PHIL 120A
Summer 2006
Weeks 6-9

Although the propositional calculus covers a lot of territory, it cannot account for the internal structure of propositions (pp. 92-3). Consider the well-known argument: 'Socrates is a man. All men are mortal. Therefore, Socrates is mortal.' This is a valid argument – in other words, it is not possible for the premises to be true and the conclusion false. But in the propositional calculus, the argument has the form '$P, Q \vdash R$' – a form which is not valid. Thus Lemmon introduces the predicate calculus as the focus of Ch. 3-4. In the predicate calculus, the argument above may be symbolized, '$Fm, (x)(Fx \rightarrow Gx) \vdash Gm$' – a valid form (as we shall see, pp. 105-6).

The predicate calculus is an enlargement of the propositional calculus (p. 140). All symbols and wffs of the propositional calculus are also symbols and wffs of the predicate calculus. But the predicate calculus includes many additional symbols and wffs.

Definitions (pp. 138-9)	**Proper Names**	'm', 'n', 'o', etc.	Infinitely many
	Arbitrary Names	'a', 'b', 'c', etc.	Infinitely many
	Individual Variables (or plain **variables**)	'x', 'y', 'z', etc.	Infinitely many
	Predicate-letters	'F', 'G', 'H', etc.	Infinitely many
	Reverse-E	'∃'	Exactly one
	Terms	Names (both proper and arbitrary) 'a', 'b', 'c', etc. 'm', 'n', 'o', etc.	Infinitely many
Along with brackets and logical connectives and propositional variables, these are the symbols of the predicate calculus (no more, no less)			
A formula of the predicate calculus is any sequence of these symbols			

Lemmon introduces new metalogical variables for the predicate calculus (p. 139). 'P' stands in for predicate-letters; 't_1', 't_2', 't_3', etc. stand in for terms; and 'v_1', 'v_2', 'v_3', etc. stand in for variables.

Using these metalogical variables, Lemmon defines an **atomic sentence**. Suppose that t_1 through t_n represent any *n* terms, where *n* is a finite number, zero or greater (and where the terms may be repeated). And suppose that P is any predicate-letter. Then '$Pt_1 \ldots t_n$' is an atomic sentence. [Some examples: 'Fm', 'Fa', 'Faa', 'Fab', 'Fmm', 'Fmn', 'Gam', 'Gnb', 'Fmanb', and so on, but not 'Fx' or 'Fxa' or 'Fxm'. When *n* (the number of terms) is zero, Lemmon recommends using propositional variables instead of predicate letters (p. 140).]

A **well-formed formula** (wff) of the predicate calculus is defined by the following nine clauses (p. 140):

1) Any atomic sentence is a wff;
2) If A is a wff, then –A is a wff;

3) If A and B are wffs, then $(A \rightarrow B)$ is a wff;
4) If A and B are wffs, then $(A \& B)$ is a wff;
5) If A and B are wffs, then $(A v B)$ is a wff;
6) If A and B are wffs, then $(A \leftrightarrow B)$ is a wff;
7) Let A(t) be a wff containing a term t, and let v be some variable not occurring in A(t); let A(v) be a formula resulting from A(t) by replacing at least one occurrence of t by v; then (v)A(v) is a wff;
8) Let v be some variable and A(v) be a formula as described in the preceding clause; then $(\exists v)A(v)$ is a wff;
9) If a formula is not a wff in virtue of those 8 clauses, then it is not a wff.

Two kinds of quantifiers (p. 143):
Universal quantifier – '(v)' (where 'v' is a variable). Examples: '(x)', '(y)', '(z)'
Existential quantifier – '$(\exists v)$' (where 'v' is a variable). Examples: '$(\exists x)$', '$(\exists y)$', '$(\exists z)$'

A **propositional function** is either a wff or else one of the A(v) formulas described in clause (7) of the definition of a wff (p. 143).

Scope (p. 143) – the scope of an occurrence of a logical connective in a propositional function is the shortest propositional function in which it occurs. The scope of an occurrence of a quantifier in a propositional function is the shortest propositional function in which it occurs.

