Particle filters in the bearings-only tracking problem

Billy D. Jones

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Applied Mathematics, Inc.	1999-present
TRIUMF theory group	1997-1999
Ohio State nuclear theory	1992-1997

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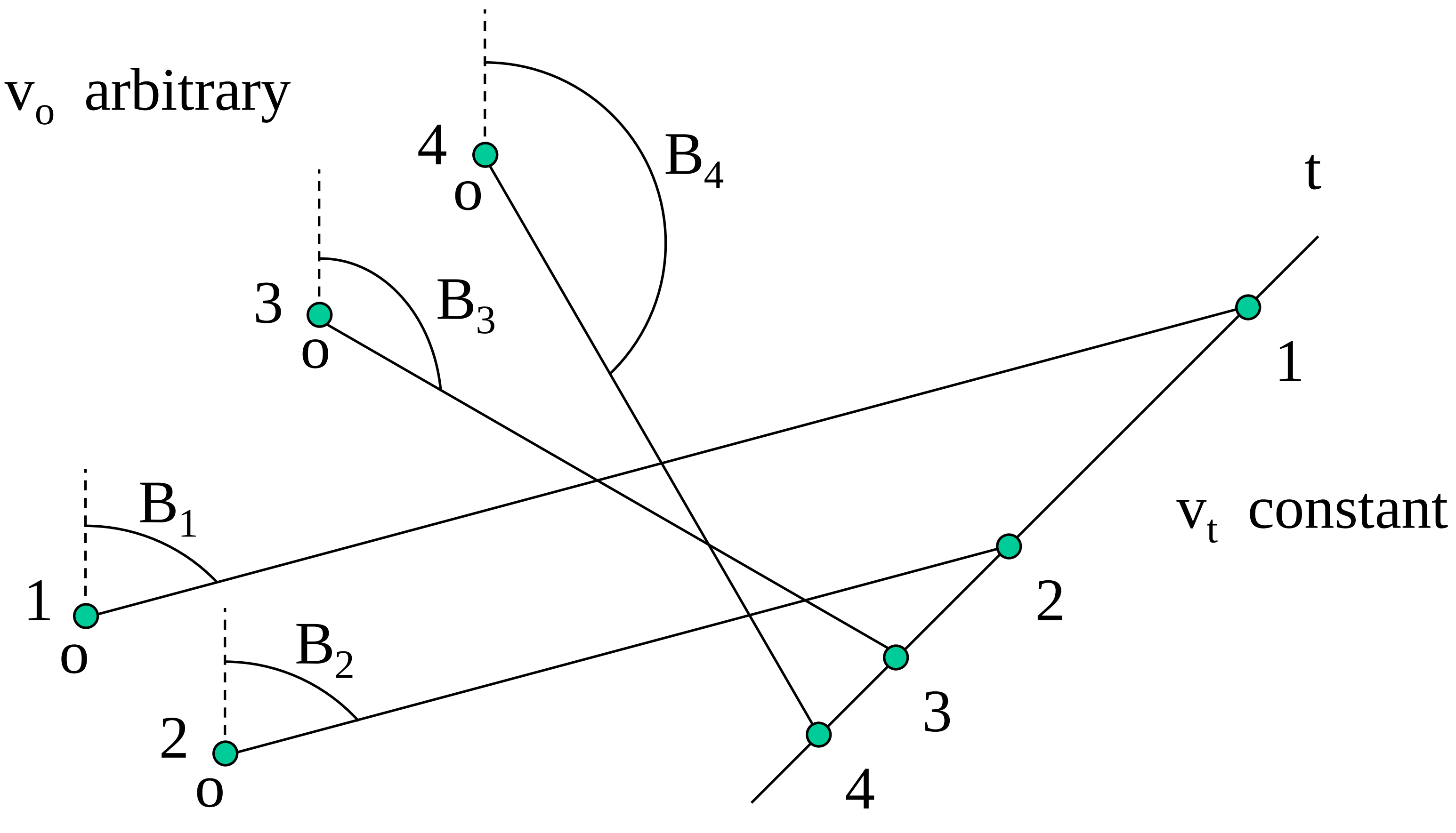
Bearings-only tracking



Four-bearing solution

- Ownship motion arbitrary (but known):

  $\mathbf{x}_{oi} = (x_{oi}, y_{oi})$, $i = \{1, 2, 3, 4\}$

- Coordinate system origin:

  $t_4 = 0$, $\mathbf{x}_{o4} = 0$

- Target velocity constant:

  $\mathbf{x}_{ti} = \mathbf{x}_{t4} + \mathbf{v}_t \, t_i$, $i = \{1, 2, 3, 4\}$

  $\mathbf{x}_{ti} = (x_{ti}, y_{ti})$, $\mathbf{v}_t = (v_{tx}, v_{ty})$

Four-bearing cont.

