

Scales Associated with DWT Coefficients

- jth level of algorithm transforms scale 2^{j-1} averages into
  - differences of averages on scale 2^{j-1}, i.e., wavelet coefficients $\mathbf{W}_j$
  - averages on scale $2 \times 2^{j-1} = 2^j$, i.e., scaling coefficients $\mathbf{V}_j$
- $\tau_j \equiv 2^{j-1}$ denotes scale associated with $\mathbf{W}_j$
  - for $j = 1, \ldots, J$, takes on values $1, 2, 4, \ldots, N/4, N/2$
- $\lambda_j \equiv 2^j = 2\tau_j$ denotes scale associated with $\mathbf{V}_j$
  - takes on values $2, 4, 8, \ldots, N/2, N$

Matrix Description of Pyramid Algorithm: I

- form $\frac{N}{2^j} \times \frac{N}{2^{j-1}}$ matrix $\mathcal{B}_j$ in same way as $\frac{N}{2} \times N$ matrix $\mathcal{W}_1$
- when $L = 4$ and $N/2^{j-1} = 16$, have

$$\mathcal{B}_j = \begin{bmatrix}
h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 \\
h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0
\end{bmatrix}$$

- matrix gets us jth level wavelet coefficients via $\mathbf{W}_j = \mathcal{B}_j \mathbf{V}_{j-1}$

Matrix Description of Pyramid Algorithm: II

- form $\frac{N}{2^j} \times \frac{N}{2^{j-1}}$ matrix $\mathcal{A}_j$ in same way as $\frac{N}{2} \times N$ matrix $\mathcal{V}_1$
- when $L = 4$ and $N/2^{j-1} = 16$, have

$$\mathcal{A}_j = \begin{bmatrix}
g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 \\
g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0
\end{bmatrix}$$

- matrix gets us jth level scaling coefficients via $\mathbf{V}_j = \mathcal{A}_j \mathbf{V}_{j-1}$

Matrix Description of Pyramid Algorithm: III

- if we define $\mathbf{V}_0 = \mathbf{X}$ and let $j = 1$, then

  $\mathbf{W}_j = \mathcal{B}_j \mathbf{V}_{j-1}$ reduces to $\mathbf{W}_1 = \mathcal{B}_1 \mathbf{V}_0 = \mathcal{B}_1 \mathbf{X} = \mathcal{W}_1 \mathbf{X}$

  because $\mathcal{B}_1$ has the same definition as $\mathcal{W}_1$
- likewise, when $j = 1$,

  $\mathbf{V}_j = \mathcal{A}_j \mathbf{V}_{j-1}$ reduces to $\mathbf{V}_1 = \mathcal{A}_1 \mathbf{V}_0 = \mathcal{A}_1 \mathbf{X} = \mathcal{V}_1 \mathbf{X}$

  because $\mathcal{A}_1$ has the same definition as $\mathcal{V}_1$

Formation of Submatrices of $\mathcal{W}$: I

- using $\mathbf{V}_j = \mathcal{A}_j \mathbf{V}_{j-1}$ repeatedly and $\mathbf{V}_1 = \mathcal{A}_1 \mathbf{X}$, can write

$$\begin{aligned}\mathbf{W}_j &= \mathcal{B}_j \mathbf{V}_{j-1} \\ &= \mathcal{B}_j \mathcal{A}_{j-1} \mathbf{V}_{j-2} \\ &= \mathcal{B}_j \mathcal{A}_{j-1} \mathcal{A}_{j-2} \mathbf{V}_{j-3} \\ &= \mathcal{B}_j \mathcal{A}_{j-1} \mathcal{A}_{j-2} \cdots \mathcal{A}_1 \mathbf{X} \equiv \mathcal{W}_j \mathbf{X},\end{aligned}$$

  where $\mathcal{W}_j$ is $\frac{N}{2^j} \times N$ submatrix of $\mathcal{W}$ responsible for $\mathbf{W}_j$
- likewise, can get $1 \times N$ submatrix $\mathcal{V}_J$ responsible for $\mathbf{V}_J$

$$\begin{aligned}\mathbf{V}_J &= \mathcal{A}_J \mathbf{V}_{J-1} \\ &= \mathcal{A}_J \mathcal{A}_{J-1} \mathbf{V}_{J-2} \\ &= \mathcal{A}_J \mathcal{A}_{J-1} \mathcal{A}_{J-2} \mathbf{V}_{J-3} \\ &= \mathcal{A}_J \mathcal{A}_{J-1} \mathcal{A}_{J-2} \cdots \mathcal{A}_1 \mathbf{X} \equiv \mathcal{V}_J \mathbf{X}\end{aligned}$$