We will define the **main quantifier or main connective** as the quantifier or connective in a wff whose scope is the entire wff.

In every wff, every quantifier controls at least one occurrence of its variable; and every variable is within the scope of a quantifier. For every quantifier in every wff or propositional function, there are at least two occurrences of its variable within its scope (one of them being in the quantifier itself); and within that scope, no other quantifier will use the same variable (p. 144).

Rules of derivation for the predicate calculus: All rules of the propositional calculus are allowed and may be applied to wffs of the predicate calculus. In addition, four new rules are introduced – UE, EI, UI, and EE (p. 145) (see separate notes). This system of rules is safe and complete (pp. 148-159). The predicate calculus assumes that at least one object exists (p. 156).

The proof of the Socrates argument above is as follows (p. 105):

$Fm, (x)(Fx \rightarrow Gx) \vdash Gm$

1	(1)	Fm	A
2	(2)	$(x)(Fx \rightarrow Gx)$	A
2	(3)	$Fm \rightarrow Gm$	2 UE
1,2	(4)	Gm	1,3 MPP

Translation: In that example, 'Fm' stands for 'Socrates is a man'. The proper name 'm' stands for Socrates, while the predicate-letter 'F' stands for the predicate 'is a man'. The predicate-letter 'G' stands for the predicate 'is mortal'. So, 'Gm' stands for 'Socrates is mortal'. The wff

‘$(x)(Fx \rightarrow Gx)$’ stands for ‘All men are mortal’. It is a universally quantified statement, with the idea that for all x, if x is a man, then x is mortal.

Here are some well-known standardized translation forms (p. 98):

F: is a man G: is mortal			
All men are mortal	No men are mortal	Some men are mortal	Some men are not mortal
$(x)(Fx \rightarrow Gx)$	$(x)(Fx \rightarrow - Gx)$	$(\exists x)(Fx \,\&\, Gx)$	$(\exists x)(Fx \,\&\, - Gx)$
Note that any variable could have been used (it did not have to be ‘x’) Also note that ‘some’ means ‘at least one’ (p. 97 n1)			

These standard translation forms will cover many English sentences. Be very careful to use these forms when translating sentences like those in the table above. The first two wffs are universally quantified conditionals, while the next two are existentially quantified conjunctions.

Note that $(\exists x)(Fx \rightarrow Gx)$ and $(x)(Fx \,\&\, Gx)$ will almost always be incorrect translations of English sentences.

Note also that ‘only’ tends to have a reversing effect (pp. 101-2). For example, ‘Only fish are green’ is translated ‘$(x)(Gx \rightarrow Fx)$’, while ‘All fish are green’ would be ‘$(x)(Fx \rightarrow Gx)$’.

Be aware that the wff , ‘$(x)Fx \rightarrow (x)Gx$’, is a conditional statement and not a universally quantified statement. The UE rule cannot be used on this wff.

There are two-place predicates, three-place predicates, and so on. Take, for example, the two-place predicate ‘loves’, which we may symbolize by ‘F’.

F: loves; m: Mary; n: Norman	
Mary loves Norman	Fmn
Somebody loves Norman	$(\exists x)Fxn$
Mary loves somebody	$(\exists x)Fmx$
Somebody loves somebody	$(\exists x)(\exists y)Fxy$
Somebody is loved by somebody	$(\exists y)(\exists x)Fxy$
Somebody loves herself	$(\exists x)Fxx$
Everybody loves Norman	$(x)Fxn$
Mary loves everybody	$(x)Fmx$
Everybody loves somebody (not necessarily the same somebody)	$(x)(\exists y)Fxy$
Somebody in particular loves everybody	$(\exists x)(y)Fxy$
Everybody is loved by somebody (not necessarily the same somebody)	$(y)(\exists x)Fxy$
Somebody in particular is loved by everybody	$(\exists y)(x)Fxy$
Everybody loves everybody	$(x)(y)Fxy$
Everybody is loved by everybody	$(y)(x)Fxy$
Everybody loves herself	$(x)Fxx$

As for the three-place predicate ‘is between’, which we may symbolize by ‘G’, consider the statement ‘Madrid is between Norway and Ontario’, which may be translated ‘Gmno’.