- Bearings $\{B_1, B_2, B_3, B_4\}$ assumed known.
- Four equations to solve:

  $\tan B_i = (x_{ti} - x_{oi})/(y_{ti} - y_{oi})$, $i = \{1, 2, 3, 4\}$

- Target state to solve for:

  $u = (R_4, v_{tx}, v_{ty})^T$

  where

  $R_4 = \sqrt{(x_{t4})^2 + (y_{t4})^2}$

Four-bearing cont.

After some algebra, this reduces to the following matrix equation:

$$A\,u = b$$

where

$$A = \begin{bmatrix} \tan B_1 \cos B_4 - \sin B_4 & -t_1 & t_1 \tan B_1 \\ \tan B_2 \cos B_4 - \sin B_4 & -t_2 & t_2 \tan B_2 \\ \tan B_3 \cos B_4 - \sin B_4 & -t_3 & t_3 \tan B_3 \end{bmatrix}$$

and

$$u = \begin{bmatrix} R_4 \\ v_{tx} \\ v_{ty} \end{bmatrix} \quad , \quad b = \begin{bmatrix} y_{o1} \tan B_1 - x_{o1} \\ y_{o2} \tan B_2 - x_{o2} \\ y_{o3} \tan B_3 - x_{o3} \end{bmatrix}$$

Four-bearing cont.

If a solution exists, it is given by

$$u = A^{-1}\, b$$

By Cramer's rule, this can be written out explicitly as

$$R_4 = \frac{\det A\,|_{\text{1st col}\to b}}{\det A}$$

$$v_{tx} = \frac{\det A\,|_{\text{2nd col}\to b}}{\det A}$$

$$v_{ty} = \frac{\det A\,|_{\text{3rd col}\to b}}{\det A}$$

We will focus on the range solution at time t_4: R_4

Range solution R_4

Writing out the determinants, and then noticing the symmetry in the equations, the previous reduces to

$$\boxed{R_4 = \frac{\varepsilon_{ijk} b_i t_j t_k \tan B_j}{\varepsilon_{i'j'k'} b_{i'4} t_{j'} t_{k'} \tan B_{j'}}}$$ (General four-bearing solution)

where repeat Latin indices are summed from 1 to 3, the non-vanishing components of the Levi-Civita symbol are given by

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{132} = -\varepsilon_{213} = -\varepsilon_{321} = 1 \ ,$$

$$b_{i4} = \cos B_4 \tan B_i - \sin B_4 \ ,$$

and

$$b_i = y_{oi} \tan B_i - x_{oi}$$

When does det A vanish?

$$R_4 = \frac{\varepsilon_{ijk} b_i t_j t_k \tan B_j}{\varepsilon_{i'j'k'} b_{i'4} t_{j'} t_{k'} \tan B_{j'}}$$

(General four-bearing solution)

If ownship velocity is constant between times t_1 to t_4 , then and only then, b_i and b_{i4} are proportional to t_i, and due to the antisymmetric properties of ε_{ijk} , this form of the four-bearing solution easily shows that the matrix A is singular.

That is, if ownship has constant velocity, the target state is unobservable, even given the four bearings B_1 through B_4.

Bearing bias error

Let Greek indices range from 1 to 4. Write the four-bearing solution as

$$R_4(B_\mu) \quad , \quad B_\mu = (B_1, B_2, B_3, B_4)$$

Average $R_4(B_\mu)$ over a 4-dimensional ball of radius ε

centered about the exact four-bearing, $\tilde{B}_\mu$, of a

given measurement:

$$\langle R_4 \rangle = \frac{\int_\varepsilon R_4(B_\mu) dB_\mu}{\int_\varepsilon dB_\mu}$$

Bearing bias error cont.