- $\mathcal{V}_J$ is the last row of $\mathcal{W}$, & all its elements are equal to $1/\sqrt{N}$

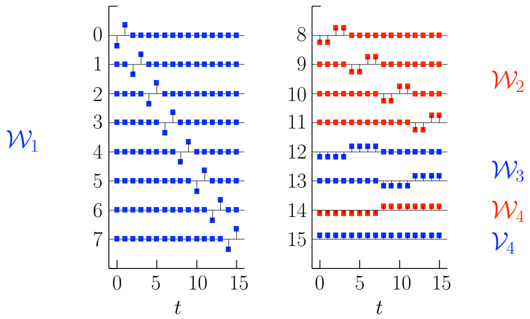
Formation of Submatrices of $\mathcal{W}$: II

- have now constructed all of DWT matrix:

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{W}_2 \\ \mathcal{W}_3 \\ \mathcal{W}_4 \\ \vdots \\ \mathcal{W}_j \\ \vdots \\ \mathcal{W}_J \\ \mathcal{V}_J \end{bmatrix} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \mathcal{A}_1 \\ \mathcal{B}_3 \mathcal{A}_2 \mathcal{A}_1 \\ \mathcal{B}_4 \mathcal{A}_3 \mathcal{A}_2 \mathcal{A}_1 \\ \vdots \\ \mathcal{B}_j \mathcal{A}_{j-1} \cdots \mathcal{A}_1 \\ \vdots \\ \mathcal{B}_J \mathcal{A}_{J-1} \cdots \mathcal{A}_1 \\ \mathcal{A}_J \mathcal{A}_{J-1} \cdots \mathcal{A}_1 \end{bmatrix}$$

Examples of $\mathcal{W}$ and its Partitioning: I

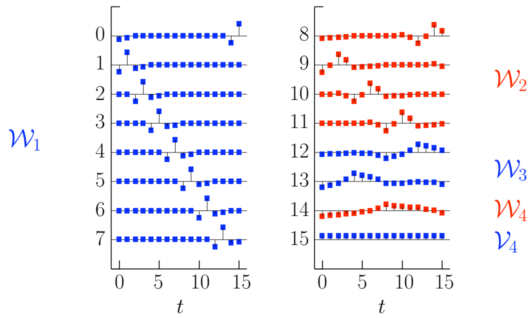
- $N = 16$ case for Haar DWT matrix $\mathcal{W}$



- above agrees with qualitative description given previously

Examples of $\mathcal{W}$ and its Partitioning: II

- $N = 16$ case for D(4) DWT matrix $\mathcal{W}$



- note: elements of last row equal to $1/\sqrt{N} = 1/4$, as claimed

Partial DWT: I

- J repetitions of pyramid algorithm for $\mathbf{X}$ of length $N = 2^J$ yields 'complete' DWT, i.e., $\mathbf{W} = \mathcal{W}\mathbf{X}$
- can choose to stop at $J_0 < J$ repetitions, yielding a 'partial' DWT of level J_0:

$$\begin{bmatrix} \mathcal{W}_1 \\ \mathcal{W}_2 \\ \vdots \\ \mathcal{W}_j \\ \vdots \\ \mathcal{W}_{J_0} \\ \mathcal{V}_{J_0} \end{bmatrix} \mathbf{X} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2\mathcal{A}_1 \\ \vdots \\ \mathcal{B}_j\mathcal{A}_{j-1}\cdots\mathcal{A}_1 \\ \vdots \\ \mathcal{B}_{J_0}\mathcal{A}_{J_0-1}\cdots\mathcal{A}_1 \\ \mathcal{A}_{J_0}\mathcal{A}_{J_0-1}\cdots\mathcal{A}_1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_j \\ \vdots \\ \mathbf{W}_{J_0} \\ \mathbf{V}_{J_0} \end{bmatrix}$$

- $\mathcal{V}_{J_0}$ is $\frac{N}{2^{J_0}} \times N$, yielding $\frac{N}{2^{J_0}}$ coefficients for scale $\lambda_{J_0} = 2^{J_0}$