Generally speaking, as Lemmon puts it (p. 102), flexibility of mind is required for translation, as there are no firm rules and practice is needed. English sentences are often ambiguous, while the predicate calculus lacks ambiguity. In addition, the kind of translation done will depend on how much of the internal structure of propositions needs to be revealed in order to assess validity of arguments (p. 167).

Consider, for example, ‘Every frog greets an idle halibut’. Here is one likely way to translate it.
F: is a frog
G: greets (2-place predicate)
H: is a halibut
I: is idle

$(x)(Fx \rightarrow (\exists y)((Iy \,\&\, Hy) \,\&\, Gxy))$
This says that for every frog, there is some idle halibut that it greets (not necessarily the same idle halibut for every frog).

If we think that the English sentence says that there is some particular idle halibut that every frog greets, then the translation would look like this:
$(\exists x)((Ix \,\&\, Hx) \,\&\, (y)(Fy \rightarrow Gyx))$
This says that there is some particular idle halibut that every frog greets.

Notice that in both translations, the quantifiers conform to the standard translation forms seen above. Each universal quantifier has an arrow as the main connective within its scope, and each existential quantifier has an ampersand as the main connective within its scope. This will almost always be the case when translating English sentences into symbols.

Consider, ‘There’s a sucker born every minute.’ We might take this to mean that there is some particular sucker who is born, over and over again, every minute. Or we might take it to mean that for every minute, some sucker (not necessarily the same one) is born. Here are those two translations.
F: is a sucker
G: is a minute
H: is born during (2-place predicate)
The same sucker every minute: $(\exists x)(Fx \,\&\, (y)(Gy \rightarrow Hxy))$
Not necessarily the same sucker: $(x)(Gx \rightarrow (\exists y)(Fy \,\&\, Hyx))$

To take another example, consider the famous line, ‘Everybody loves a lover.’ What does this English sentence mean? Most likely, it means that everybody loves every lover. And what is a lover? Anybody who loves somebody. So, to say that everybody loves every lover is to say, ‘Everybody who loves someone is loved by everyone.’ This requires three variables.
F: loves (2-place predicate)
$(y)((\exists z)Fyz \rightarrow (x)Fxy)$

Consider the statement, “Every guppy head is a fish head.” What does this English sentence mean? Most likely, that every head of a guppy is the head of a fish. Or in other words, everything is such that if it is the head of a guppy, then it is the head of a fish.
F: is a fish
G: is a guppy
H: is the head of (2-place predicate)
(x) ((∃y)(Gy & Hxy) → (∃z)(Fz & Hxz))
Notice once again the scopes of the main connectives within each quantifier’s scope.

Equivalences: Since we don’t have truth tables for the predicate calculus, it is useful to find some other way for seeing whether wffs are equivalent. It is a fact that propositional functions bear the same logical equivalence relations to each other as do wffs of the propositional calculus. For example, we know that ‘P→Q’ is logically equivalent to ‘- P v Q’. By the same token, the wff ‘(x)(Fx → Gx)’ is logically equivalent to ‘(x)(- Fx v Gx)’. There are no rules of derivation that allow us to directly make this move in a proof, but we can look and see that it is so, and this foreknowledge might help us in doing some proofs or translations. Similar comments apply to other sequents on the sequent sheet. There are also equivalences within the predicate calculus that we may be aware of and exploit for some purposes, even though ***we are not allowed to directly make these moves in a proof***. In addition to the one just given, here are some more useful equivalences that might be worth learning:

P → Q ┤├ - Q → - P	(sequent 9+)
(P & Q) → R ┤├ P → (Q → R)	(sequent 30)
(x)(Fx → Gx) ┤├ -(∃x)(Fx & -Gx)	(sequent 117)
(x)(Fx → - Gx) ┤├ -(∃x)(Fx & Gx)	(3.4.1 e)
(x)(Fx → P) ┤├ (∃x)Fx → P	(sequent 118)
(∃x)(Fx → P) ┤├ (x)Fx → P	(3.4.3 f)
(∃x)(P → Fx) ┤├ P → (∃x)Fx	(sequent 119)
(x)(P → Fx) ┤├ P → (x)Fx	(3.4.3 a)
(x)(y)Fxy ┤├ (y)(x)Fxy	(sequent 120)
(∃x)(∃y)Fxy ┤├ (∃y)(∃x)Fxy	(sequent 121)
(∃x)(P & Fx) ┤├ P & (∃x)Fx	(3.4.3 c)
(x)-Fx ┤├ - (∃x)Fx	(3.4.1 c)
(∃x)-Fx ┤├ - (x)Fx	(3.4.1 d)
(∃x)Fx ┤├ - (x)-Fx	(sequent 113)
(x)Fx ┤├ - (∃x)-Fx	(sequent 114)

In normal English usage, existentially quantified conditionals do not occur. To show an example of what happens when there is an existentially quantified conditional, consider the Drunkard’s Theorem: There is someone such that, if she is drunk, then everyone is drunk.
F: is drunk
(∃x)(Fx → (y)Fy)
This is a theorem of logic. Perhaps it will be easy to see why if we consider what we know to be a logically equivalent wff:
(∃x)(-Fx v (y)Fy)
On this rendering, it should be clear. There is someone such that, either she is not drunk, or else everyone is drunk. In other words, if there is no one who is not drunk, then everyone is drunk.

Notice that the English version of the Drunkard's Theorem is not a sentence that is readily understood by English speakers. This goes to show that existentially quantified conditionals, where the variable occurs in the antecedent, are not regular English expressions.

Identity– finally, let us notice that Lemmon expands the predicate calculus by adding a special predicate and two new rules (p. 161). The predicate is called 'identity' and is symbolized by the equals sign. (See separate handout for the two identity rules.)

Unlike predicate-letters, which appear before their terms or variables, the identity sign is placed between terms or variables (this is known as *inline notation*). So, instead of writing 'm is identical to n' as something like 'Emn' or '=mn', it is written as 'm=n'. There is also a special symbol for non-identity. When symbolizing 'm is not identical to n', instead of something like '-Emn' or '- =mn' or even '-m=n', we write 'm≠n'. Note that 'm=n' is an atomic sentence, and 'm≠n' is the negation of an atomic sentence. <u>For the purposes of this course, these kinds of expressions are written without brackets around them</u> (in the same way that we don't put brackets around 'Fmn' or '-Fmn').

The identity predicate allows us to symbolize statements about numbers of things. The existential quantifier says that there is at least one thing, but with the identity predicate we may also say that there is at most one thing, exactly one thing, at least two things, at most two things, exactly two things, and so on for higher numbers. Here are some examples adapted from Marc Cohen:

F: is a frog	
There is at least one frog.	$(\exists x)Fx$
There is at most one frog.	$(x)(y)((Fx\&Fy) \rightarrow x=y)$
There is exactly one frog.	$(\exists x)(y)(Fy \leftrightarrow y=x)$
There are at least two frogs.	$(\exists x)(\exists y)((Fx \& Fy) \& x \neq y)$
There are at most two frogs.	$(x)(y)(z)((((Fx \& Fy) \& Fz)) \rightarrow ((x=y \text{ v } y=z) \text{ v } x=z))$
There are exactly two frogs.	$(\exists x)(\exists y)(x \neq y \& (z)(Fz \leftrightarrow (z=x \text{ v } z=y)))$
There are at least three frogs.	$(\exists x)(\exists y)(\exists z)(((Fx \& Fy) \& Fz)\& ((x \neq y \& x \neq z) \& y \neq z))$
And so on	