Let there be a constant bearing bias error (of either sign):

$$B_\mu = \tilde{B}_\mu \pm \frac{\varepsilon}{\sqrt{4}}(1,1,1,1)$$

Forming a Taylor series of the average integral, the first derivative term vanishes by symmetry and the leading surviving term is the generalized Laplacian:

$$\langle R_4 \rangle - R_4(\tilde{B}_\mu) \sim \frac{1}{\sqrt{4}^2\, 2!} \frac{\int \varepsilon^2 d\varepsilon}{\int d\varepsilon} \sum_{\alpha,\beta=1}^{4} \partial_\alpha \partial_\beta R_4(\tilde{B}_\mu)$$

Bearing bias error cont.

Skipping several pages of algebra, the four-bearing solution satisfies (exactly)

$$\sum_{\alpha,\beta=1}^{4} \partial_\alpha \partial_\beta R_4(\tilde{B}_\mu) = -R_4(\tilde{B}_\mu)$$

Thus the average integral's leading term becomes

$$\langle R_4 \rangle - R_4(\tilde{B}_\mu) \sim -\frac{\varepsilon^2}{24} R_4(\tilde{B}_\mu)$$

and one can say with leading order confidence:
Given constant bearing bias error, on average,
the four-bearing solution exactly underranges,
and it is a quadratic effect.

What next?

- So, what is the question we should be asking ourselves now?
- What about random noise?
  - Process noise (model error, target state error)
  - Measurement noise
- This leads to particle filters . . . .

Nonlinear filtering

A recursive solution to the nonlinear filtering problem is sought. Nonlinear problems are complicated and (usually) lead to Monte Carlo (MC) methods being used in practice.

The usual inverse square-root of MC sample size dependence of the solution variance appears. To alleviate this slow convergence rate, importance sampling (IS) techniques are used to try and zoom in on the regions of sample space that are accessible and yet close enough to the true distribution to be useful. This reduces the solution variance.

Particle filter setup

Let the target state vector be $\mathbf{x}_i$ with discrete time index $i \geq 0$, and the measurement vector be $\mathbf{z}_i$. Denote the initial target probability density by $p(\mathbf{x}_0)$, and assume that the transition density and likelihood function are known for this assumed first-order Markov process:

$$p(\mathbf{x}_i \mid \mathbf{x}_{i-1}) \quad , \quad \text{transition density}$$

$$p(\mathbf{z}_i \mid \mathbf{x}_i) \quad , \quad \text{likelihood function}$$

For the filtering problem, the desired target posterior density is

$$p(\mathbf{x}_i \mid \mathbf{Z}_i) \quad , \quad \mathbf{Z}_i = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_i\}$$

Particle filter setup cont.

Assuming target prior density $p(\mathbf{x}_{i-1}|\mathbf{Z}_{i-1})$, first $\mathbf{x}_{i-1}$ is updated via the Chapman-Kolmogorov equation:

$$p(\mathbf{x}_i \mid \mathbf{Z}_{i-1}) = \int p(\mathbf{x}_i \mid \mathbf{x}_{i-1}) p(\mathbf{x}_{i-1} \mid \mathbf{Z}_{i-1}) d\mathbf{x}_{i-1}$$

Then $\mathbf{Z}_{i-1}$ is updated via Bayes' rule and the assumption that measurements are independent:

$$\begin{aligned} p(\mathbf{x}_i \mid \mathbf{Z}_i) &= p(\mathbf{x}_i \mid \mathbf{z}_i, \mathbf{Z}_{i-1}) \\ &= \frac{p(\mathbf{z}_i \mid \mathbf{x}_i) p(\mathbf{x}_i \mid \mathbf{Z}_{i-1})}{p(\mathbf{z}_i \mid \mathbf{Z}_{i-1})} \end{aligned}$$

where the denominator of the last formula is just the integral of the numerator over $\mathbf{x}_i$.

Sequential importance sampling (SIS)

SIS is a Monte Carlo of the basic particle filtering equations of the previous slide, with the inclusion of an importance (or proposal) density $q(\mathbf{x}_i^{(j)}|...)$, where the index $j \in [1, N]$ denotes particle number, or Monte Carlo sample. In this context, a particle is a discrete uniform sample of a density, in this case the target posterior density.

