Shallow-water acoustics

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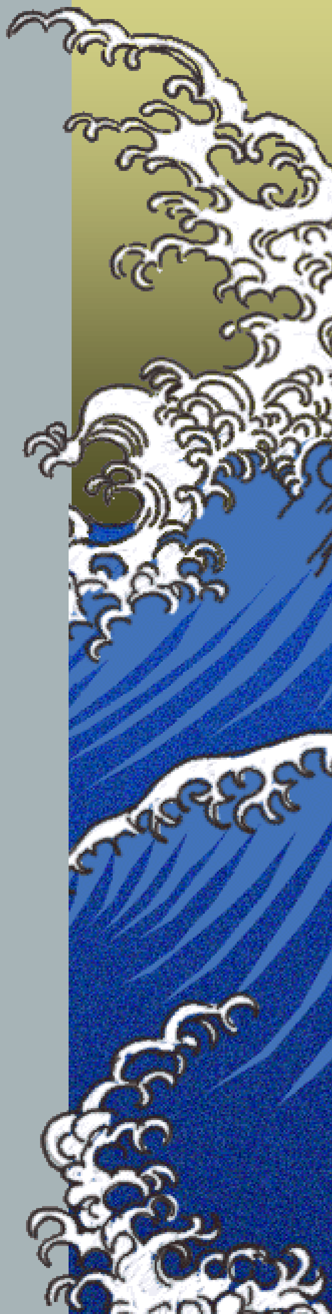


Physics Today 10/2004 “Shallow-water acoustics”

“**Cover:** In 1826, an experiment measured the speed of sound in the waters of Lake Geneva, Switzerland, as memorialized in this sketch…”

References

- Pierce, *Acoustics*, 1989
  - "The Jackson of acoustics"
- DeSanto, *Boundary value problems for scalar waves*, 1989
- Brekhovskikh and Lysanov, *Fundamentals of ocean acoustics*, 2003
- Jensen et al., *Computational ocean acoustics*, 2000
- Katsnelson and Petnikov, *Shallow-water acoustics*, 2002

Compressible fluid equations

- Equation of continuity (conservation of mass)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Equation of motion (Euler equation)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{-\nabla p}{\rho} + \mathbf{g}$$

- Equation of state (Laplace hypothesis)

$$\frac{dp}{dt} = c(\mathbf{x})^2 \frac{d\rho}{dt} \quad , \quad \text{where} \quad c(\mathbf{x}) \equiv \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

Sound

- Small oscillations in a compressible fluid:

$$\mathbf{v} = \mathbf{v_o} + \mathbf{v}'$$

$$\rho = \rho_{\mathbf{o}} + \rho'$$

$$p = p_{\mathbf{o}} + p'$$

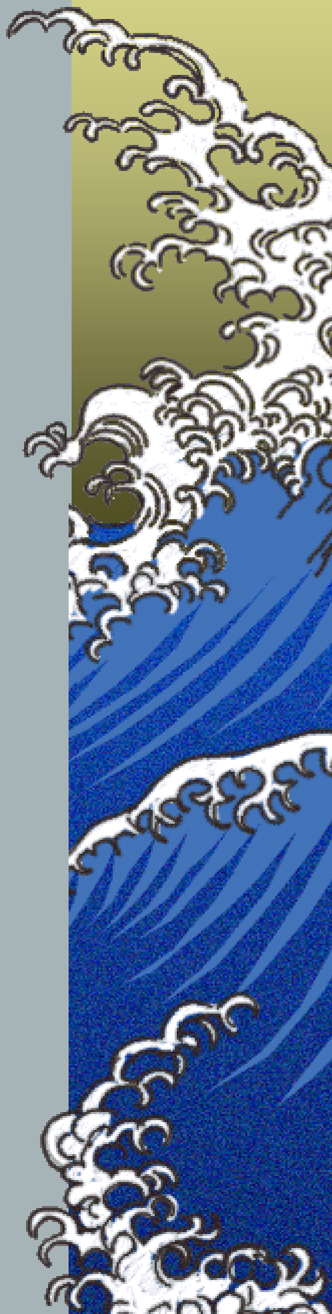
- Zeroth order fluid equations (given $\mathbf{v_o} = 0$):

$$\frac{\partial \rho_{\mathbf{o}}}{\partial t} = 0$$

$$\nabla p_o(\mathbf{x}) = \rho_o(\mathbf{x})\,\mathbf{g}$$

$$\frac{\partial p_o}{\partial t} = c(\mathbf{x})^2 \frac{\partial \rho_{\mathbf{o}}}{\partial t} \to 0$$

A static but inhomogeneous ambient background field.



