PHYSICS REPORTS

ELSEVIER

Physics Reports 348 (2001) 77–89

www.elsevier.com/locate/physrep

Flow equations for Hamiltonians

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Contents

Abstract

A recently developed method to diagonalize or block-diagonalize Hamiltonians by means of an appropriate continuous unitary transformation is reviewed. The main aspects will be discussed: (i) Elimination of off-diagonal matrix elements at different energy scales and (ii) problems and advantages of this method. Two applications in condensed matter physics are given as examples: the interaction of an n-orbital model of fermions in the limit of large n is brought to block-diagonal form, and the generation of the effective attractive two-electron interaction due to the elimination of electron–phonon interaction is given. The advantage of this method in particular in comparison with conventional perturbation theory is pointed out. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 11.10.Gh; 71.10.−w; 74.20.−z

Keywords: Renormalization; Many-electron systems; Superconductivity

0370-1573/01/$ - see front matter
PII: S0370-1573(00)00136-8

1. Introduction

In this contribution I describe a method for diagonalizing or block-diagonalizing Hamiltonians, which has been developed [1] in Heidelberg and has there been mainly applied to models in solid-state physics. Since this is a workshop on the renormalization group I would like to emphasize that I will not consider critical phenomena here. However, some renormalization group ideas enter into the method I am going to describe.

(i) A very general observation from the renormalization group is that if a problem is to hard to solve in one step, then one may still obtain a solution if one breaks it down to many small (infinitesimal) steps. Therefore we introduce a continuous flow parameter l and as a function of this parameter the Hamiltonian $H(l)$ will be diagonalized from the original Hamiltonian $H(0)=H$ to the (block-)diagonal Hamiltonian $H(\infty)$ by means of a unitary transformation:

$$H(l)=U(l)HU^{\dagger}(l) \tag{1}$$

(U is unitary). Differentiation yields

$$\frac{\mathrm{d}H(l)}{\mathrm{d}l}=[\eta(l),H(l)] \tag{2}$$

with the generator

$$\eta(l)=\frac{\mathrm{d}U(l)}{\mathrm{d}l}U^{\dagger}(l)=-\eta^{\dagger}(l)\,. \tag{3}$$

Obviously, $U(l)$ or actually $\eta(l)$ has to be chosen in an appropriate way.

(ii) Renormalization group compares systems at different length and energy scales. The same is true here. By means of an appropriate choice of η off-diagonal matrix elements will be removed between states of energy difference $\Delta\varepsilon$ as soon as $\Delta\varepsilon \approx 1/\sqrt{l}$. Thus the smaller the energy difference the larger l must be until the off-diagonal matrix element is removed. This comes naturally out of our choice of η. Independently, Wilson and Głazek [2,3] introduced a scheme, where they introduce an energy parameter which decreases and they require the off-diagonal matrix elements to disappear exactly as soon as this parameter is smaller than the energy difference. Procedure I introduced yields, however, a smooth cutoff, that is a smooth disappearance of the diagonal matrix elements. They apply their scheme called similarity renormalization mainly in light-front physics [4,5].

The idea to diagonalize a Hamiltonian by considering it at different *discretely* chosen energy and length scales dates back to the seventies. Two main streams were the solution of the Kondo model by Wilson [6] and the investigation of Anderson localization by Licciardello and Thouless [7], Wegner [8], Lee [9,10], Domany and Sarker [11,12], and Bender [13].

In the following, I will first suggest a choice for the generator η of the unitary transformation. Although this choice can serve as a guideline it is often useful or even necessary to modify η in order to obtain reasonable results. This will be shortly explained by the application of this method to an n-orbital model of interacting electrons. In this case one better does not require diagonalization but block diagonalization. Another example is the diagonalization of the spin-boson model [14] where a modification of the choice for η yields equations which can be solved more easily.