SIS basic algorithm

- Loop over particles $j \in [1, N]$ (initialize norm = 0)
  - Draw $\mathbf{x}_i^{(j)} \sim q(\mathbf{x}_i \mid \mathbf{x}_{i-1}^{(j)}, \mathbf{z}_i)$
  - Update particle weight (keep track of norm)

$$\tilde{w}_i^{(j)} = \tilde{w}_{i-1}^{(j)} \frac{p(\mathbf{z}_i \mid \mathbf{x}_i^{(j)}) p(\mathbf{x}_i^{(j)} \mid \mathbf{x}_{i-1}^{(j)})}{q(\mathbf{x}_i^{(j)} \mid \mathbf{x}_{i-1}^{(j)}, \mathbf{z}_i)}$$

$$\text{norm} = \text{norm} + \tilde{w}_i^{(j)}$$

- Loop over particles $j \in [1, N]$
  - Normalize particle weights

$$w_i^{(j)} = \frac{\tilde{w}_i^{(j)}}{\text{norm}}$$

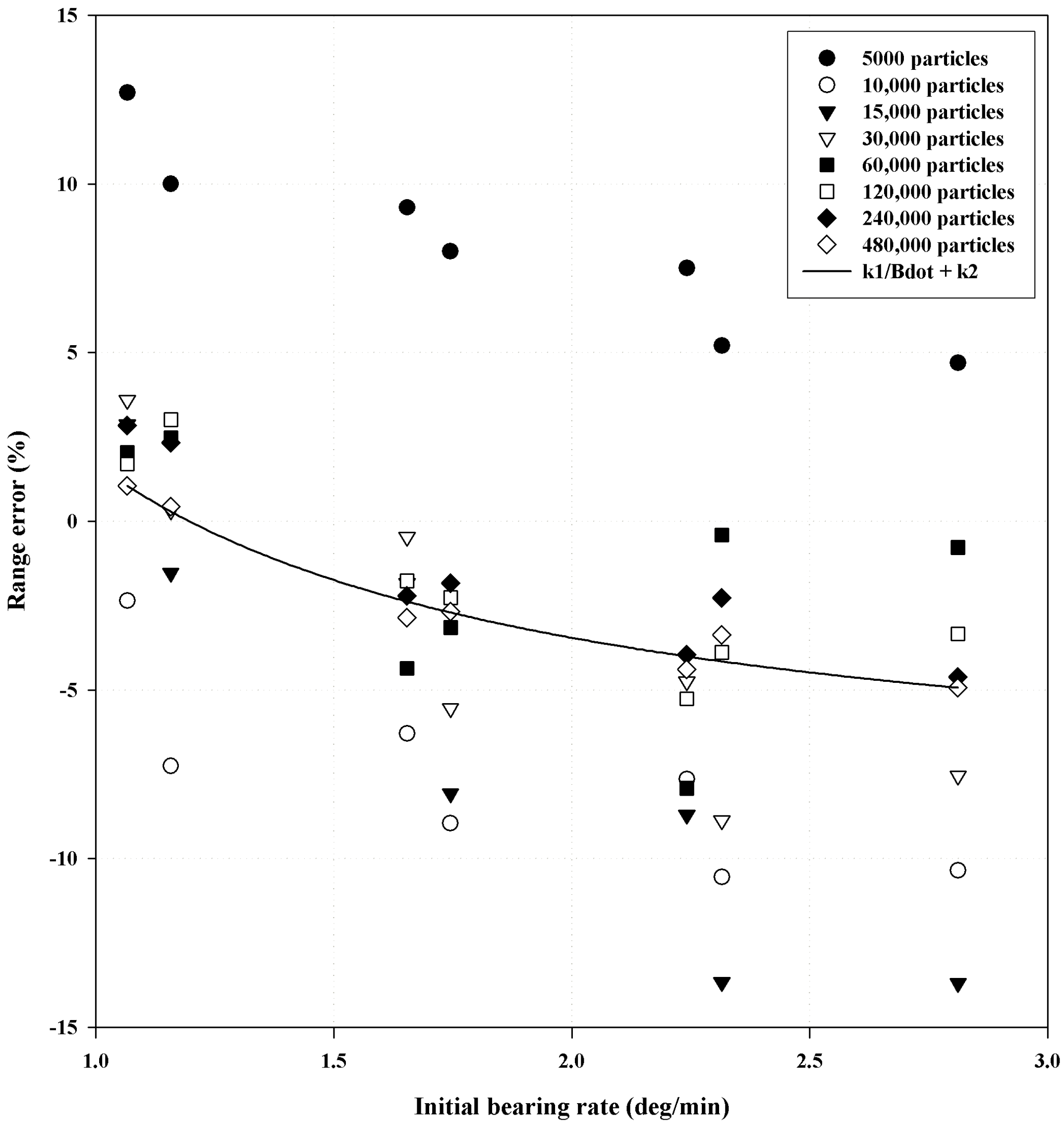
Sampling importance resampling (SIR)

- SIR = SIS + resampling
- SIS is guaranteed to fail as $t \to \infty$ (See Ref. 3). Basically the system moves to the state where all but one of the particles have negligible weight. This is the so-called degeneracy phenomenon.
- Resampling attempts to fix the degeneracy problem by redistributing the particles according to their weight: High weight particles are multiplied, and low weight particles are deleted. (One popular, efficient resampling algorithm is Kitagawa's systematic resampling.)