Sound: first order

$$\text{(A)} \quad \frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_{\mathbf{o}} v') = 0$$

$$\text{(B)} \quad \frac{\partial v'}{\partial t} = -\frac{\nabla p'}{\rho_o} + \frac{\rho' \nabla p_o}{\rho_o^2} \xrightarrow{\text{zeroth}} -\frac{\nabla p'}{\rho_o} + \frac{g\,\rho'}{\rho_o}$$

This last term is negligible for oscillations satisfying

$$\omega >> \frac{g}{c} \rightarrow f >> 1\text{mHz}$$

$$\text{(C)} \quad \frac{\partial p'}{\partial t} = c^2 \left(\frac{\partial \rho'}{\partial t} + (v' \cdot \nabla)\rho_o \right)$$

Sound: wave equation

Taking $\partial(A)/\partial t$ and substituti ng in (B) gives

$$\frac{\partial^2 \rho'}{\partial t^2} = -\nabla \cdot \left(\rho_o \frac{\partial \mathrm{v}'}{\partial t} \right) = \nabla^2 p'$$

Taking $\partial(C)/\partial t$ and using the previous gives

$$\frac{1}{c(\mathbf{x})^2} \frac{\partial^2 p'}{\partial t^2} - \rho_o(\mathbf{x}) \nabla \cdot \left(\frac{1}{\rho_o(\mathbf{x})} \nabla p' \right) = 0$$

the Pekeris equation (1948).

Where's the Helmholtz equation?

Define

$$P(\mathbf{x},t) = P(\mathbf{x})e^{-i\omega t} = \frac{p'(\mathbf{x},t)}{\sqrt{\rho_o(\mathbf{x})}}$$

Then the Pekeris equation reduces to

$$\left(\nabla^2 + k_{eff}^2(\mathbf{x})\right)P(\mathbf{x}) = 0$$

where the effective wavenumber is given by

$$k_{eff}^2(\mathbf{x}) = \frac{\omega^2}{c^2(\mathbf{x})} - \sqrt{\rho_o(\mathbf{x})}\,\nabla^2\left(\frac{1}{\sqrt{\rho_o(\mathbf{x})}}\right)$$

Perturbation theory

- Rewrite the effective Helmholtz equation as

$$\left(H_o + V\right)P(\mathbf{x}) = 0$$

where

$$H_o \equiv \nabla^2 + k_o^2(\mathbf{x})$$

and

$$V \equiv k_{eff}^2(\mathbf{x}) - k_o^2(\mathbf{x})$$

with

$$|V| << |H_o|$$

Recall

$$k_{eff}^2(\mathbf{x}) = \frac{\omega^2}{c^2(\mathbf{x})} - \sqrt{\rho_o(\mathbf{x})}\,\nabla^2\left(\frac{1}{\sqrt{\rho_o(\mathbf{x})}}\right)$$

Solution types

- Depth dependent:

$$c(\mathbf{x}) \rightarrow c(z)$$
$$\rho_o(\mathbf{x}) \rightarrow \rho_o(z)$$

- Depth and weakly range dependent:

$$c(\mathbf{x}) \rightarrow c(r,z)$$
$$\frac{\partial c}{\partial z} >> \frac{\partial c}{\partial r} >> \frac{\partial c}{r\partial \varphi} \approx 0$$
$$\rho_o(\mathbf{x}) \rightarrow \rho_o(r,z)$$
$$\frac{\partial \rho_o}{\partial z} >> \frac{\partial \rho_o}{\partial r} >> \frac{\partial \rho_o}{r\partial \varphi} \approx 0$$