Partial DWT: II

- only requires N to be integer multiple of 2^{J_0}
- partial DWT more common than complete DWT
- choice of J_0 is application dependent
- multiresolution analysis for partial DWT:

$$\mathbf{X} = \sum_{j=1}^{J_0} \mathcal{D}_j + \mathcal{S}_{J_0}$$

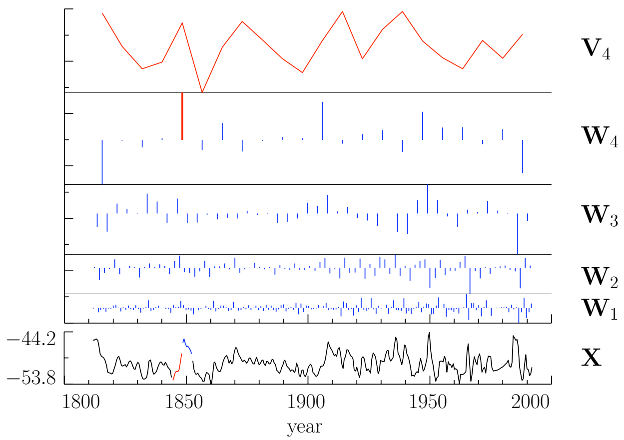
$\mathcal{S}_{J_0}$ represents averages on scale $\lambda_{J_0} = 2^{J_0}$ (includes $\overline{X}$)

- analysis of variance for partial DWT:

$$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{j=1}^{J_0} \|\mathbf{W}_j\|^2 + \frac{1}{N}\|\mathbf{V}_{J_0}\|^2 - \overline{X}^2$$

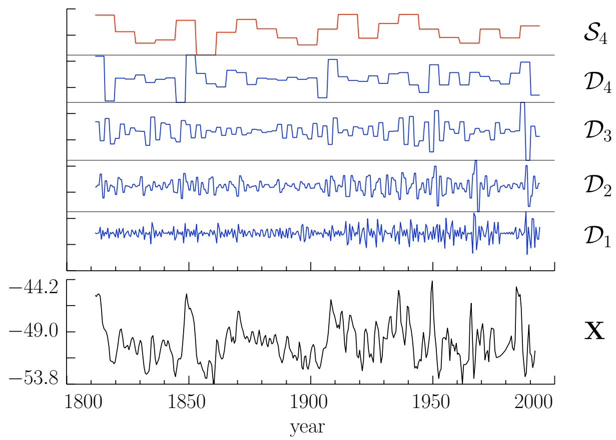
Example of $J_0 = 4$ Partial Haar DWT

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core



Example of MRA from $J_0 = 4$ Partial Haar DWT

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core



Example of Variance Decomposition

- decomposition of sample variance from $J_0 = 4$ partial DWT

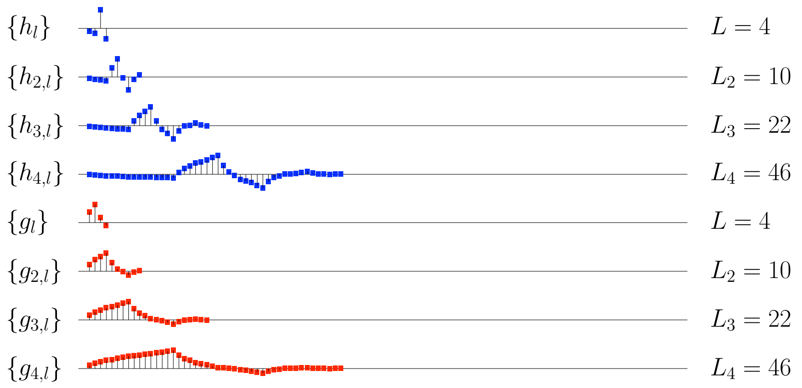
$$\hat{\sigma}_X^2 \equiv \frac{1}{N}\sum_{t=0}^{N-1}\left(X_t - \overline{X}\right)^2 = \sum_{j=1}^{4}\frac{1}{N}\|\mathbf{W}_j\|^2 + \frac{1}{N}\|\mathbf{V}_4\|^2 - \overline{X}^2$$

