

Why study logic? Some general remarks

Argument (p.1) – a collection of statements, including premises (zero or more) and one conclusion; the conclusion is supposed to follow from the premises

Statements are true or false; arguments are valid or invalid (pp. 2, 52). An argument is valid if and only if it is impossible for all its premises to be true and its conclusion false.

Logic defined (see p. 5) – we will use symbols

The Propositional Calculus (the focus of Ch. 1-2)

Definitions (pp. 43-4)	Two kinds of <u>brackets</u>	Five <u>logical connectives</u>	Infinitely many <u>propositional variables</u>
	Left and right	Arrow, negation, ampersand, wedge, and double-arrow	Capital letters, beginning with ‘P’
	‘(’, ‘)’	‘ \rightarrow ’, ‘ \neg ’, ‘ $\&$ ’, ‘ \vee ’, ‘ \leftrightarrow ’	‘P’, ‘Q’, ‘R’, ‘S’ etc.
These are the symbols of the propositional calculus (no more, no less)			
A formula of the propositional calculus is any sequence of these symbols			

Propositional variables stand in for statements. **Metalogical variables** (capital letters A and B, with or without subscripts: ‘A’, ‘A₁’, ‘A₂’, ... ‘A_n’, ‘B’, etc.), stand in for symbols or sequences of symbols in the propositional calculus (p. 49).

A **well-formed formula** (wff) of the propositional calculus is defined by the following seven clauses (pp. 44-5, 49):

- 1) Any propositional variable is a wff;
- 2) If A is a wff, then $\neg A$ is a wff;
- 3) If A and B are wffs, then $(A \rightarrow B)$ is a wff;
- 4) If A and B are wffs, then $(A \& B)$ is a wff;
- 5) If A and B are wffs, then $(A \vee B)$ is a wff;
- 6) If A and B are wffs, then $(A \leftrightarrow B)$ is a wff;
- 7) If a formula is not a wff in virtue of those 6 clauses, then it is not a wff.

It is OK (as a practical matter) to drop outermost brackets (p. 46), but only if those brackets are paired with each other as the first and last symbols in the wff. For the purposes of this course, no other brackets may be dropped.

Sequent (p.12) – an argument-frame containing a (possibly empty) set of assumptions and one conclusion which is claimed to follow from them

Assertion-sign (pp. 11, 48): ‘ \vdash ’

Sequent-expression (pp. 48-9) – any finite number of wffs (including zero), with commas between them, followed by the assertion-sign, followed by a wff

$A_1, A_2, \dots A_n \vdash B$

(Note that metalogical variables, commas, and assertion-signs are not symbols of the propositional calculus. In other words, they do not appear in any formula or wff.)

Scope (p. 47) – the scope of an occurrence of a connective in a wff is the shortest wff in which that occurrence appears

Main connective (p. 48) – the connective in a wff whose scope is the whole wff

Truth-values (pp. 64-5) – true and false, symbolized ‘T’ and ‘F’ – the only two possibilities - assigned to propositional variables and connectives in wffs

Connectives defined by truth-tables (p.65) (memorize – compare to multiplication tables):

Negation				Arrow					Ampersand					Wedge							
P		-	P	P	Q		P	\rightarrow	Q	P	Q		P	&	Q	P	Q		P	\vee	Q
T			F	T	T		T	T	T	T	T		T	T	T	T	T		T	T	T
T			T	T	F		T	F	F	T	F		T	F	F	T	F		T	T	F
F			T	F	T		F	T	T	F	T		F	F	T	F	T		F	T	T
F			F	F	F		F	T	F	F	F		F	F	F	F	F		F	F	F

Double-Arrow					
P	Q		P	\leftrightarrow	Q
T	T		T	T	T
T	F		T	F	F
F	T		F	F	T
F	F		F	T	F

The five connectives are examples of truth-functions (pp. 70-1, 82). We will use them to translate English expressions, but they do not correspond exactly. There are, for example, paradoxes of material implication (pp. ix, 60, 82).

Negation	- P	Not P; It is not the case that P		
Arrow	$P \rightarrow Q$ (conditional statement)	If P, then Q; P only if Q; Q if P; P is a sufficient condition for Q; Q is a necessary condition for P	P is the antecedent Q is the consequent	Material Implication
Ampersand	$P \& Q$ (conjunction)	P and Q; Both P and Q; P but Q; Etc.	P and Q are conjuncts	
Wedge	$P \vee Q$ (disjunction)	P or Q; Either P or Q; P unless Q; Unless P, Q	P and Q are disjuncts	Inclusive disjunction
Double-Arrow	$P \leftrightarrow Q$ (biconditional)	P if and only if Q; P just in case Q; P is a necessary and sufficient condition for Q		Material equivalence

- ‘- (P \vee Q)’ may be translated as ‘Neither P nor Q.’
- ‘- P \vee - Q’ may be translated as ‘(Either) Not P or not Q.’
- ‘- (P $\&$ Q)’ may be translated as ‘Not both P and Q.’
- ‘- P $\&$ - Q’ may be translated as ‘(Both) Not P and not Q.’

Truth-table test (pp. 66, 75) – an evaluation of a wff or sequent-expression for every possible assignment of truth-values to its variables

If there are n propositional variables, there will be 2^n rows in the truth-table, with truth-values assigned in a standard order (pp. 66-7). The main column of the truth-table is beneath the main connective (p. 66), or else beneath the only symbol of the wff if the wff has no connective.