The price we have to pay for the use of the flow equations is that it will generate complicated interactions in general. Starting from a problem with two-particle interactions the transformation will generate three-particle, four-particle, etc. interactions. In the original paper [1] an n-orbital model was considered in the limit $n \to \infty$. Although in this limit these many-particle interactions are generated, it turns out that the one-particle energies are independent of l and the equations for the two-particle interaction are closed in themselves, so that an explicit calculation can be performed to a large extent. Another approach is to truncate the equations. Kehrein and Mielke observed that for systems with impurities (Anderson impurity model [15,16] and spin-boson model [14,17]) it is sufficient to keep only rather simple contributions to the Hamiltonian in order to obtain good results. Finally one can perform a perturbation expansion. This will be done in Section 4 for the elimination of the electron–phonon interaction [18]. One might expect that the result agrees with Fröhlich's [19] which can be found in all textbooks of theoretical solid-state physics. His effective interaction has an energy denominator (which can vanish) and gives rise to both attraction and repulsion of the electron pairs. With the present procedure we obtain in a simple way an attractive interaction between all pairs which yields very good agreement [20,21] with more sophisticated methods [22,23,29]. The permanent adjustment of the infinitesimal unitary transformation to the Hamiltonian yields a smoother effective interaction than conventional perturbation theory.

An interesting question is how interactions between nearly degenerate states do disappear and influence the final diagonal interaction. Apparently such matrix elements will decay slowly if at all. Kehrein et al. [14] have found for the spin-boson model that the off-diagonal matrix elements between states which are degenerate in the limit $l \to \infty$ decay (although slowly), so that the off-diagonal matrix elements are completely eliminated. The same applies for the electron–phonon interaction, if we take the change of the phonon energies with l into account. This is obtained in an approximation beyond perturbation theory. Quite generally, it is of interest to consider the behaviour of states with degenerate or nearly degenerate states of various interacting systems and to investigate in this way the infra-red physics of these systems. It may be that eventually one finds different classes of systems with different characteristic infra-red behaviour.

In the three following sections we consider the choice of the generator η of the unitary transformation (Section 2), some aspects of the n-orbital model (Section 3) and the elimination of the electron–phonon interaction (Section 4). In the final Section 5 concluding remarks are given and a few other systems treated with this method are cited.

2. Generator η of the unitary transformation

The generator η of the unitary transformation should be chosen in such a way that the off-diagonal matrix elements decay. For a finite matrix

$$\eta = [H_{\mathrm{d}}, H] \tag{4}$$

is a good choice, where H_d is the diagonal part of the Hamiltonian. If the diagonal matrix elements are denoted by ε, then

$$\eta_{k,n} = (\varepsilon_k - \varepsilon_n)H_{k,n} . \tag{5}$$

A simple calculation yields

$$\frac{dH_{k,n}}{dl} = \sum_m (\varepsilon_k + \varepsilon_n - 2\varepsilon_m)H_{k,m}H_{m,n} \tag{6}$$

and

$$\frac{d}{dl}\sum_{k,n,k\neq n} H_{k,n}H_{n,k} = -\frac{d}{dl}\sum_k \varepsilon_k^2 = -2\sum_{k,n}(\varepsilon_k - \varepsilon_n)^2 H_{k,n}H_{n,k} . \tag{7}$$

Thus, the sum of the squares of the off-diagonal matrix elements is indeed negative or zero. The procedure comes to an end when all off-diagonal matrix elements vanish. It may be, however, that off-diagonal matrix elements survive if the corresponding energies ε are degenerate. Since the diagonal matrix elements themselves vary as a function of l, it may happen that even the off-diagonal matrix elements between asymptotically degenerate states vanish. An example will be given in Section 4.3.

In order to illustrate the decay of the off-diagonal matrix elements let us consider a two-particle system where for conserved total momentum the relative momentum between the two particles is k. Suppose the diagonal matrix element increases linearly with k, whereas the interaction depends only on the difference $|k - k'|$. Then one can easily verify that

$$\frac{d\varepsilon_k}{dl} = 0 , \tag{8}$$

$$\frac{dH_{k,k'}}{dl} = -(\varepsilon_k - \varepsilon_{k'})^2 H_{k,k'} . \tag{9}$$

All other terms in the sums vanish. Thus the ε_k stay constant and the off-diagonal matrix elements decay like

$$H_{k,k'}(l) = H_{k,k'}(0)\exp(-(\varepsilon_k - \varepsilon_{k'})^2 l) . \tag{10}$$

Therefore $H_{k,k'}(l)$ becomes small, when $l \gg (\varepsilon_k - \varepsilon_{k'})^2$. Here and in many cases it turns out that the uncertainty of the basis states decays under the flow like $1/\sqrt{l}$.