- Haar-based example for oxygen isotope records
  - 0.5 year changes: $\frac{1}{N}\|\mathbf{W}_1\|^2 \doteq 0.295$ ($\doteq$ 9.2% of $\hat{\sigma}_X^2$)
  - 1.0 years changes: $\frac{1}{N}\|\mathbf{W}_2\|^2 \doteq 0.464$ ($\doteq$ 14.5%)
  - 2.0 years changes: $\frac{1}{N}\|\mathbf{W}_3\|^2 \doteq 0.652$ ($\doteq$ 20.4%)
  - 4.0 years changes: $\frac{1}{N}\|\mathbf{W}_4\|^2 \doteq 0.846$ ($\doteq$ 26.4%)
  - 8.0 years averages: $\frac{1}{N}\|\mathbf{V}_4\|^2 - \overline{X}^2 \doteq 0.947$ ($\doteq$ 29.5%)
  - sample variance: $\hat{\sigma}_X^2 \doteq 3.204$

Haar Equivalent Wavelet & Scaling Filters

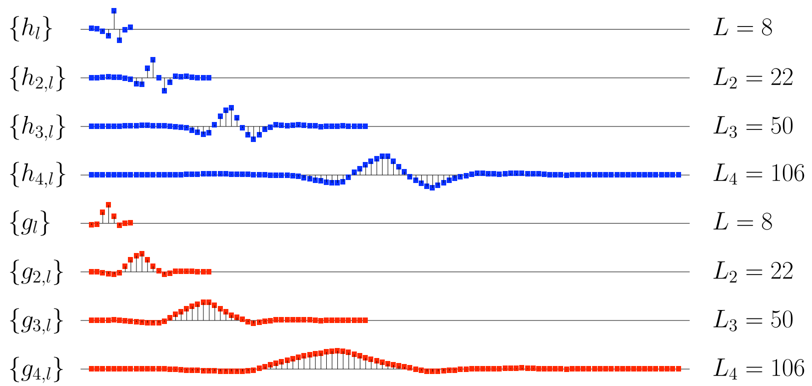
- $L_j = 2^j$ is width of $\{h_{j,l}\}$ and $\{g_{j,l}\}$
- note: convenient to define $\{h_{1,l}\}$ to be same as $\{h_l\}$

D(4) Equivalent Wavelet & Scaling Filters



- L_j dictated by general formula $L_j = (2^j - 1)(L - 1) + 1$, but can argue that *effective* width is 2^j (same as Haar L_j)

LA(8) Equivalent Wavelet & Scaling Filters

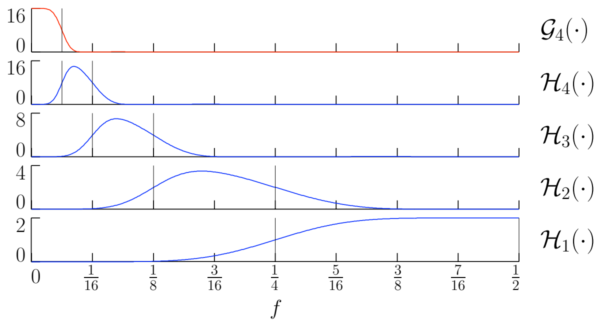


Squared Gain Functions for Filters

- squared gain functions give us frequency domain properties:

$$\mathcal{H}_j(f) \equiv |H_j(f)|^2 \text{ and } \mathcal{G}_j(f) \equiv |G_j(f)|^2$$

- example: squared gain functions for LA(8) $J_0 = 4$ partial DWT



Maximal Overlap Discrete Wavelet Transform

- abbreviation is MODWT (pronounced 'mod WT')
- transforms very similar to the MODWT have been studied in the literature under the following names:
  - undecimated DWT (or nondecimated DWT)
  - stationary DWT
  - translation invariant DWT
  - time invariant DWT
  - redundant DWT
- also related to notions of 'wavelet frames' and 'cycle spinning'
- basic idea: use values removed from DWT by downsampling