Classification of wffs in the propositional calculus (p. 68)

- 1) Tautologous – a wff that has nothing but T under its main connective
- 2) Inconsistent – a wff that has nothing but F under its main connective
- 3) Contingent – a wff that has at least one T and at least one F under its main connective
- 4) Consistent – a wff that is either tautologous or contingent (at least one T)

Seven logical relationships between two wffs A and B in the propositional calculus (pp. 69-70)

- 1) A and B are contrary if they never both have T on the same row in their main columns
- 2) A and B are subcontrary if they never both have F on the same row in their main columns
- 3) A implies B if there is no row where A has T in its main column where B has F
- 4) A is implied by B if there is no row where B has T in its main column where A has F

- 5) A and B are equivalent if their main columns match exactly
- 6) A and B are contradictory if their main columns disagree in truth-value on every row
- 7) A and B are independent if none of those six relations holds between them

Tautologous sequent-expression (p. 75) corresponds to ***valid argument*** – there is no assignment of truth-values for which all the premises are true and the conclusion false. An argument is ***invalid*** (at the level of the propositional calculus) if there is some assignment for which all the premises are true and the conclusion is false (pp. 81-2).

The classification of wffs or of their relationships can sometimes determine validity.

If there are no premises, then all the premises are true. If A is a tautologous wff, then ‘ $\vdash A$ ’ is a valid sequent-expression. If A is a non-tautologous wff, then ‘ $\vdash A$ ’ represents an invalid argument (at the level of the propositional calculus).

Every sequent-expression has a corresponding conditional (p. 76). A sequent-expression represents a valid argument (at the level of the propositional calculus) if and only if its corresponding conditional is tautologous.

Sequent-Expression	Corresponding Conditional
$A_1 \vdash B$	$A_1 \rightarrow B$
$A_1, A_2 \vdash B$	$A_1 \rightarrow (A_2 \rightarrow B)$
$A_1, A_2, A_3 \vdash B$	$A_1 \rightarrow (A_2 \rightarrow (A_3 \rightarrow B))$
$A_1, A_2, A_3, A_4 \vdash B$	$A_1 \rightarrow (A_2 \rightarrow (A_3 \rightarrow (A_4 \rightarrow B)))$
And so on	

Rules of derivation (p. 8) may be formulated, where each rule authorizes derivations from a (possibly empty) set of premises to a conclusion. Any system of such rules may be either safe or unsafe. A ***safe*** system of rules authorizes the derivation of no invalid arguments, while an ***unsafe*** system does authorize the derivation of at least one invalid argument (p. 75).

Similarly, a system of rules may be either complete or incomplete, with respect to a specified kind of expression (we will specify *valid sequent-expressions*). A ***complete*** system of rules authorizes the derivation of every valid argument, while an ***incomplete*** system does not authorize the derivation of every valid argument (p. 90).

A ***theorem*** is a wff that is the conclusion of a derivable sequent (according to some system of rules of derivation), where the number of assumptions is zero (p. 50). In other words, if ‘ $\vdash A$ ’ represents a derivable sequent, then wff A is a theorem.

When a system of rules of derivation for the propositional calculus is safe and complete, then every derivable sequent is valid, and every valid sequent is derivable. There are many possible systems of rules that are safe and complete. Lemmon proves (pp. 75-91) that the system of rules he has adopted is safe and complete. In the next part of this course, we will turn to Lemmon’s system of rules of derivation for the propositional calculus.

Some Useful Logical Info

An argument is valid if and only if it is impossible for all of its premises to be true while its conclusion is false.

If A and B are wffs, then ...

Wffs of the following forms are tautologous:

$A \vee \neg A$	$A \rightarrow A$	$\neg (A \& \neg A)$	$(A \rightarrow B) \vee (B \rightarrow A)$
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(Note that every tautologous wff is implied by any and every wff. Furthermore, every argument that has a tautologous wff as conclusion is valid.)

Wffs of the following forms are inconsistent:

$A \& \neg A$	$\neg (A \vee \neg A)$	$A \leftrightarrow \neg A$
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(Note that every inconsistent wff implies any and every wff. Furthermore, every argument that has an inconsistent wff as a premise is valid. By the same token, any argument with contrary or contradictory premises is valid.)

Pairs of wffs of the following forms (separated by commas) are logically equivalent:

$A, \neg \neg A$	$A \vee B, \neg A \rightarrow B$	$A \rightarrow B, \neg A \vee B$	$\neg (A \rightarrow B), A \& \neg B$
$\neg (A \& B), \neg A \vee \neg B$	$\neg (A \vee B), \neg A \& \neg B$	$A \rightarrow B, \neg B \rightarrow \neg A$	

Sequents of the following forms are tautologous (i.e. they correspond to valid arguments):

$A \& \neg A \vdash B$	$\neg A \vdash A \rightarrow B$	$B \vdash A \rightarrow B$
$A \vee B, \neg A \vdash B$	$\neg (A \& B), A \vdash \neg B$	$A \rightarrow B, A \vdash B$
$A \rightarrow B, \neg B, \vdash \neg A$	$A \vdash A \vee B$	

In a safe system of rules, every derivable sequent is valid (i.e. no invalid sequent is derivable). If an invalid sequent is derivable according to some system of rules, then that system is unsafe.

In a complete system of rules, every valid sequent is derivable. If a valid sequent is not derivable according to some system of rules, then that system is incomplete.

In a safe and complete system of rules, every derivable sequent is valid and every valid sequent is derivable.