Boundary conditions

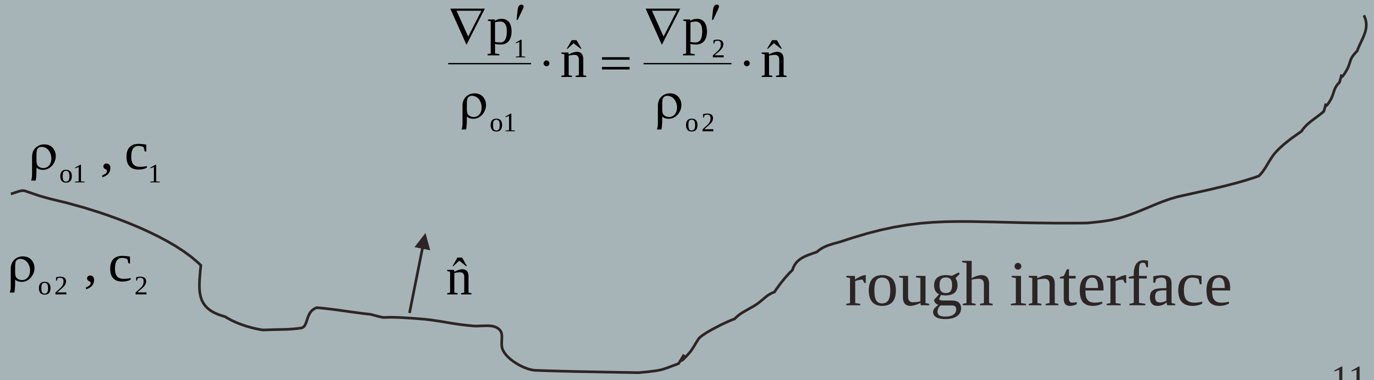
- Continuity of acoustic pressure and the normal component of acoustic velocity across interfaces:

$$p'_1 = p'_2$$

$$v'_1 \cdot \hat{n} = v'_2 \cdot \hat{n}$$

- From the first-order Euler equation, for a time harmonic source, this second condition is equivalent to

$$\frac{\nabla p'_1}{\rho_{o1}} \cdot \hat{n} = \frac{\nabla p'_2}{\rho_{o2}} \cdot \hat{n}$$



Depth-dependent solution

- The effective Helmholtz equation for a time harmonic point source in a depth-dependent background is

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\cancelto{0}{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}}+\frac{\partial^2}{\partial z^2}+k_{\mathrm{eff}}^2(z)\right)P(r,z)=\frac{S_\omega \delta(r)\delta(z-z_s)}{2\pi r}$$

- Taking a Hankel transform of this equation:

$$P(r,z)=\int_0^\infty \tilde{P}(k_r,z)\,J_0(k_r r)\,k_r\,dk_r$$

gives

$$\left(\frac{\partial^2}{\partial z^2}+k_{\mathrm{eff}}^2(z)-k_r^2\right)\tilde{P}(k_r,z)=\frac{S_\omega \delta(z-z_s)}{2\pi}$$

Depth-dependent solution cont.

- Given a constant effective wavenumber, the Weyl-Sommerfeld representation of a point source is relevant:

$$\frac{e^{ik_{eff}R}}{4\pi R} = \frac{i}{4\pi}\int_0^\infty \frac{e^{i\sqrt{k_{eff}^2 - k_r^2}\,|z-z_s|}}{\sqrt{k_{eff}^2 - k_r^2}} J_0(k_r r) k_r dk_r$$

and the so-called depth-dependent Green function becomes

$$\tilde{P}(k_r, z) = -\frac{iS_\omega}{4\pi}\frac{e^{i\sqrt{k_{eff}^2 - k_r^2}\,|z-z_s|}}{\sqrt{k_{eff}^2 - k_r^2}}$$

Weak range dependence

- In the original Pekeris equation, assume harmonic time dependence and insert the ansatz

$$p'(r,z) = \sum_m \Phi_m(r)\Psi_m(r,z)$$

with

$c(r,z)$ and $\rho_o(z)$

Define the local modes $\Psi_m(r,z)$ through

$$\rho_o(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho_o(z)}\frac{\partial \Psi_m(r,z)}{\partial z}\right) + \left(\frac{\omega^2}{c^2(r,z)} - k_{rm}^2(r)\right)\Psi_m(r,z) = 0$$

After a little algebra...

$$\sum_m \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(\Phi_m\Psi_m)}{\partial r}\right) + \sum_m k_{rm}^2(r)\Phi_m\Psi_m = \frac{S_\omega \delta(r)\delta(z-z_s)}{2\pi r}$$

Weak range dependence cont.