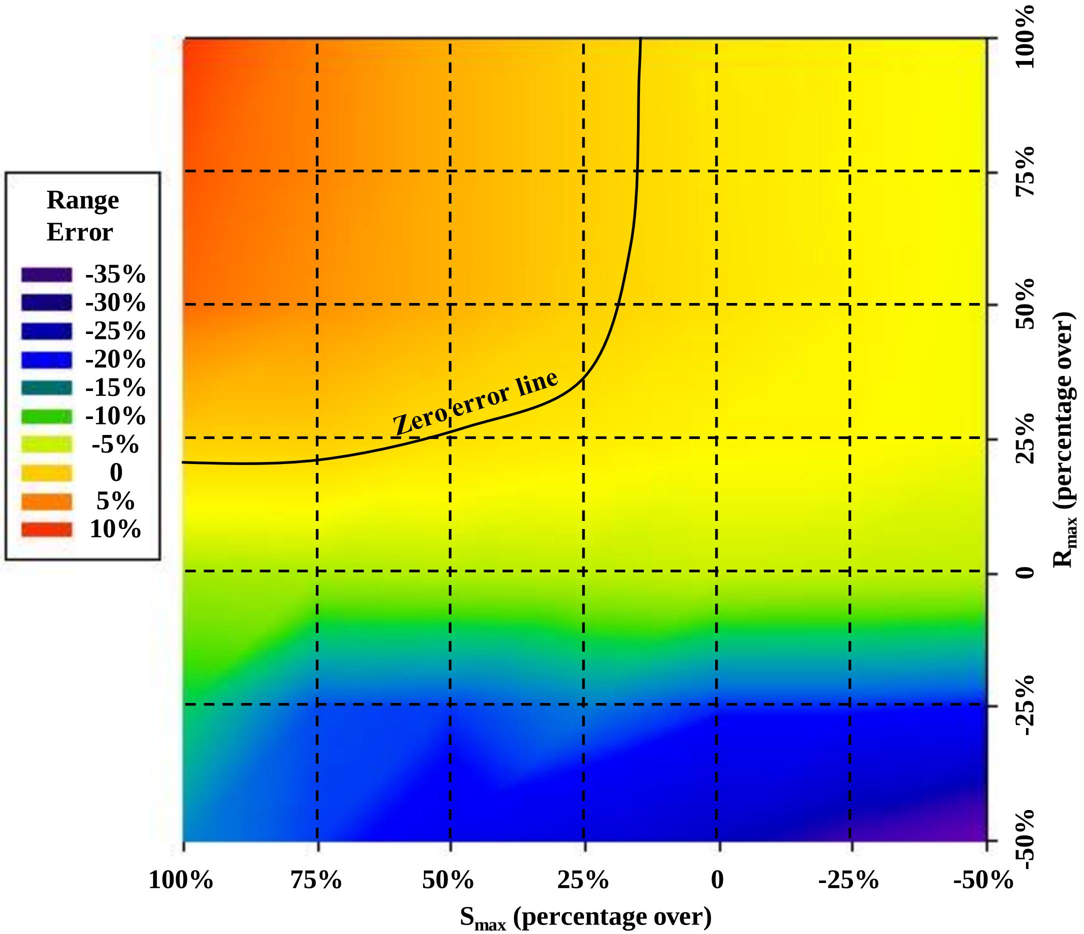
Systematic resampling

- For the full algorithm see Ref. 1 (or 2).
- Basically, the algorithm is as follows:
  - Generate the cumulative sum of weights (CSW).
  - Draw a uniform U[0,1/N]. This is a starting point for N systematically incremented points on the unit interval, where the increment is 1/N.
  - Count the number of respective points that fall in a particular weight bin of the CSW. If there are no points in a particular bin, then that particle gets deleted; 1 point, then just survives; n + 1 points, then n particles are created with that same index.