Quick Comparison of the MODWT to the DWT

- unlike the DWT, MODWT is not orthonormal (in fact MODWT is highly redundant)
- unlike the DWT, MODWT is defined naturally for all samples sizes (i.e., N need not be a multiple of a power of two)
- similar to the DWT, can form multiresolution analyses (MRAs) using MODWT with certain additional desirable features; e.g., unlike the DWT, MODWT-based MRA has details and smooths that shift along with $\mathbf{X}$ (if $\mathbf{X}$ has detail $\widetilde{\mathcal{D}}_j$, then $\mathcal{T}^m\mathbf{X}$ has detail $\mathcal{T}^m\widetilde{\mathcal{D}}_j$, where $\mathcal{T}^m$ circularly shifts $\mathbf{X}$ by m units)
- similar to the DWT, an analysis of variance (ANOVA) can be based on MODWT wavelet coefficients
- unlike the DWT, MODWT discrete wavelet power spectrum same for $\mathbf{X}$ and its circular shifts $\mathcal{T}^m\mathbf{X}$

Definition of MODWT Coefficients: I

- define MODWT filters $\{\tilde{h}_{j,l}\}$ and $\{\tilde{g}_{j,l}\}$ by renormalizing the DWT filters:

$$\tilde{h}_{j,l} = h_{j,l}/2^{j/2} \text{ and } \tilde{g}_{j,l} = g_{j,l}/2^{j/2}$$

- level j MODWT wavelet and scaling coefficients are *defined* to be output obtaining by filtering $\mathbf{X}$ with $\{\tilde{h}_{j,l}\}$ and $\{\tilde{g}_{j,l}\}$:

$$\mathbf{X} \longrightarrow \boxed{\{\tilde{h}_{j,l}\}} \longrightarrow \widetilde{\mathbf{W}}_j \text{ and } \mathbf{X} \longrightarrow \boxed{\{\tilde{g}_{j,l}\}} \longrightarrow \widetilde{\mathbf{V}}_j$$

- compare the above to its DWT equivalent:

$$\mathbf{X} \longrightarrow \boxed{\{h_{j,l}\}} \underset{\downarrow 2^j}{\longrightarrow} \mathbf{W}_j \text{ and } \mathbf{X} \longrightarrow \boxed{\{g_{j,l}\}} \underset{\downarrow 2^j}{\longrightarrow} \mathbf{V}_j$$

- level J_0 MODWT consists of $J_0 + 1$ vectors, namely,

$$\widetilde{\mathbf{W}}_1, \widetilde{\mathbf{W}}_2, \ldots, \widetilde{\mathbf{W}}_{J_0} \text{ and } \widetilde{\mathbf{V}}_{J_0},$$

each of which has length N

Definition of MODWT Coefficients: II

- MODWT of level J_0 has $(J_0 + 1)N$ coefficients, whereas DWT has N coefficients for any given J_0
- whereas DWT of level J_0 requires N to be integer multiple of 2^{J_0}, MODWT of level J_0 is well-defined for *any* sample size N
- when N is divisible by 2^{J_0}, we can write

$$W_{j,t} = \sum_{l=0}^{L_j-1} h_{j,l} X_{2^j(t+1)-1-l \bmod N} \quad \& \quad \widetilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N},$$

and we have the relationship

$$W_{j,t} = 2^{j/2}\widetilde{W}_{j,2^j(t+1)-1} \text{ \&, likewise, } V_{J_0,t} = 2^{J_0/2}\widetilde{V}_{J_0,2^{J_0}(t+1)-1}$$

(here $\widetilde{W}_{j,t}$ & $\widetilde{V}_{J_0,t}$ denote the tth elements of $\widetilde{\mathbf{W}}_j$ & $\widetilde{\mathbf{V}}_{J_0}$)

Properties of the MODWT

- as was true with the DWT, we can use the MODWT to obtain
  - a scale-based additive decomposition (MRA):

$$\mathbf{X} = \sum_{j=1}^{J_0} \widetilde{\mathcal{D}}_j + \widetilde{\mathcal{S}}_{J_0}$$

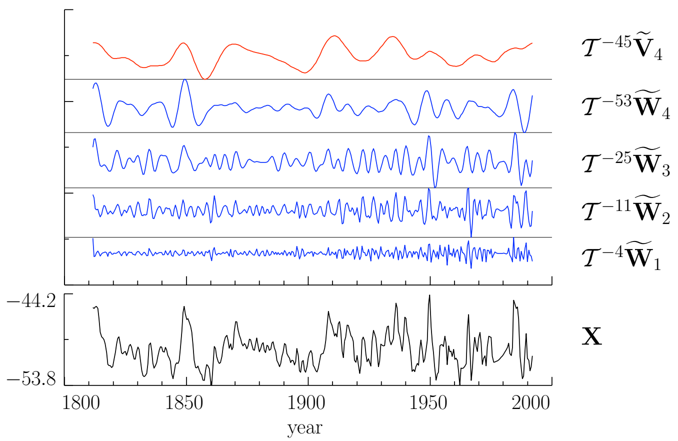
  - a scale-based energy decomposition (basis for ANOVA):

$$\|\mathbf{X}\|^2 = \sum_{j=1}^{J_0} \|\widetilde{\mathbf{W}}_j\|^2 + \|\widetilde{\mathbf{V}}_{J_0}\|^2$$