- Applying the operator $\int (\cdot) \frac{\Psi_n(r,z)}{\rho_o(z)} dz$ and using orthonormality on the previous gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Phi_n}{\partial r} \right) + k_{rn}^2(r)\Phi_n + 2\sum_m B_{mn} \frac{\partial \Phi_m}{\partial r} +$$

$$+ \sum_m A_{mn}\Phi_m = \frac{S_\omega \delta(r) \Psi_n(0,z_s)}{2\pi r \rho_o(z_s)}$$

where

$$A_{mn} \equiv \int \frac{1}{r}\frac{\Psi_n}{\rho_o}\frac{\partial}{\partial r}\left(r \frac{\partial \Psi_m}{\partial r} \right) dz$$

and

$$B_{mn} \equiv \int \frac{\Psi_n}{\rho_o}\frac{\partial \Psi_m}{\partial r} dz$$

Weak range dependence final

- The Born-Oppenheimer (adiabatic) approximation drops these coupling matrices, A_{mn} and B_{mn}, and we are left with

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_n}{\partial r}\right) + k_{rn}^2(r)\Phi_n = \frac{S_\omega \delta(r)\Psi_n(0,z_s)}{2\pi r \rho_o(z_s)}$$

which in the WKB approximation becomes

$$\Phi_n(r) = \frac{A}{\sqrt{r}} \frac{e^{i\int_0^r k_{rn}(r')dr'}}{\sqrt{k_{rn}(r)}}$$

where A is a constant fixed by matching to the range independent solution in the far field ($k_r r >> 1$).

A nice analogy (Dashen et al.)

Application of the Foldy–Wouthuysen transformation to the reduced wave equation in range-dependent environments

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The Foldy–Wouthuysen transformation can be used to reduce the relativistic Klein–Gordon equation to the nonrelativistic Schrödinger equation. This technique is modified and applied to the problem of wave propagation through media with a range-dependent index of refraction. The forward and backward propagating components of the field are decoupled order-by-order to produce a perturbative expansion of the range-dependent parabolic equation. The result includes energy-conserving correction terms that can be associated with a rapid fluctuation of energy between forward and backward propagating solutions of the Helmholtz equation. The approach selects out physical processes which accumulate over the entire range of propagation, distinguishing them from effects which depend solely on the initial and final values of the index of refraction and its derivatives. It is also shown that the corresponding backscatter mechanism is fundamentally nonperturbative, so that the parabolic equation technique as applied to the problem of propagation through range-dependent media generates an asymptotic expansion of the exact solution. This procedure has been applied to long-distance low-frequency propagation through a sound channel with internal waves. For this application, the expansion parameters are typically very small, so the propagation distances must be very large for the effect to be detectable. [S0001-4966(97)02203-0]

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A nice analogy cont.

- Klein-Gordon to Schrodinger equation:
  hyperbolic to parabolic equation

$$-(\partial_t)^2 \rightarrow \pm\partial_t$$

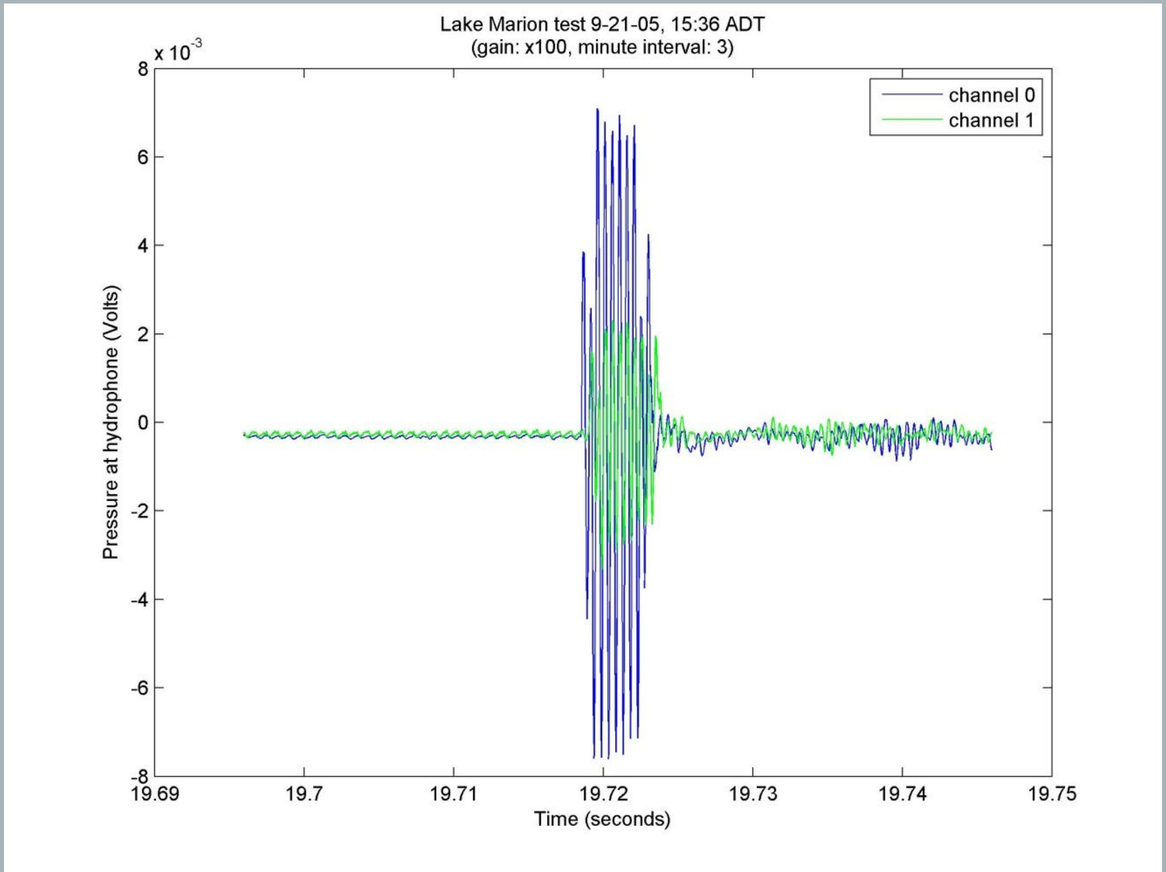
- Acoustic Helmholtz to parabolic equation (PE):
  elliptic to parabolic equation