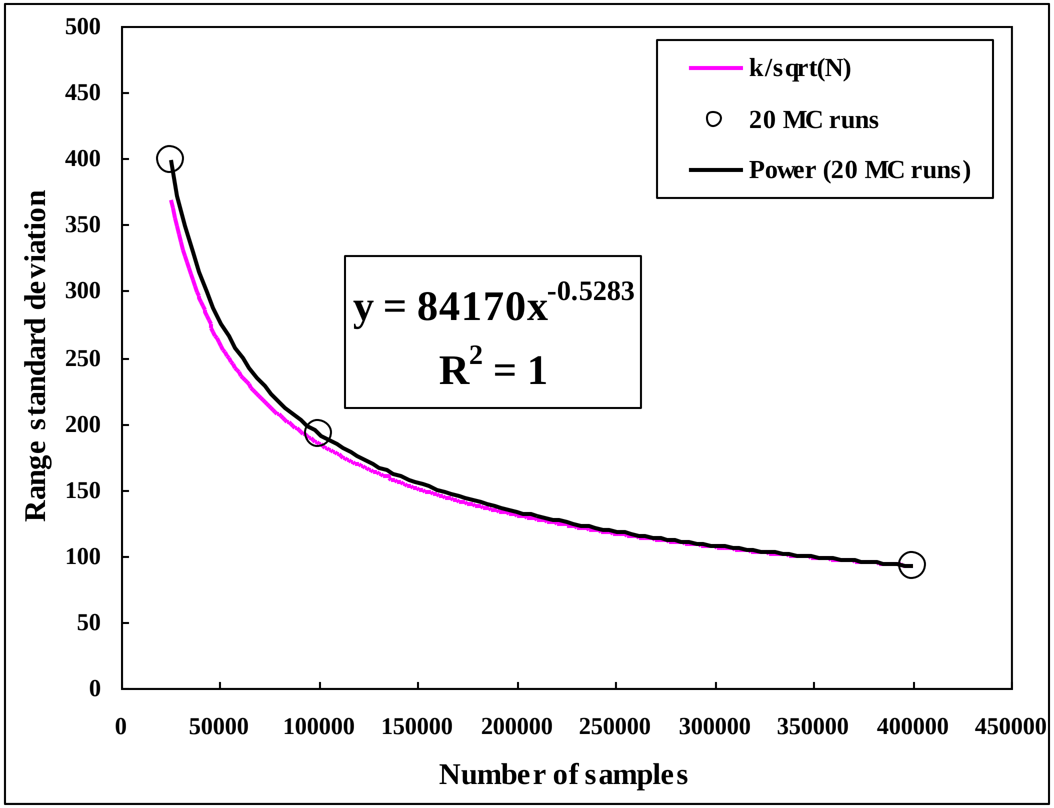
SIR particle filter sample size dependence