- in addition, the MODWT can be computed efficiently via a pyramid algorithm

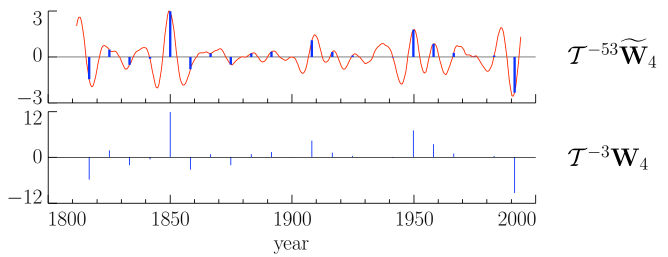
Example of $J_0 = 4$ LA(8) MODWT

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core



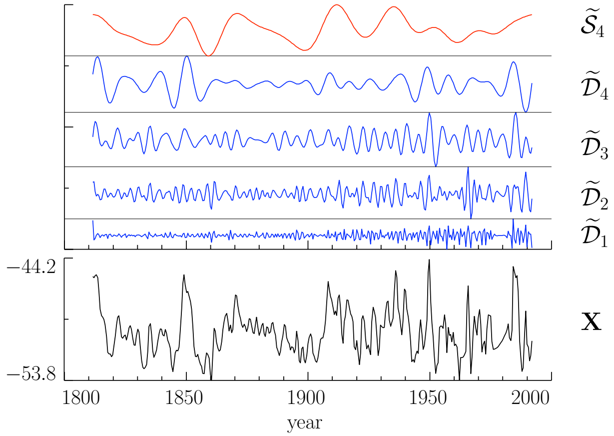
Relationship Between MODWT and DWT

- bottom plot shows $\mathbf{W}_4$ from DWT after circular shift $\mathcal{T}^{-3}$ to align coefficients properly in time
- top plot shows $\widetilde{\mathbf{W}}_4$ from MODWT and subsamples that, upon rescaling, yield $\mathbf{W}_4$ via $W_{4,t} = 4\widetilde{W}_{4,16(t+1)-1}$



Example of $J_0 = 4$ LA(8) MODWT MRA

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core



Example of Variance Decomposition

- decomposition of sample variance from MODWT

$$\hat{\sigma}_X^2 \equiv \frac{1}{N}\sum_{t=0}^{N-1}\left(X_t - \overline{X}\right)^2 = \sum_{j=1}^{4}\frac{1}{N}\|\widetilde{\mathbf{W}}_j\|^2 + \frac{1}{N}\|\widetilde{\mathbf{V}}_4\|^2 - \overline{X}^2$$

- LA(8)-based example for oxygen isotope records
  - 0.5 year changes: $\frac{1}{N}\|\widetilde{\mathbf{W}}_1\|^2 \doteq 0.145$ ($\doteq$ 4.5% of $\hat{\sigma}_X^2$)
  - 1.0 years changes: $\frac{1}{N}\|\widetilde{\mathbf{W}}_2\|^2 \doteq 0.500$ ($\doteq$ 15.6%)
  - 2.0 years changes: $\frac{1}{N}\|\widetilde{\mathbf{W}}_3\|^2 \doteq 0.751$ ($\doteq$ 23.4%)
  - 4.0 years changes: $\frac{1}{N}\|\widetilde{\mathbf{W}}_4\|^2 \doteq 0.839$ ($\doteq$ 26.2%)
  - 8.0 years averages: $\frac{1}{N}\|\widetilde{\mathbf{V}}_4\|^2 - \overline{X}^2 \doteq 0.969$ ($\doteq$ 30.2%)
  - sample variance: $\hat{\sigma}_X^2 \doteq 3.204$