$$(\partial_r)^2 \rightarrow \pm\partial_r$$

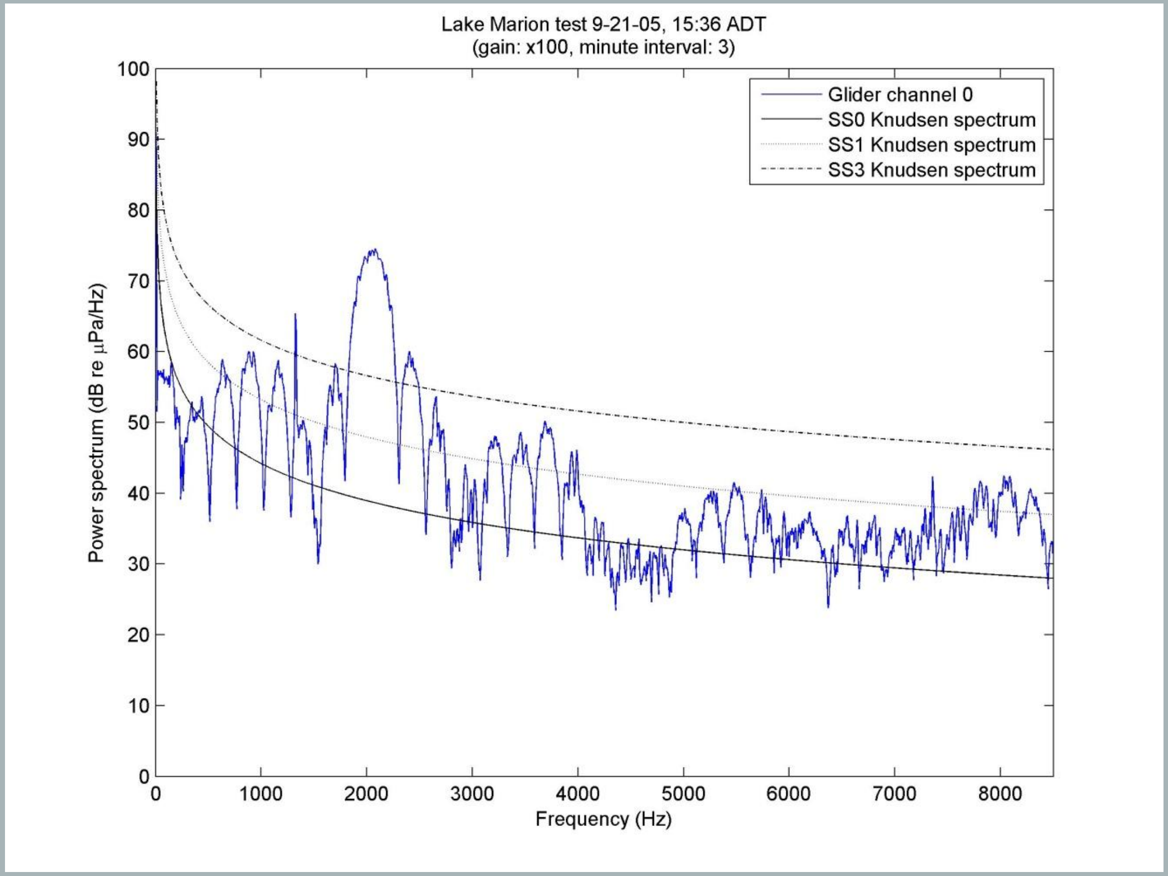
- Another way of putting it: The valence QCD (Woloshyn et al.) and ubiquitous acoustic PE method are analogous approximations. In the former, backward propagation in time is suppressed, while in the latter it is the backward propagation in horizontal range that is being suppressed. Both are nonperturbative systematic simplifications.