3. *n*-orbital model

3.1. The model and its flow equations

If applied to realistic many-particle systems the flow equations will in general generate complicated interactions. Starting from a problem with two-particle interactions the transformation generates three-particle, four-particle, etc. interactions. In the original paper [1] an n-orbital model was considered in the limit $n \to \infty$. Although in this limit these many-particle interactions will be generated, it turns out that the one-particle energies are independent of l and the equations for the two-particle interaction are closed in themselves, so that an explicit calculation can be performed to a large extent.

Let us shortly consider this model of interacting electrons. The electrons carry a momentum and in addition a quantum number (flavor) s, which runs from 1 to n. Finally, we are interested in the limit n going to ∞.

The Hamiltonian will be expressed in terms of the operators

$$N_{p,q} = \frac{1}{n}\sum_s c^\dagger_{p,s} c_{q,s} \tag{11}$$

with creation and annihilation operators $c^\dagger$ and c and momenta p and q. All operators will be expressed as normal-ordered polynomials of the $N_{p,q}$. Thus $N_{p,p} = n_p + : N_{p,p} :$, where n_p is the occupation number for the groundstate of the free system $n_p = \theta(k_{\mathrm{F}} - |p|)$. One can convince oneself that the commutator between two operators yields in leading order in $1/n$

$$[: A :, : B :] = [A,B]_1 + [A,B]_2 + \mathrm{O}(n^{-2}) , \tag{12}$$

$$n[A,B]_1 = \sum_{p,q,r} : \frac{\partial A}{\partial N_{p,q}} \frac{\partial B}{\partial N_{q,r}} N_{p,r} : - \sum_{p,q,r} : \frac{\partial A}{\partial N_{p,q}} \frac{\partial B}{\partial N_{r,p}} N_{r,q} : , \tag{13}$$

$$n[A,B]_2 = \sum_{p,q} : \frac{\partial A}{\partial N_{p,q}} \frac{\partial B}{\partial N_{q,p}} : (n_p - n_q) . \tag{14}$$

Let us consider a Hamiltonian

$$H = H_1 + H_2 + \cdots , \tag{15}$$

$$H_1 = n \sum_q \varepsilon_q : N_{q,q} : , \tag{16}$$

$$H_2 = \frac{n}{2\Omega} \sum_{\delta,Q,K} v_{\delta,Q,K} : N_{Q+\delta/2,Q-\delta/2} N_{K-\delta/2,K+\delta/2} : , \tag{17}$$

where δ is the momentum transfer and Ω the volume of the system and H_k contains the k-particle interaction. Similarly, one has

$$\eta = \eta_2 + \cdots , \tag{18}$$

$$\eta_2 = \frac{n}{2\Omega} \sum_{\delta,Q,K} \eta_{\delta,Q,K} : N_{Q+\delta/2,Q-\delta/2} N_{K-\delta/2,K+\delta/2} : . \tag{19}$$

In leading order in $1/n$, that is in the limit n going to ∞ one obtains

$$\eta_2 = [H_1, H_2 - H_{2d}]_1 + [H_{2d}, H_2]_2 , \tag{20}$$

$$\frac{\partial H_1}{\partial l} = 0 , \tag{21}$$

$$\frac{\partial H_2}{\partial l} = [\eta_2, H_1]_1 + [\eta_2, H_2]_2 . \tag{22}$$

Thus H_1 is constant in leading order in $1/n$. Although more-particle contributions H_k with $k > 2$ will be generated (which I did not write down), they do not couple back into the equation for H_2. Thus the equations for H_2 are closed in themselves.

3.2. Solution

In a first attempt, I have used the flow equations literally, i.e. as the diagonal part H_d I chose the contributions diagonal in momentum representation. It turns out that in the thermodynamic limit, i.e. for Ω approaching ∞, the diagonal part reduces to H_1 and the diagonal part of H_2 becomes negligible. The solutions of the corresponding equations do not yield a converging behavior. We assume that it is not allowed to neglect many-particle interactions in H_d.

Therefore I made a different choice for the diagonal part. I considered all terms to be diagonal, which conserve the number of quasiparticles, i.e. the number of electrons above plus the number of holes below the Fermi edge. Then H_0, which is a constant in addition to the Hamiltonian already written down, becomes the energy of (hopefully) the groundstate, H_1 contains the energies of the one-quasiparticle excitations. H_2 contains the interaction between two quasiparticles, etc. Thus to obtain the two-quasiparticle states, it is sufficient to diagonalize a two-particle problem, which is of course much easier than to solve the N-particle problem.