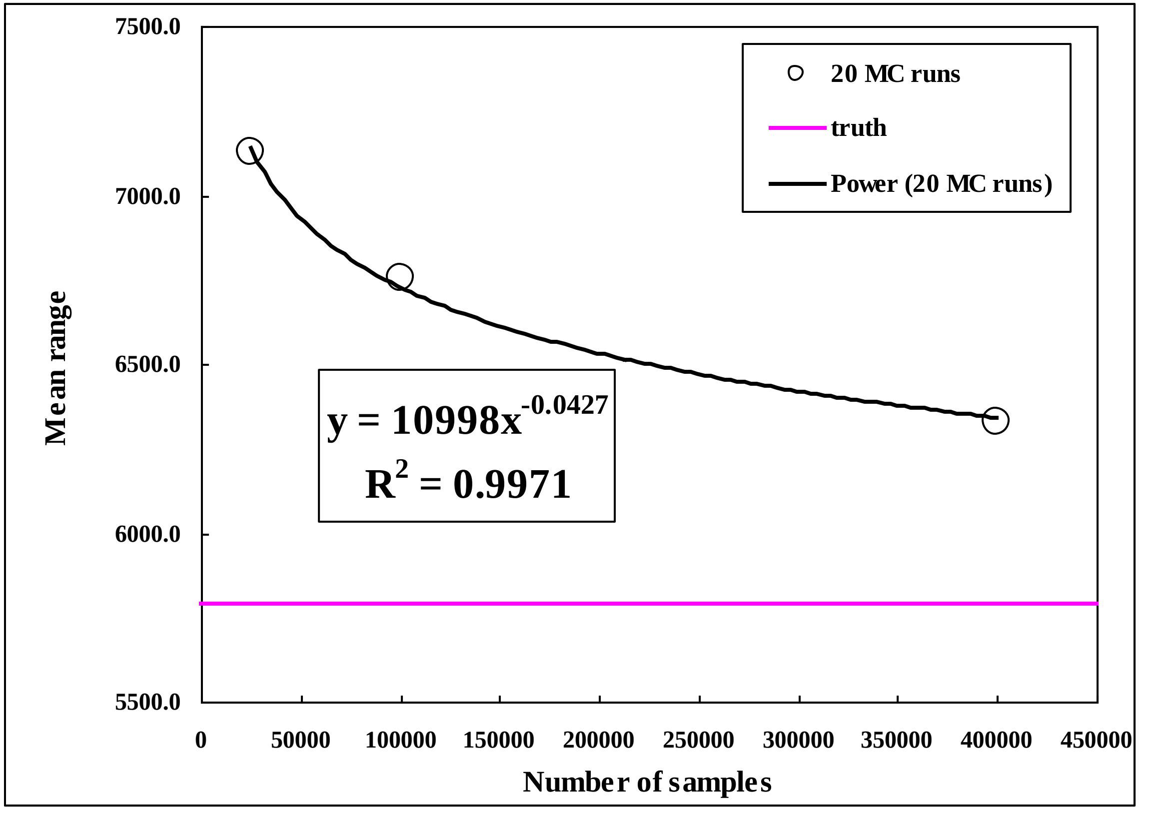
SIR particle filter:
Initial distribution dependence



SIR particle filter: Precision



SIR particle filter: "Accuracy"



Due to systematic errors, a precise converged solution may not be an accurate one.