The application

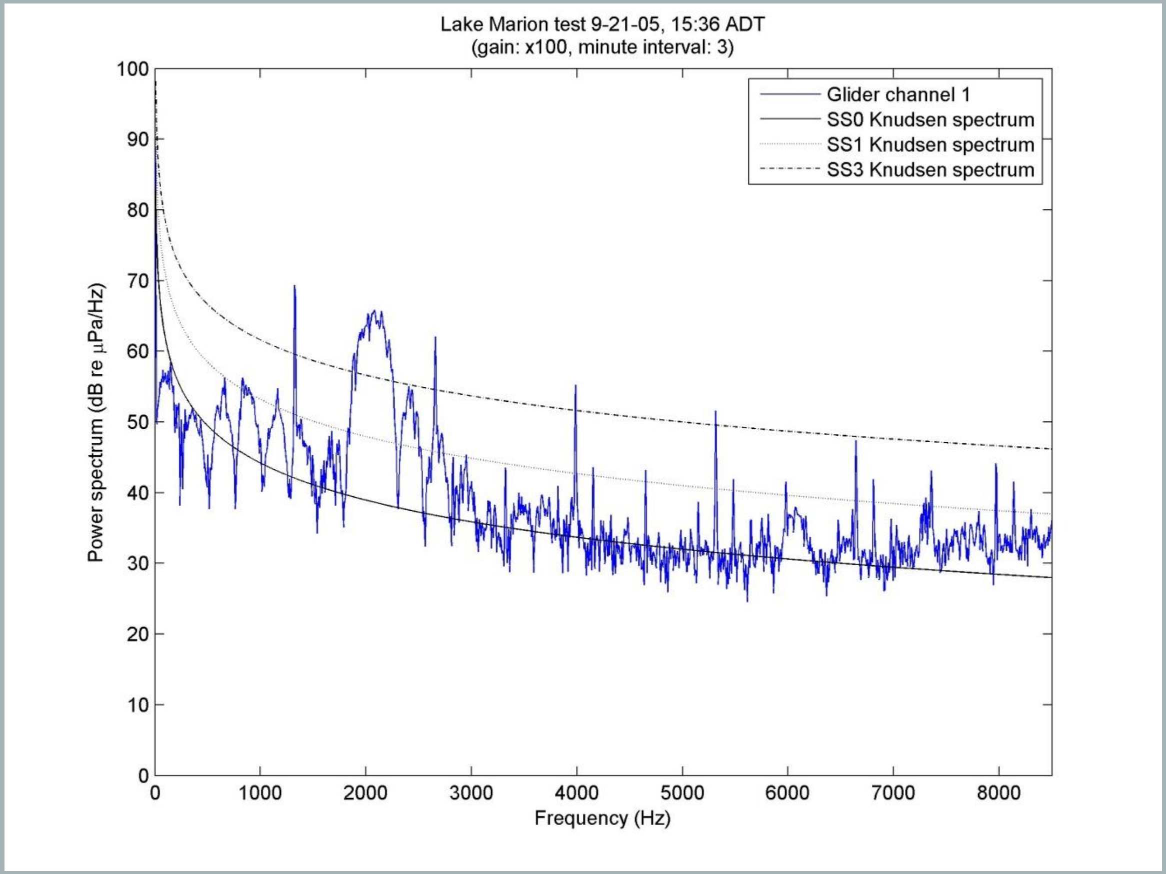
Shallow-water pressure signal



Shallow-water spectral density: channel 0



Shallow-water spectral density: channel 1



Summary

- Introduced Pekeris equation:

  Wave equation for acoustic pressure
  in a static inhomogeneous medium valid for
  frequencies much greater than a milliHertz

- Discussed methods of solution

- Presented example shallow-water spectra