With this in mind and

$$s_{\delta,k} = n_{k-\delta/2} - n_{k+\delta/2} \tag{23}$$

the terms $v_{\delta,Q,K} : N_{Q+\delta/2,Q-\delta/2} N_{K-\delta/2,K+\delta/2} :$ belong to H_d, if $s_{\delta,Q} = s_{\delta,K}$.

With

$$\varepsilon_k = k^2/2 \tag{24}$$

one obtains

$$\begin{aligned}\eta_{\delta,Q,K} = {} & (Q-K)\delta v_{\delta,Q,K}(1-\delta_{s_{\delta,Q},s_{\delta,K}}) \\ & + \frac{1}{\Omega}\sum_P s_{\delta,P}(\delta_{s_{\delta,P},s_{\delta,Q}} - \delta_{s_{\delta,P},s_{\delta,K}}) v_{\delta,Q,P} v_{\delta,P,K} ,\end{aligned} \tag{25}$$

$$\frac{\partial v_{\delta,Q,K}}{\partial l} = (K-Q)\delta\eta_{\delta,Q,K} + \frac{1}{\Omega}\sum_P s_{\delta,P}(\eta_{\delta,Q,P} v_{\delta,P,K} - v_{\delta,Q,P}\eta_{\delta,P,K}) . \tag{26}$$

One observes that only matrix elements with the same momentum transfer δ are connected to each other. Further inspection shows that even the equations for matrix elements $v_{\delta,Q,K}$ for which $s_{\delta,Q}$ and $s_{\delta,K}$ are different from zero close into themselves.

For small momentum transfer there are only small regions of K and Q around k_F, in which s is different from zero. We assume, that in this limit v does not depend essentially on $|K|$ and $|Q|$, but only on the direction of these vectors.

Let us now restrict to the one-dimensional case. If the momenta lie in the interval of size δ, in which s equals $+1$ and -1, then I write instead of the momenta $+$ and $-$, respectively. The equations for $v_{\pm,\pm}$ are closed,

$$\frac{\mathrm{d}v_{+,+}}{\mathrm{d}l} = \frac{\mathrm{d}v_{-,-}}{\mathrm{d}l} = -\frac{\delta^2}{\pi^2} A v_{+,-} v_{-,+} , \tag{27}$$

$$\frac{\mathrm{d}v_{+,-}}{\mathrm{d}l} = -\frac{\delta^2}{\pi^2} A^2 v_{+,-} , \tag{28}$$

$$A = 2\pi k_{\mathrm{F}} + \tfrac{1}{2}(v_{+,+} + v_{-,-}) . \tag{29}$$

One immediately sees that $v_{+,+} - v_{-,-}$ and $B = A^2 - v_{+,-}v_{-,+}$ are constants and the solutions can be given in the form

$$A^2(l) = \frac{BA^2(0)}{A^2(0) - v_{+,-}(0)v_{-,+}(0)\exp(-\gamma l)} , \tag{30}$$

$$v_{+,-}(l)v_{-,+}(l) = \frac{Bv_{+,-}(0)v_{-,+}(0)}{A^2(0) - v_{+,-}(0)v_{-,+}(0)\exp(-\gamma l)} , \tag{31}$$

$$\gamma = \frac{2\delta^2 B}{\pi^2} . \tag{32}$$

Apparently, the $v_{\pm,\pm}$ are smooth functions of l. For positive B the off-diagonal matrix elements $v_{+,-}$ and $v_{-,+}$ vanish as l goes to ∞. If B is negative, then the diagonal matrix elements $v_{+,+}$ and $v_{-,-}$ vanish, whereas the off-diagonal matrix elements approach a finite limit. In this latter case, however, the system is unstable.