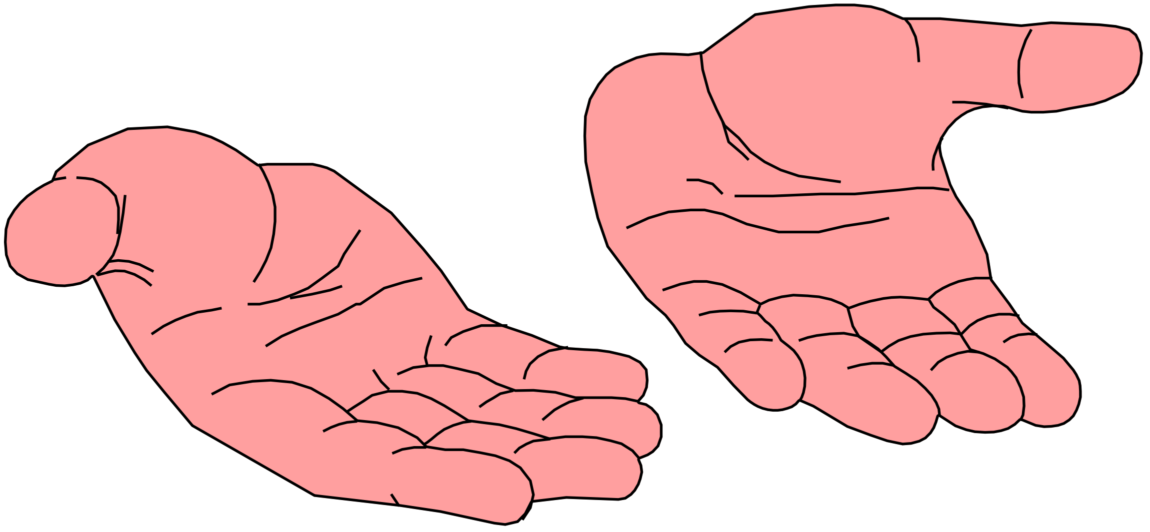
Summary

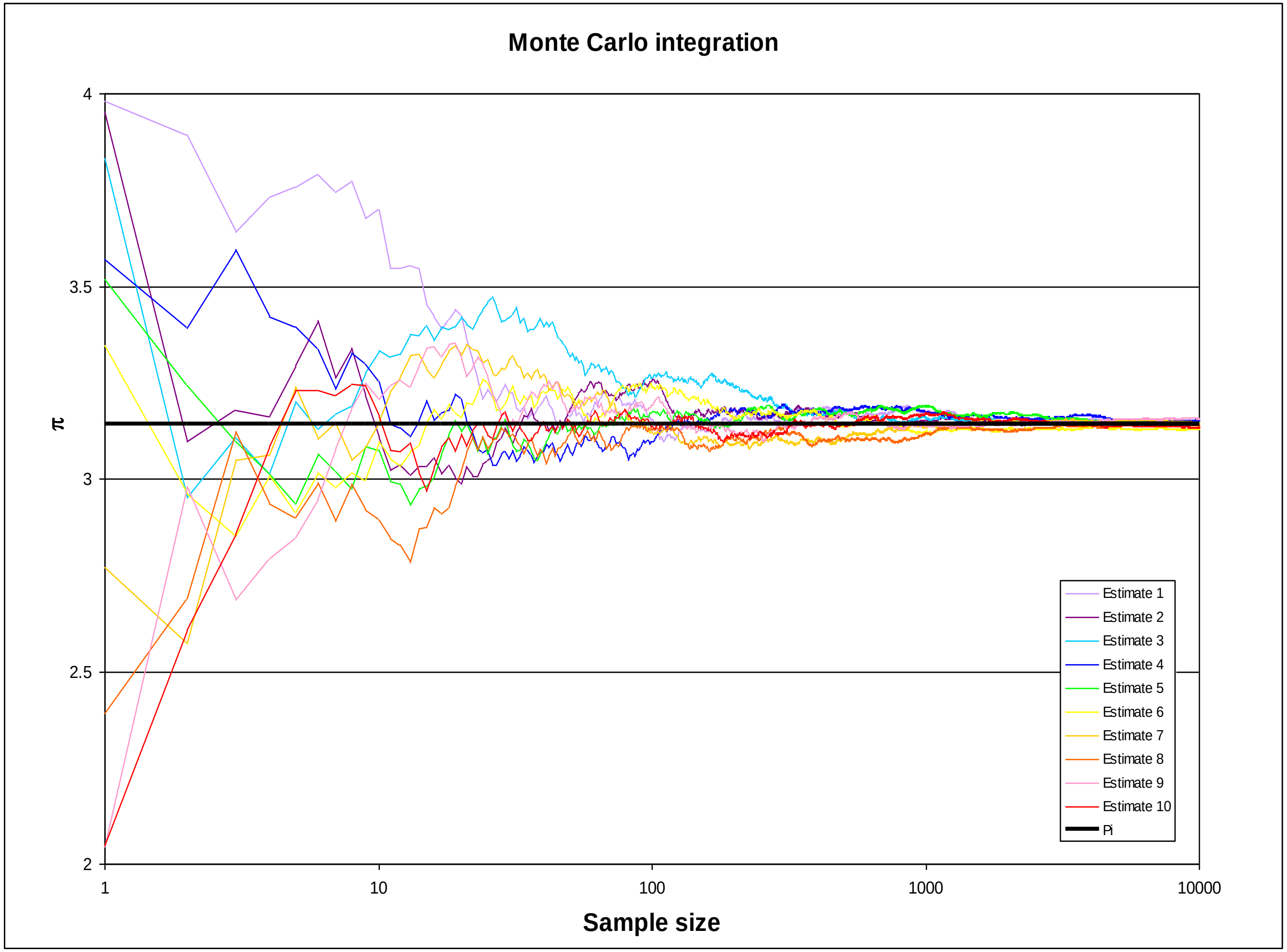
- Four-bearing solution reviewed to gain an understanding of the (un)observability issues in the bearings-only tracking problem.
  - Four bearings and at least one maneuver required
  - Constant bearing bias error leads to underranging on average.
- Overview of particle filters given. Characteristic results for the SIR particle filter given.

References

- B. Ristic, S. Arulampalam, N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*. Boston: Artech House, 2004.
- A. Doucet, N. de Freitas, and N. J. Gordon, eds., *Sequential Monte Carlo Methods in Practice*. New York: Springer, 2001.
- A. Doucet, S. Godsill, and C. Andrieu, “On sequential Monte Carlo sampling methods for Bayesian filtering,” *Statistics and Computing*, vol. 10, no. 3, pp. 197-208, 2000.

Questions?




Monte Carlo integration
π
4
3.5
3
2.5
2
1
10
100
1000
10000
Sample size
Estimate 1
Estimate 2
Estimate 3
Estimate 4
Estimate 5
Estimate 6
Estimate 7
Estimate 8
Estimate 9
Estimate 10
Pi