Summary of Key Points about the DWT: I

- the DWT $\mathcal{W}$ is orthonormal, i.e., satisfies $\mathcal{W}^T\mathcal{W} = I_N$
- construction of $\mathcal{W}$ starts with a wavelet filter $\{h_l\}$ of even length L that by definition
  1. sums to zero; i.e., $\sum_l h_l = 0$;
  2. has unit energy; i.e., $\sum_l h_l^2 = 1$; and
  3. is orthogonal to its even shifts; i.e., $\sum_l h_l h_{l+2n} = 0$
- 2 and 3 together called orthonormality property
- wavelet filter defines a scaling filter via $g_l = (-1)^{l+1}h_{L-1-l}$
- scaling filter satisfies the orthonormality property, but sums to $\sqrt{2}$ and is also orthogonal to $\{h_l\}$; i.e., $\sum_l g_l h_{l+2n} = 0$
- while $\{h_l\}$ is a high-pass filter, $\{g_l\}$ is a low-pass filter

Summary of Key Points about the DWT: II

- $\{h_l\}$ and $\{g_l\}$ work in tandem to split time series $\mathbf{X}$ into
  - wavelet coefficients $\mathbf{W}_1$ (related to changes in averages on a unit scale) and
  - scaling coefficients $\mathbf{V}_1$ (related to averages on a scale of 2)
- $\{h_l\}$ and $\{g_l\}$ are then applied to $\mathbf{V}_1$, yielding
  - wavelet coefficients $\mathbf{W}_2$ (related to changes in averages on a scale of 2) and
  - scaling coefficients $\mathbf{V}_2$ (related to averages on a scale of 4)
- continuing beyond these first 2 levels, scaling coefficients $\mathbf{V}_{j-1}$ at level $j-1$ are transformed into wavelet and scaling coefficients $\mathbf{W}_j$ and $\mathbf{V}_j$ of scales $\tau_j = 2^{j-1}$ and $\lambda_j = 2^j$

Summary of Key Points about the DWT: III

- after J_0 repetitions, this 'pyramid' algorithm transforms time series $\mathbf{X}$ whose length N is an integer multiple of 2^{J_0} into DWT coefficients $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{J_0}$ and $\mathbf{V}_{J_0}$ (sizes of vectors are $\frac{N}{2}, \frac{N}{4}, \ldots, \frac{N}{2^{J_0}}$ and $\frac{N}{2^{J_0}}$, for a total of N coefficients in all)
- DWT coefficients lead to two basic decompositions
- first decomposition is additive and is known as a multiresolution analysis (MRA), in which $\mathbf{X}$ is reexpressed as

$$\mathbf{X} = \sum_{j=1}^{J_0} \mathcal{D}_j + \mathcal{S}_{J_0},$$

where $\mathcal{D}_j$ is a time series reflecting variations in $\mathbf{X}$ on scale τ_j, while $\mathcal{S}_{J_0}$ is a series reflecting its λ_{J_0} averages

Summary of Key Points about the DWT: IV

- second decomposition reexpresses the energy (squared norm) of $\mathbf{X}$ on a scale by scale basis, i.e.,

$$\|\mathbf{X}\|^2 = \sum_{j=1}^{J_0} \|\mathbf{W}_j\|^2 + \|\mathbf{V}_{J_0}\|^2,$$

leading to an analysis of the sample variance of $\mathbf{X}$:

$$\begin{aligned}\hat{\sigma}_X^2 &= \frac{1}{N}\sum_{t=0}^{N-1}\left(X_t - \overline{X}\right)^2 \\ &= \frac{1}{N}\sum_{j=1}^{J_0}\|\mathbf{W}_j\|^2 + \frac{1}{N}\|\mathbf{V}_{J_0}\|^2 - \overline{X}^2\end{aligned}$$

Summary of Key Points about the MODWT

- similar to the DWT, the MODWT offers
  - a scale-based multiresolution analysis
  - a scale-based analysis of the sample variance
  - a pyramid algorithm for computing the transform efficiently
- unlike the DWT, the MODWT is
  - defined for all sample sizes (no 'power of 2' restrictions)
  - unaffected by circular shifts to $\mathbf{X}$ in that coefficients, details and smooths shift along with $\mathbf{X}$
  - highly redundant in that a level J_0 transform consists of $(J_0 + 1)N$ values rather than just N
- MODWT can eliminate 'alignment' artifacts, but its redundancies are problematic for some uses