Next one considers the off-diagonal matrix element $v_{\pm,K}$ and obtains

$$\frac{\partial v_{+,K}}{\partial l} = -\frac{\delta^2}{4\pi^2}(A + c_K)^2 v_{+,K} - \frac{\delta^2}{4\pi^2}(3A - c_K)v_{+,-}v_{-,K} , \tag{33}$$

$$c_K = \tfrac{1}{2}(v_{+,+} - v_{-,-}) - 2\pi K . \tag{34}$$

Together with a similar equation for $v_{-,K}$ this is a system of coupled linear differential equations. Since for large l $v_{+,-}$ tends to zero, the equations become decoupled and it is the negative prefactor in front of the $v_{+,K}$ which guarantees asymptotically an exponential decay. There is one exception: if $A + c_K$ vanishes, which happens for a special value $K_{c,\pm}$, then $v_{K,\pm}$ does not decay. In a different model we will see below that a change of the diagonal matrix elements as a function of l may be sufficient, so that matrix elements even at such resonances go to zero.

Finally, one writes down the flow equation for $v_{Q,K}$ and obtains asymptotically

$$\begin{aligned} v_{Q,K}(l) = {} & v_{Q,K,\mathrm{reg}} \\ & + \frac{(2K_{c,+} - K - Q)v_{K_{c,+},+}(\infty)v_{+,K_{c,+}}(\infty)}{2\pi[(K_{c,+} - K)^2 + (K_{c,+} - Q)^2]} \\ & \times (1 - \exp(-\delta^2[(K_{c,+} - K)^2 + (K_{c,+} - Q)^2]l)) \end{aligned}$$

$$+\frac{(2K_{c,-}-K-Q)v_{K_{c,-},-}(\infty)v_{-,K_{c,-}}(\infty)}{2\pi[(K_{c,-}-K)^2+(K_{c,-}-Q)^2]}$$

$$\times(1-\exp(-\delta^2[(K_{c,-}-K)^2+(K_{c,-}-Q)^2]l))\,. \tag{35}$$

These couplings tend to a finite value unless both Q and K are at K_c.

The observation that the flow equations converge or yield only mild divergencies in the interaction, when we restrict ourselves to block diagonalization is important. In particular if the eigenstates are far from being reminiscent to extended states as in the case of the bound states of positronium it is hardly possible to perform the diagonalization completely. The elimination of the off-diagonal interaction works well down to an energy uncertainty of the order of Rydberg [24]. Below this uncertainty one has to use other approaches or one restricts oneself (from the beginning) to the elimination of the coupling to the photon field, that is to the terms which do not conserve the number of photons. In this case one can carry through the scheme to arbitrary small energy differences (at least in second order in the coupling) [25,26].

3.3. Further results

Starting from this transformation one can determine expectation values and correlation functions. In order to do this, the operators have to be subject to the same unitary transformations as the Hamiltonian, i.e. the same flow equation is applied to operators. After transformation of the operators one can evaluate them in the $l=\infty$ basis. This has been done for the average occupation number in the n-orbital model [1], which yields a correction in order $1/n$. The result is compatible with that for the Luttinger model [27], which can easily be generalized to n orbitals. In the Luttinger model one finds a power-law behavior of the occupation number both above and below the Fermi edge. The occupation number itself is a continuous function of the energy at the Fermi energy.

Similar calculations in dimensions $d>1$ [28] indicate that there is a jump of the occupation number at least for small interactions in agreement with the idea of a Landau liquid.

4. Elimination of the electron–phonon coupling

4.1. The effective electron–electron interaction

This section is to a large extent based on the diploma thesis of Peter Lenz and on Ref. [18]. Our aim is to calculate the effective electron–electron interaction responsible for the superconductivity. The Hamiltonian of an electronic system coupled to phonons consists of three contributions

$$H=H_0+H_{\mathrm{e-ph}}+H_{\mathrm{e-e}}\,. \tag{36}$$

Here H_0 is the free part

$$H_0 = \sum_q \omega_q a_q^\dagger a_q + \sum_k \varepsilon_k : c_k^\dagger c_k : , \tag{37}$$

where ω_q and ε_k are the phonon and electron energies, respectively. The electron–phonon interaction is given by

$$H_{\mathrm{e-ph}} = \sum_{k,q} M_{k,q} a_{-q}^\dagger c_{k+q}^\dagger c_k + \mathrm{h.c} . \tag{38}$$

and the electron–electron interaction by

$$H_{\mathrm{e-e}} = \sum_{k,k',q} V_{k,k',q} : c_{k+q}^\dagger c_{k'-q}^\dagger c_{k'} c_k : \tag{39}$$

Here k and k' include the z-component of the spin s and s', respectively. In all contributions of the Hamiltonian we have used normal ordering. All terms which after normal ordering are not of these types will be neglected. Here we proceed similarly as before, since we consider those terms to be diagonal which conserve the number of particles. This applies to H_0 and $H_{\mathrm{e-e}}$, whereas $H_{\mathrm{e-ph}}$ does not conserve the number of phonons and thus is considered to be off-diagonal. Furthermore we will assume the electron–electron interaction to be small and only the second-order contribution from $H_{\mathrm{e-ph}}$ to $H_{\mathrm{e-e}}$ will be calculated. Thus, we choose

$$\eta = [H_0, H_{\mathrm{e-ph}}] = \sum_{k,q} M_{k,q} \alpha_{k,q} a_{-q}^\dagger c_{k+q}^\dagger c_k - \mathrm{h.c.} \tag{40}$$

with the energy difference

$$\alpha_{k,q} = \varepsilon_{k+q} - \varepsilon_k + \omega_q . \tag{41}$$

This generator yields several contributions to $\mathrm{d}H/\mathrm{d}l = [\eta, H]$. The contribution to the change of M results from $[\eta, H_0]$

$$\frac{\partial M_{k,q}(l)}{\partial l} = -\alpha_{k,q}^2 M_{k,q}(l) \tag{42}$$

with the solution

$$M_{k,q}(l) = M_q \exp(-\alpha_{k,q}^2 l) , \tag{43}$$

where M_q is the initial electron–phonon coupling. The contribution to the electron–phonon coupling is obtained from $[\eta, H_{\mathrm{e-ph}}]$. The terms which describe the interaction between the electron pairs (with zero momentum) obey

$$\frac{\partial V_{k,-k,q}(l)}{\partial l} = -(\alpha_{k,q} + \alpha_{-k-q,-q}) M_{k,q}(l) M_{-k-q,q}(l) \tag{44}$$

with the solution

$$V_{k,-k,q}(\infty) = V_{k,-k,q}(0) - M_q^2 \frac{\omega_q}{\omega_q^2 + (\varepsilon_{k+q} - \varepsilon_k)^2} . \tag{45}$$

Several remarks are in order, since Fröhlich's result differs from this one by a minus sign between the two squares in the denominator:

(i) The interaction from the electron–phonon coupling is attractive for all values of k and q.

(ii) Mielke has also obtained an effective attractive interaction [20] without pole by means of Głazek and Wilson's similarity transformation [3]. The critical temperature determined from this interaction yields values very similar [20] to those determined with the method by MacMillan and Dynes [23,29] based on the Eliashberg theory [22] and close to the experimental values [21]. In contrast to the Eliashberg theory which works with a retarded effective interaction our interaction is instantaneous.

(iii) Similar sums of two squares in the denominator appear in the matrix elements $v_{K,Q}$ of the n-orbital model (35) and in a revised treatment of the Schrieffer–Wolf transformation [30] with the present scheme [16].

(iv) For on-shell matrix elements V, i.e. for those which obey $\varepsilon_{k+q}+\varepsilon_{k'-q}=\varepsilon_k+\varepsilon_{k'}$, Fröhlich's result and ours coincide.

(v) We observe that the permanent adjustment of $\eta(l)$ to the current $H(l)$ yields smoother interactions than conventional perturbation theory.

(vi) Perturbation theory for Hamiltonians is not uniquely defined. The reason is that within the blocks with fixed particle numbers there can be arbitrary unitary transformations.

4.2. Comparison with Fröhlich's treatment

Let us now compare Fröhlich's and our treatment. Fröhlich introduces a transformation

$$\begin{aligned} H^{\mathrm{Fr}} &= e^{-S} H e^{S} = H + [H,S] + \tfrac{1}{2}[[H,S],S] + \cdots \\ &= H_0 + H_{\mathrm{e-ph}} + [H_0,S] + [H_{\mathrm{e-ph}},S] + \tfrac{1}{2}[[H_0,S],S] + \cdots \ . \end{aligned} \tag{46}$$

He assumes S to be of the order of the electron–phonon coupling and requires the contribution in first order, i.e. $H_{\mathrm{e-ph}} + [H_0,S]$, to vanish. This yields

$$S^{\mathrm{Fr}} = -\sum_{k,q} M_q \left(\frac{a^\dagger_{-q}}{\alpha_{k,q}} - \frac{a_q}{\alpha_{k+q,q}} \right) c^\dagger_{k+q} c_k \ . \tag{47}$$

Our treatment yields

$$\exp(-S^{\mathrm{LW}}) = T_l \exp\left(\int \mathrm{d}l\, \eta(l) \right) \tag{48}$$

where T_l is an ordering of l, since η's with different argument do not commute. An expansion in powers of η yields

$$S^{\mathrm{LW}} = -\int_0^\infty \mathrm{d}l\, \eta(l) - \frac{1}{2} \int_0^\infty \mathrm{d}l \int_0^l \mathrm{d}l' [\eta(l), \eta(l')] + \cdots \ . \tag{49}$$

The first term agrees with Fröhlich's S^{Fr}. The second term yields the difference due to the permanent adjustment of η to H.

4.3. Asymptotics of $\omega_q(l)$

By now we have not considered a variation of the one-particle energies with l. For $\omega(l)$ one has the flow equation

$$\frac{\partial \omega_q(l)}{\partial l} = 2\sum_k M_{k,q}^2(l)\alpha_{k,q}(l)(n_{k+q} - n_k)\,, \tag{50}$$

$$M_{k,q}(l) = M_q \exp\left(-\int_0^l \mathrm{d}l'\, \alpha_{k,q}^2(l')\right)\,. \tag{51}$$

This yields a nonlinear integro-differential equation for $\omega_q(l)$. Comparable equations were obtained before by Kehrein et al. [14] for the tunneling frequency. They obtained the asymptotic behaviour $\Delta(l) = \Delta(\infty) + 1/(2\sqrt{l})$. Lenz found that the general asymptotic behavior obeys $\omega(l) = \omega(\infty) + c(l)/\sqrt{l}$ for the electron–phonon system [18], where $c(l)$ is a periodic function in $\ln(l)$ and the average of c^2 equals 1/4. This decay to the asymptotic value is sufficiently slow, so that even the offdiagonal matrix elements M, for which $\alpha(\infty) = 0$, vanish.

5. Concluding remarks and other applications

In two applications, the n-orbital model and the electron–phonon coupling in superconductivity, it has been shown, how the idea of flow equations for Hamiltonians can be applied to condensed matter physics in order to obtain effective block-diagonal interactions. As already mentioned, this scheme has also been applied to two impurity models, the Anderson impurity model and the spin-boson model. An interesting subject only briefly mentioned for the operator of occupancy in the n-orbital model is the question what happens to operators under the flow equations. It has been investigated for the spin-boson model [14,17]. The interesting observation is that the spin operator completely transforms into a linear combination of boson creation and annihilation operators. From this one sees explicitely that quasi-particles may be quite different from bare particles. This transformation allows the calculation of the time-dependent spin-correlation function. The results for intermediate time scales is in good agreement with NIBA, whereas for longer time scales (where NIBA fails to predict the correct asymptotic behavior) they are in good agreement with the Shiba relation. Thus this procedure covers ranges otherwise only covered by different approaches. In the case of the electron–phonon coupling [31] the phonon-creation operator transforms into an electron–hole pair which allows the determination of the phonon damping. Kehrein and Mielke have investigated the question of dissipation in the framework of the flow equations [32,33].

Some other applications are: the elimination of the coupling between states with different double occupancy in the Hubbard model [34] and the diagonalization of the Hamiltonian of the Heisenberg antiferromagnet near the classical limit [35] by Stein. Here a different generator of the unitary transformation is used, which orders the states according to their eigenvalues, a method also used by Mielke for band matrices [36], which he applied to the Lipkin model and the spin-boson model. Other applications to the Lipkin model are by Pirner and Friman [37] and by Stein [38]. Some further applications are to a Dirac particle in an external magnetic field [39], to the Henon-Heiles Hamiltonian [40], the crossover from the weak to strong coupling in the sine-Gordon model [41], to the spin-Peierls transition [42], to dimerized and frustrated $S = 1/2$ chains [43], and to the derivation of the Ruderman–Kittel–Yosida interaction [44]. An instructive example of renormalization is a two-dimensional particle in a contact potential [45]. I will not list all the contributions to light-front physics. Apart from those quoted here [4,5,25,26], I refer the reader to the review by Perry [46], which summarizes some of the results.

Acknowledgements

The author acknowledges the pleasant hospitality extended to him at the RG2000 workshop in Taxco. This work was supported in part by the German Science Foundation DFG.

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